SYNTHETIC CATEGORY THEORY - EXERCISE SHEET 3

October 28, 2026

Exercise 3.1 (Diagrammatic pullbacks)**.** For every strict category $\mathcal{C}$, prove that the following commutative squares in $\mathcal{CAT}_s$ are pullback squares

$$\begin{array}{ccc} \mathrm{Fun}([2],\mathcal{C}) & \xrightarrow{\langle 0,1\rangle^*} & \mathrm{Fun}([1],\mathcal{C}) \\ {\scriptstyle \langle 1,2\rangle^*}\downarrow & \lrcorner & \downarrow{\scriptstyle \langle 1\rangle^*} \\ \mathrm{Fun}([1],\mathcal{C}) & \xrightarrow[\langle 0\rangle^*]{} & \mathrm{Fun}([0],\mathcal{C}) \cong \mathcal{C} \end{array} \qquad \begin{array}{ccc} \mathrm{Fun}([1]\times[1],\mathcal{C}) & \xrightarrow{\langle (0,0),(0,1),(1,1)\rangle^*} & \mathrm{Fun}([2],\mathcal{C}) \\ {\scriptstyle \langle (0,0),(1,0),(1,1)\rangle^*}\downarrow & \lrcorner & \downarrow{\scriptstyle \langle 0,2\rangle^*} \\ \mathrm{Fun}([2],\mathcal{C}) & \xrightarrow[\langle 0,2\rangle^*]{} & \mathrm{Fun}([1],\mathcal{C}) \end{array}$$

The above squares are induced by precomposition with the following commutative squares

$$\begin{array}{ccc} [0] & \xrightarrow{\langle 1\rangle} & [1] \\ {\scriptstyle \langle 0\rangle}\downarrow & & \downarrow{\scriptstyle \langle 0,1\rangle} \\ [1] & \xrightarrow[\langle 1,2\rangle]{} & [2] \end{array} \qquad \begin{array}{ccc} [1] & \xrightarrow{\langle 0,2\rangle} & [2] \\ {\scriptstyle \langle 0,2\rangle}\downarrow & & \downarrow{\scriptstyle \langle (0,0),(0,1),(1,1)\rangle} \\ [2] & \xrightarrow[\langle (0,0),(1,0),(1,1)\rangle]{} & [1]\times[1] \end{array}$$

where we use $\langle f(0),\dots,f(n)\rangle$ to denote an order-preserving map of posets $f\colon [n]\to\mathcal{P}$.

Exercise 3.2 (Pullback cancellation)**.** Let $\mathcal{C}$ be a strict category and consider a commutative diagram in $\mathcal{C}$,

$$\begin{array}{ccccc} s & \xrightarrow{q} & t & \xrightarrow{\pi_d} & d \\ {\scriptstyle p}\downarrow & & {\scriptstyle \pi_c}\downarrow & \lrcorner & \downarrow{\scriptstyle h} \\ b & \xrightarrow[f]{} & c & \xrightarrow[g]{} & e \end{array}$$

where the right-hand square is a pullback square. Prove that the left-hand square is a pullback square if and only if the outer rectangle is a pullback square.

Exercise 3.3 (Localizations)**.** Let $\mathcal{C}$ be a strict category.

(a) Let $W\subseteq \mathcal{CAT}_s([1],\mathcal{C})$ be a subset of the morphisms of $\mathcal{C}$ containing the identity morphisms of $\mathcal{C}$ and closed under composition. Prove that the strict category $\mathcal{C}[W^{-1}]$ together with the canonical functor $\gamma\colon \mathcal{C}\to\mathcal{C}[W^{-1}]$ satisfies the universal property of the localization of $\mathcal{C}$ at W.

(b) Let $W=\mathcal{CAT}_s([1],\mathcal{C})$ be the set of morphisms in $\mathcal{C}$. Prove that $\Pi\mathcal{C}\coloneqq\mathcal{C}[W^{-1}]$ is a strict groupoid.

Exercise 3.4 (Strict groupoids)**.** Prove that the following adjunctions hold

(a) the underlying groupoid functor $(-)^{\simeq}\colon \mathcal{C}at_s\to\mathcal{G}pd_s$ is a right adjoint to the inclusion $\iota\colon \mathcal{G}pd_s\hookrightarrow\mathcal{C}at_s$;

(b) the groupoidification functor $\Pi\colon \mathcal{C}at_s\to\mathcal{G}pd_s$ is a left adjoint to the inclusion $\iota\colon \mathcal{G}pd_s\hookrightarrow\mathcal{C}at_s$.

Hint: Use the pointwise criterion for adjunctions and its dual version to prove that $\iota\colon \mathcal{G}pd_s\hookrightarrow\mathcal{C}at_s$ has both a right and a left adjoint.

Exercise 3.5 (Representation theorem)**.** Let $\mathcal{C}$ be a locally small strict category.

(a) Prove that, for an object $c\in\mathcal{C}$, the object $(c,\mathrm{id}_c)\in\mathcal{C}_{/c}$ is terminal.

(b) Prove that a functor $F\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{S}et$ is representable if and only if the strict category $\int^{\mathcal{C}} F$ has a terminal object.

(c) Prove that, for objects $c, d \in \mathcal{C}$, the pullback $\mathcal{C}_{/c} \times_{\mathcal{C}} \mathcal{C}_{/d}$ has a terminal object if and only if the product of c and d exists in $\mathcal{C}$.

(d) Suppose that $\mathcal{C}$ has all binary products. Construct the product functor $- \times -\colon \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ as the right adjoint of the diagonal functor $\mathrm{diag}\colon \mathcal{C} \to \mathcal{C} \times \mathcal{C}$ using the pointwise formula for adjunctions.