SYNTHETIC CATEGORY THEORY - EXERCISE SHEET 2

October 21, 2026

Exercise 2.1 (Opposite category)**.**

(a) Construct a functor $(-)^{\mathrm{op}} \colon \mathcal{CAT}_s \to \mathcal{CAT}_s$ sending a strict category $\mathcal{C}$ to its opposite $\mathcal{C}^{\mathrm{op}}$.

(b) Prove that, for every strict category $\mathcal{C}$, we have

$$(\mathcal{C}^{\mathrm{op}})^{\mathrm{op}} = \mathcal{C}.$$

(c) Prove that a natural transformation $\alpha \colon F \Rightarrow G$ between two functors $F, G \colon \mathcal{C} \to \mathcal{D}$ induces a natural transformation $\alpha^{\mathrm{op}} \colon G^{\mathrm{op}} \Rightarrow F^{\mathrm{op}}$ between the functors $G^{\mathrm{op}}, F^{\mathrm{op}} \colon \mathcal{C}^{\mathrm{op}} \to \mathcal{D}^{\mathrm{op}}$.

Exercise 2.2 (Adjunctions)**.**

(a) Prove that the following adjunctions hold:

- the free-forgetful adjunctions $(-)_+ \colon \mathcal{S}et \rightleftarrows \mathcal{S}et_\bullet \,: U$ and $\bigoplus_{(-)} k \colon \mathcal{S}et \rightleftarrows \mathcal{V}ect_k \,: U$;
- the discrete–forgetful–indiscrete adjunctions $D \colon \mathcal{S}et \rightleftarrows \mathcal{T}op \,: U$ and $U \colon \mathcal{T}op \rightleftarrows \mathcal{S}et \,: I$.

(b) If $L \colon \mathcal{C} \rightleftarrows \mathcal{D} \,: R$ is an adjunction between strict categories with unit $\eta \colon \mathrm{id}_{\mathcal{C}} \Rightarrow RL$ and counit $\varepsilon \colon LR \Rightarrow \mathrm{id}_{\mathcal{D}}$, then $R^{\mathrm{op}} \colon \mathcal{D}^{\mathrm{op}} \rightleftarrows \mathcal{C}^{\mathrm{op}} \,: L^{\mathrm{op}}$ is an adjunction with unit $\varepsilon^{\mathrm{op}} \colon \mathrm{id}_{\mathcal{D}^{\mathrm{op}}} \Rightarrow L^{\mathrm{op}} R^{\mathrm{op}}$ and counit $\eta^{\mathrm{op}} \colon R^{\mathrm{op}} L^{\mathrm{op}} \Rightarrow \mathrm{id}_{\mathcal{C}^{\mathrm{op}}}$.

(c) If $L \colon \mathcal{C} \rightleftarrows \mathcal{D} \,: R$ is an adjunction between strict categories, prove the following:

(1) the left adjoint L is fully faithful if and only if the unit $\eta \colon \mathrm{id}_{\mathcal{C}} \Rightarrow RL$ is a natural isomorphism;

(2) the right adjoint R is fully faithful if and only if the counit $\varepsilon \colon LR \Rightarrow \mathrm{id}_{\mathcal{D}}$ is a natural isomorphism. **Hint:** Deduce (2) from (1) using (b)

(d) Which of the functors in (a) are fully faithful?

Definition. A **strict groupoid** is a strict category $\mathcal{G}$ in which every morphism is an isomorphism.

Exercise 2.3 (Equivalences)**.** We will study several properties of strict categories and determine whether they are equiv-invariant or equiv-sensitive.

(a) Prove that a strict category $\mathcal{C}$ is equivalent to a strict groupoid if and only if it is a strict groupoid.

(b) Prove that a strict category $\mathcal{C}$ is equivalent to a poset if and only if, for all objects $x, y \in \mathcal{C}$, the set of morphisms $\mathcal{C}(x, y)$ is either empty or a singleton.

(c) Give an example of a strict category that is equivalent to the poset $[1] = \{0 < 1\}$ but is not a poset.

(d) Prove that a strict groupoid $\mathcal{G}$ is equivalent to a set if and only if, for all objects $x, y \in \mathcal{G}$, the set of morphisms $\mathcal{G}(x, y)$ is either empty or a singleton.

(e) Give an example of a strict groupoid that is equivalent to a set but is not a set.

(f) Prove that a strict category $\mathcal{C}$ is equivalent to $\emptyset$ if and only if $\mathcal{C} = \emptyset$.

Exercise 2.4 (Binary products)**.** Let $\mathcal{C}$ be a strict category and $c, d \in \mathcal{C}$ be objects. Prove that the following hold.

(a) (Uniqueness) Binary products are unique up to a unique isomorphism commuting with the canonical projections, i.e., if $t, t' \in \mathcal{C}$ are products of c and d with canonical projections $\pi_c \colon t \to c$, $\pi_d \colon t \to d$ and $\pi'_c \colon t' \to c$, $\pi'_d \colon t' \to d$, there is a unique isomorphism $\varphi \colon t \cong t'$ such that $\pi'_c \circ \varphi = \pi_c$ and $\pi'_d \circ \varphi = \pi_d$.

(b) (Representability) An object $t \in \mathcal{C}$ is a product of c and d if and only if, for every object $x \in \mathcal{C}$, there is a bijection

$$\mathcal{C}(x,t) \cong \mathcal{C}(x,c) \times \mathcal{C}(x,d)$$

which is natural in x.

(c) (Products with terminal objects) If d is terminal in $\mathcal{C}$, then c is a product of c and d.

(d) (Iso-invariance) If t is a product of c and d in $\mathcal{C}$ and $t \cong t'$ in $\mathcal{C}$, then t' is a product of c and d in $\mathcal{C}$.

(e) (Equiv-invariance) Equivalences preserve and reflect binary products, i.e., if $F\colon \mathcal{C} \xrightarrow{\simeq} \mathcal{D}$ is an equivalence between strict categories, then an object $t \in \mathcal{C}$ is a product of c and d if and only if $Ft \in \mathcal{D}$ is a product of Fc and Fd.

(f) Given strict categories $\mathcal{C}$ and $\mathcal{D}$, construct a strict category $\mathcal{C} \times \mathcal{D}$ satisfying the universal property of the product of $\mathcal{C}$ and $\mathcal{D}$ in $\mathcal{CAT}_s$ and a strict category $\mathcal{C} \amalg \mathcal{D}$ satisfying the universal property of the coproduct of $\mathcal{C}$ and $\mathcal{D}$ in $\mathcal{CAT}_s$.