SYNTHETIC CATEGORY THEORY - EXERCISE SHEET 1

October 14, 2026

Definition. A functor $F\colon \mathcal{C} \to \mathcal{D}$ between strict categories is **fully faithful** if, for all objects $x, y \in \mathcal{C}$, the induced function

$$\mathcal{C}(x, y) \to \mathcal{D}(Fx, Fy)$$

is a bijection of sets.

Exercise 1.1 (Isomorphisms)**.** Let $\mathcal{C}$ and $\mathcal{D}$ be strict categories.

(a) Prove that any functor $F\colon \mathcal{C} \to \mathcal{D}$ preserves isomorphisms.

(b) Prove that, if $F\colon \mathcal{C} \to \mathcal{D}$ is a fully faithful functor, then F also reflects isomorphisms. In this case, a morphism f is an isomorphism in $\mathcal{C}$ if and only if Ff is an isomorphism in $\mathcal{D}$.

(c) Prove that a morphism $f\colon c \to d$ in $\mathcal{C}$ is an isomorphism if and only if the corresponding morphism $f^{\mathrm{op}}\colon d \to c$ is an isomorphism in $\mathcal{C}^{\mathrm{op}}$.

(d) Construct the strict category $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ of functors $\mathcal{C} \to \mathcal{D}$ and natural transformations between them.

(e) Prove that a morphism in $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ is an isomorphism if and only if it is a natural isomorphism.

(f) Prove that a functor $F\colon \mathcal{C} \to \mathcal{D}$ is an isomorphism in $\mathcal{CAT}_s$ if and only if it is fully faithful and the function $\mathrm{Ob}(\mathcal{C}) \to \mathrm{Ob}(\mathcal{D})$ is a bijection.

Exercise 1.2 (Posets)**.** Let $(\mathcal{P}, \leq)$ and $(\mathcal{Q}, \leq)$ be posets.

(a) Construct a strict category, also denoted by $\mathcal{P}$, whose objects are the elements of $\mathcal{P}$ and whose morphisms are induced by the relations $x \leq y$ in $\mathcal{P}$.

(b) Prove that functors $\mathcal{P} \to \mathcal{Q}$ are in one-to-one correspondence with order-preserving functions $\mathcal{P} \to \mathcal{Q}$.

(c) Determine when a natural transformation between two functors $\mathcal{P} \to \mathcal{Q}$ exists, and whether it is unique.

(d) Consider the poset $[n] \coloneqq \{0 < 1 < \ldots < n\}$ for $n \in \mathbb{N}$ and a strict category $\mathcal{C}$. Describe functors $[0] \to \mathcal{C}$, $[1] \to \mathcal{C}$, $[2] \to \mathcal{C}$, and natural transformations between them in terms of objects and morphisms in $\mathcal{C}$.

(e) Prove that there is an isomorphism of categories $\mathrm{Fun}([1], [1]) \cong [2]$.

Exercise 1.3 (Smallness)**.** Prove that the strict categories $\mathcal{S}et$, $\mathcal{S}et_\bullet$, $\mathcal{T}op$, and $\mathcal{C}at_s$ are locally small.

Exercise 1.4 (Yoneda lemma)**.** Let $\mathcal{C}$ be a locally small strict category.

(a) Prove the **Yoneda lemma** for $\mathcal{C}$: For all objects $c \in \mathcal{C}$ and functors $F\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{S}et$, there is a bijection of sets

$$\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}et)(\mathcal{C}(-, c), F) \cong Fc$$

which is natural in both c and F.

(b) Construct a functor $\mathbf{y}_{\mathcal{C}}\colon \mathcal{C} \to \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}et)$, called the **Yoneda functor**, sending an object $c \in \mathcal{C}$ to its representable functor $\mathcal{C}(-, c)\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{S}et$.

(c) Prove that the Yoneda functor $\mathbf{y}_{\mathcal{C}}\colon \mathcal{C} \to \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}et)$ is fully faithful.