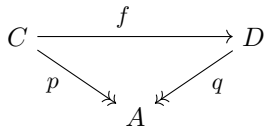
SYNTHETIC CATEGORY THEORY - EXERCISE SHEET 5

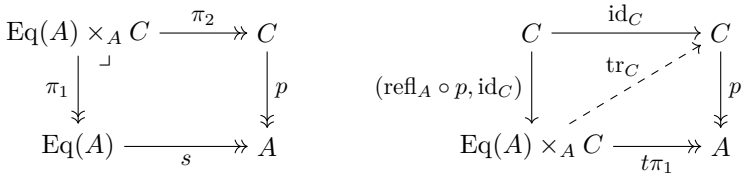
November 11, 2026

Exercise 5.1 (Fiberwise equivalences)**.** Let $p\colon C \twoheadrightarrow A$ and $q\colon D \twoheadrightarrow A$ be family projections and f be a functor over A.



The goal of this exercise is to prove that f is an equivalence over A if and only if it is an equivalence.

(a) Consider the pullback as below left. Construct a lift $\mathrm{tr}_C\colon \mathrm{Eq}(A) \times_A C \to C$, called the **transport**, in the below right commutative square.



(b) The identity $\mathrm{id}_{\mathrm{Eq}(C)}\colon \mathrm{Eq}(C) \to \mathrm{Eq}(C)$ is a homotopy $\mathrm{id}_{\mathrm{Eq}(C)}\colon s \sim t$, which cannot be a homotopy relative to A since $ps \neq pt$ in general. We rectify the source functor: let $\overline{s}$ be the composite

$$\overline{s}\colon \mathrm{Eq}(C) \xrightarrow{(\mathrm{Eq}(p),s)} \mathrm{Eq}(A) \times_A C \xrightarrow{\mathrm{tr}_C} C.$$

Verify that $p\overline{s} = pt$, and construct a relative homotopy $\mathtt{rect}\colon \overline{s} \sim_A t$ between functors $\mathrm{Eq}(C) \to C$.

(c) Construct a relative homotopy $\mathtt{nat}_f\colon f \circ \mathrm{tr}_C \sim_A \mathrm{tr}_D \circ (\mathrm{Eq}(A) \times_A f)$ between functors $\mathrm{Eq}(A) \times_A C \to D$.

(d) Let $g\colon D \to C$ be a functor and $\alpha\colon fg \sim \mathrm{id}_D$ be a homotopy. Prove that there is a functor $\overline{g}\colon D \to C$ over A, i.e., $p\overline{g} = q$, and a relative homotopy $\overline{\alpha}\colon f\overline{g} \sim_A \mathrm{id}_D$.

Hint: Take $\overline{g}$ to be the composite

$$\overline{g}\colon D \xrightarrow{(q\alpha,g)} \mathrm{Eq}(A) \times_A C \xrightarrow{\mathrm{tr}_C} C.$$

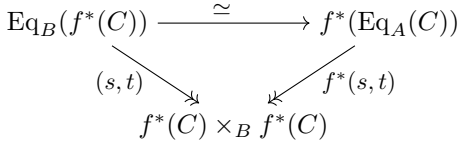
(e) Prove that f is an equivalence over A if and only if f is an equivalence.

Hint: For the converse direction, use (d) twice.

Exercise 5.2 (Pullbacks of equivalences)**.** Prove the following results.

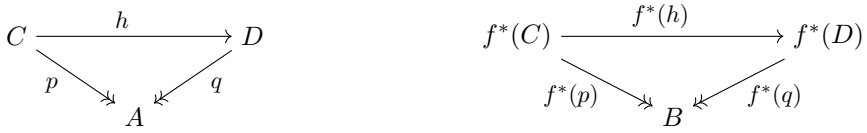
(a) Every anodyne functor is an equivalence.

(b) Let $f\colon B \to A$ be a functor and $p\colon C \twoheadrightarrow A$ be a family projection. There is an equivalence over $f^*(C) \times_B f^*(C)$



Hint: Do not construct the data of the equivalence. Construct the functor in such a way that it fits in a commutative triangle where you can use (a), the 2-out-of-3 property of equivalences, and Exercise 5.1.

(c) Let $p\colon C \twoheadrightarrow A$ and $q\colon D \twoheadrightarrow A$ be family projections. A functor $h\colon C \to D$ over A is an equivalence over A if and only if, for every functor $f\colon B \to A$, the induced functor $f^*(h)\colon f^*(C) \to f^*(D)$ is an equivalence over B.



Exercise 5.3 (Functor extensionality – mapping anima version)**.** Let C and D be categories.

(a) Prove that the functor $\mathrm{Map}_C(C, -)\colon \mathcal{F}am(C) \to \mathcal{A}n$ sends equivalences over C to equivalences.

(b) Prove that there is an equivalence over $\mathrm{Map}(C, D) \times \mathrm{Map}(C, D)$

$$\begin{array}{ccc} \mathrm{Eq}(\mathrm{Map}(C,D)) & \xrightarrow{\simeq} & \mathrm{Map}(C, \mathrm{Eq}(D)) \\ & {\scriptstyle (s,t)} \searrow \quad \swarrow {\scriptstyle (s_*, t_*)} & \\ & \mathrm{Map}(C,D) \times \mathrm{Map}(C,D) & \end{array}$$

Hint: Use functor extensionality, (a), Exercise 5.2 (b) and Exercise 5.1.

Exercise 5.4 (Coherences for equality families)**.** Let C be a category. We treat only two of the higher coherences here, since the proofs are all similar; you are encouraged to check the others too (left/right unitality, left/right inverses, etc.). See Propositions 9.12 and 9.14 of the course notes.

(a) (Associativity) Prove that there is a commutative square over $C \times C$,

$$\begin{array}{ccc} \mathrm{Eq}(C) \times_C \mathrm{Eq}(C) \times_C \mathrm{Eq}(C) & \xrightarrow{\mathrm{id} \times_C (- * -)} & \mathrm{Eq}(C) \times_C \mathrm{Eq}(C) \\ {\scriptstyle (- * -) \times_C \mathrm{id}} \downarrow & \texttt{ass} & \downarrow {\scriptstyle - * -} \\ \mathrm{Eq}(C) \times_C \mathrm{Eq}(C) & \xrightarrow[- * -]{} & \mathrm{Eq}(C) \xrightarrow{(s,t)} C \times C \end{array}$$

i.e., `ass` is a homotopy relative to $C \times C$ between the two composites. Deduce that, for all homotopies $\alpha\colon f \sim g$, $\beta\colon g \sim h$, $\gamma\colon h \sim k$ between functors $f, g, h, k\colon B \to C$, there is a relative homotopy

$$\texttt{ass}_{\alpha,\beta,\gamma}\colon \alpha * (\beta * \gamma) \sim_{C \times C} (\alpha * \beta) * \gamma.$$

(b) (Inverse of inverse) Prove that there is a commutative triangle over $C \times C$.

$$\begin{array}{ccc} \mathrm{Eq}(C) & \xrightarrow{(-)^{-1}} & \mathrm{Eq}(C) \\ & {\scriptstyle \mathrm{id}} \searrow^{\texttt{invinv}} & \downarrow {\scriptstyle (-)^{-1}} \\ & & \mathrm{Eq}(C) \xrightarrow{(s,t)} C \times C \end{array}$$

Deduce that, for every homotopy $\alpha\colon f \sim g$ between functors $f, g\colon B \to C$, there is a relative homotopy

$$\texttt{invinv}_\alpha\colon (\alpha^{-1})^{-1} \sim_{C \times C} \alpha.$$