SYNTHETIC CATEGORY THEORY - EXERCISE SHEET 4

November 4, 2026

Definition. A functor $P\colon \mathcal{E} \to \mathcal{C}$ between strict categories is a **discrete left fibration** if, for every object $x \in \mathcal{E}$ and every morphism $f\colon Px \to c$ in $\mathcal{C}$, there is a unique morphism $\overline{f}\colon x \to y$ in $\mathcal{E}$ such that $P\overline{f} = f$.

Exercise 4.1 (Discrete left fibrations)**.** Let $\mathcal{C}$ be a strict category.

(a) Prove that a functor $P\colon \mathcal{E} \to \mathcal{C}$ is a discrete left fibration if and only if the functor $P^{\mathrm{op}}\colon \mathcal{E}^{\mathrm{op}} \to \mathcal{C}^{\mathrm{op}}$ is a discrete right fibration.

(b) Deduce that the opposite isomorphism $(-)^{\mathrm{op}}\colon \mathcal{CAT}_{s/\mathcal{C}^{\mathrm{op}}} \xrightarrow{\cong} \mathcal{CAT}_{s/\mathcal{C}}$ restricts to an isomorphism

$$(-)^{\mathrm{op}}\colon \mathcal{DRF}ib(\mathcal{C}^{\mathrm{op}}) \xrightarrow{\cong} \mathcal{DLF}ib(\mathcal{C}),$$

where $\mathcal{DLF}ib(\mathcal{C})$ is the full strict subcategory of $\mathcal{CAT}_{s/\mathcal{C}}$ spanned by the discrete left fibrations with small fibers.

(c) Let the covariant Grothendieck construction be the composite

$$\textstyle\int_{\mathcal{C}}\colon \mathrm{Fun}(\mathcal{C}, \mathcal{S}et) = \mathrm{Fun}((\mathcal{C}^{\mathrm{op}})^{\mathrm{op}}, \mathcal{S}et) \xrightarrow{\int^{\mathcal{C}^{\mathrm{op}}}} \mathcal{CAT}_{s/\mathcal{C}^{\mathrm{op}}} \xrightarrow{(-)^{\mathrm{op}}} \mathcal{CAT}_{s/\mathcal{C}}$$

Describe, for every functor $F\colon \mathcal{C} \to \mathcal{S}et$, the strict category $\int_{\mathcal{C}} F$ and its projection $\pi_F\colon \int_{\mathcal{C}} F \to \mathcal{C}$.

(d) Deduce from the fact that $\int^{\mathcal{C}^{\mathrm{op}}}\colon \mathrm{Fun}((\mathcal{C}^{\mathrm{op}})^{\mathrm{op}}, \mathcal{S}et) \xrightarrow{\simeq} \mathcal{DRF}ib(\mathcal{C}^{\mathrm{op}})$ is an equivalence and part (b) that $\int_{\mathcal{C}}$ induces an equivalence $\int_{\mathcal{C}}\colon \mathrm{Fun}(\mathcal{C}, \mathcal{S}et) \xrightarrow{\simeq} \mathcal{DLF}ib(\mathcal{C})$.

(e) Prove that the forgetful functor $U\colon \mathcal{S}et_\bullet \to \mathcal{S}et$ is the *universal discrete left fibration*: it is a discrete left fibration and, for every functor $F\colon \mathcal{C} \to \mathcal{S}et$, there is a pullback square in $\mathcal{CAT}_s$:

$$\begin{array}{ccc} \int_{\mathcal{C}} F & \longrightarrow & \mathcal{S}et_\bullet \\ {\scriptstyle \pi_F}\downarrow & \lrcorner & \downarrow{\scriptstyle U} \\ \mathcal{C} & \xrightarrow[F]{} & \mathcal{S}et \end{array}$$

Exercise 4.2 (Cartesian morphisms)**.** Let $P\colon \mathcal{E} \to \mathcal{C}$ be a functor between strict categories. Prove that:

(a) The composite of two P-cartesian morphisms is P-cartesian.

(b) A morphism f in $\mathcal{E}$ is an isomorphism if and only if it is P-cartesian and Pf is an isomorphism in $\mathcal{C}$.

(c) If $f\colon x \to z$ and $g\colon y \to z$ are two P-cartesian morphisms in $\mathcal{E}$ such that $Pf = Pg$, then there is a unique isomorphism $\varphi\colon x \xrightarrow{\cong} y$ in $\mathcal{E}$ such that $P\varphi = \mathrm{id}_{Px}$ and $g\varphi = f$.

(d) Let $F\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{C}at_s$ be a functor between strict categories and let $\pi^F\colon \int^{\mathcal{C}} F \to \mathcal{C}$ denote its Grothendieck construction. Prove that a morphism $(f, u)\colon (c, x) \to (d, y)$ is π^F-cartesian if and only if $u\colon x \to Ff(y)$ is an isomorphism in Fc.

Exercise 4.3 (Functoriality of universals)**.** Let $P\colon \mathcal{E} \to \mathcal{C}$ be a cartesian fibration. Suppose that, for every object $c \in \mathcal{C}$, the fiber $\mathcal{E}_c$ has an initial object. Prove that P admits a left adjoint section $S\colon \mathcal{C} \to \mathcal{E}$.

Exercise 4.4 (Slices)**.** Let $\mathcal{C}$ be a strict category with all pullbacks and a terminal object $t \in \mathcal{C}$. Prove that:

(a) there is an isomorphism of strict categories $\mathcal{C}_{/t} \cong \mathcal{C}$;

(b) for every morphism $f\colon c \to d$ in $\mathcal{C}$, there is an adjunction

$$f_! \colon \mathcal{C}_{/c} \rightleftarrows \mathcal{C}_{/d} : f^*,$$

where $f_!$ is obtained by postcomposing with f and f^* by taking the pullback along f;

(c) for every object $c \in \mathcal{C}$, there is an adjunction

$$\pi_c \colon \mathcal{C}_{/c} \rightleftarrows \mathcal{C} : - \times c.$$

Hint: Apply (b) by taking f to be the unique morphism $!\colon c \to t$ and use (a).