Basic Localizers and Derivators

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Abstract

In this project, we examine basic localizers, a notion first introduced by Grothendieck and then developed by Maltsiniotis. Furthermore, we study basic localizers associated to derivators. We compute the basic localizer associated to the represented derivator of each cocomplete and complete category, and the one associated to the homotopy derivator of the category of simplicial sets equipped with the Quillen model structure. Among other things, this allows us to find a characterization of initial and final functors and of homotopy initial and homotopy final functors in terms of comma categories.

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1 Introduction

Universal properties in categories come in two flavours, which are dual to each other. Colimits are universal objects under diagrams that are initial with this property, while limits are universal objects over diagrams that are terminal with this property. In other words, colimits have a "mapping-out" universal properties and limits have a "mapping-in" universal property. More explicitly, let $\mathcal{C}$ be a category and A be a small category. The colimit of a diagram $F\colon A \to \mathcal{C}$ consists of a pair $(\operatorname{colim}_A F, \eta\colon F \Rightarrow \Delta \operatorname{colim}_A F)$, where $\operatorname{colim}_A F$ is an object of $\mathcal{C}$ and η is a natural transformation from the diagram F to the constant diagram at the object $\operatorname{colim}_A F$, satisfying the following universal property: for every pair $(X, \gamma\colon F \Rightarrow \Delta X)$, where X is an object of $\mathcal{C}$ and γ is a natural transformation from the diagram F to the constant diagram at the object X, there exists a unique morphism $s\colon \operatorname{colim}_A F \to X$ such that $\gamma = \Delta s \circ \eta$.

$$\begin{tikzcd}
Fa \arrow[rr] \arrow[dr, "\eta_a"] \arrow[ddr, bend right, "\gamma_a"'] & & Fa' \arrow[dl, "\eta_{a'}"'] \arrow[ddl, bend left, "\gamma_{a'}"] \\
& \operatorname{colim}_A F \arrow[d, dashed, "\exists!\, s"] & \\
& X &
\end{tikzcd}$$

Dually, the limit of a diagram $F\colon A \to \mathcal{C}$ consists of a pair $(\lim_A F, \epsilon\colon \Delta \lim_A F \Rightarrow F)$, where $\lim_A F$ is an object of $\mathcal{C}$ and ϵ is a natural transformation from the constant diagram at the object $\lim_A F$ to the diagram F, satisfying the following universal property: for every pair $(Y, \delta\colon \Delta Y \Rightarrow F)$, where Y is an object of $\mathcal{C}$ and δ is a natural transformation from the constant diagram at the object Y to the diagram F, there exists a unique morphism $t\colon Y \to \lim_A F$ such that $\delta = \epsilon \circ \Delta t$.

$$\begin{tikzcd}
& X \arrow[d, dashed, "\exists!\, t"] \arrow[ddl, bend right, "\delta_a"'] \arrow[ddr, bend left, "\delta_{a'}"] & \\
& \lim_A F \arrow[dl, "\epsilon_a"'] \arrow[dr, "\epsilon_{a'}"] & \\
Fa \arrow[rr] & & Fa'
\end{tikzcd}$$

When all colimits and limits of diagrams of shape A in $\mathcal{C}$ exist, they define two functors

$$\operatorname{colim}_A\colon \mathcal{C}^A \to \mathcal{C} \quad \text{and} \quad \lim_A\colon \mathcal{C}^A \to \mathcal{C}.$$

The universal properties of colimits and limits imply then that these functors are part of an adjunction with the diagonal functor $\Delta_A\colon \mathcal{C} \to \mathcal{C}^A$, sending an object to the constant diagram at this object. Since the universal property of colimits gives a one-to-one correspondence between the natural transformations $\gamma\colon F \Rightarrow \Delta X$ in $\mathcal{C}^A$ and the morphisms $s\colon \operatorname{colim}_A F \to X$ in $\mathcal{C}$, the colimit functor defines a left adjoint to the diagonal functor and, since the universal property of limits gives a one-to-one correspondence between the natural transformations $\delta\colon \Delta Y \Rightarrow F$ in $\mathcal{C}^A$ and the morphisms $t\colon Y \to \lim_A F$ in $\mathcal{C}$, the limit functor defines a right adjoint to the diagonal functor.

$$\begin{tikzcd}
\mathcal{C} \arrow[rr, "\Delta_A" description] & & \mathcal{C}^A \arrow[ll, bend right=50, "\operatorname{colim}_A"'] \arrow[ll, bend left=50, "\lim_A"] \arrow[ll, phantom, shift right=3ex, "\perp"] \arrow[ll, phantom, shift left=3ex, "\perp"]
\end{tikzcd}$$

More generally, when considering the object $\operatorname{colim}_A F$ as a functor $\operatorname{colim}_A F\colon \mathbb{1} \to \mathcal{C}$, where $\mathbb{1}$ denotes the terminal category, the colimit of a diagram $F\colon A \to \mathcal{C}$ corresponds to the data of the following left-hand diagram and the universal property of colimits can be summarized as in the following right-hand diagram.

$$\begin{tikzcd}
A \arrow[rr, "F"] \arrow[dr] & {} \arrow[d, phantom, "\Downarrow\eta"] & \mathcal{C} \\
& \mathbb{1} \arrow[ur, "\operatorname{colim}_A F"'] &
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
A \arrow[rr, "F"] \arrow[dr] & {} \arrow[d, phantom, "\Downarrow\gamma"] & \mathcal{C} \\
& \mathbb{1} \arrow[ur, "X"'] &
\end{tikzcd}
\;=\;
\begin{tikzcd}
A \arrow[rr, "F"] \arrow[dr] & {} \arrow[d, phantom, "\Downarrow\eta"] & \mathcal{C} \\
& \mathbb{1} \arrow[ur, "\operatorname{colim}_A F" description] \arrow[ur, bend right=50, "X"'] \arrow[ur, phantom, shift right=2ex, "\Downarrow\exists!s"] &
\end{tikzcd}$$

This says in particular that the colimit of F is the left Kan extensions of F along the unique functor $A \to \mathbb{1}$. Dually, there are similar diagrams for limits

$$\begin{tikzcd}
& \mathbb{1} \arrow[dr, "\lim_A F"] & \\
A \arrow[ur] \arrow[rr, "F"'] & {} \arrow[u, phantom, "\Downarrow\epsilon"] & \mathcal{C}
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
& \mathbb{1} \arrow[dr, "X"] & \\
A \arrow[ur] \arrow[rr, "F"'] & {} \arrow[u, phantom, "\Downarrow\delta"] & \mathcal{C}
\end{tikzcd}
\;=\;
\begin{tikzcd}
& \mathbb{1} \arrow[dr, "\lim_A F" description] \arrow[dr, bend left=50, "X"] \arrow[dr, phantom, shift left=2ex, "\Downarrow\exists!t"] & \\
A \arrow[ur] \arrow[rr, "F"'] & {} \arrow[u, phantom, "\Downarrow\epsilon"] & \mathcal{C}
\end{tikzcd}$$

and this says that the limit of F is the right Kan extension of F along the unique functor $A \to \mathbb{1}$.

These notions generalize from the special case $A \to \mathbb{1}$ to every functor $u\colon A \to B$ between small categories, replacing the colimit and limit objects by universal functors. As generalized colimits, left Kan extensions are universal functors under diagrams that are initial with this property and, as generalized limits, right Kan extensions are universal functors over diagrams that are terminal with this property. Hence one could say that left Kan extensions have a "natural mapping-out" universal property and right Kan extensions have a "natural mapping-in" universal property.

As for colimits and limits, when all left and right Kan extensions of diagrams of shape A in $\mathcal{C}$ along a functor $u\colon A \to B$ exist, they define two functors

$$\operatorname{Lan}_u\colon \mathcal{C}^A \to \mathcal{C}^B \quad\text{and}\quad \operatorname{Ran}_u\colon \mathcal{C}^A \to \mathcal{C}^B.$$

Moreover, the universal properties of left and right Kan extensions imply that these functors define left and right adjoints respectively to the precomposition functor $u^*\colon \mathcal{C}^B \to \mathcal{C}^A$, sending a diagram $F\colon B \to \mathcal{C}$ to the diagram $F \circ u\colon A \to \mathcal{C}$.

$$\begin{tikzcd}[column sep=huge]
\mathcal{C}^B \arrow[r, "u^*" description] & \mathcal{C}^A \arrow[l, bend right=50, "\mathrm{Lan}_u"'] \arrow[l, bend left=50, "\mathrm{Ran}_u"] \arrow[l, phantom, bend right=25, "\perp"] \arrow[l, phantom, bend left=25, "\perp"]
\end{tikzcd}$$

For every category $\mathcal{C}$, we can define a 2-functor $\mathcal{C}^{(-)}\colon \mathrm{Cat}^{\mathrm{op}} \to \mathrm{CAT}$, sending a small category A to the category $\mathcal{C}^A$ of diagrams of shape A, a functor $u\colon A \to B$ between small categories to the precomposition functor $u^*\colon \mathcal{C}^B \to \mathcal{C}^A$, and a natural transformation $\alpha\colon u \Rightarrow v$ to the natural transformation $\alpha^*\colon u^* \Rightarrow v^*$ defined by $\alpha^*_F = F\alpha$, for every diagram $F\colon B \to \mathcal{C}$. In particular, when the category $\mathcal{C}$ is cocomplete and complete, the left and right Kan extension functors along every functor $u\colon A \to B$ exist and we say that the 2-functor $\mathcal{C}^{(-)}\colon \mathrm{Cat}^{\mathrm{op}} \to \mathrm{CAT}$ admits Kan extensions. In this case, the Kan extensions are moreover computed pointwise, which means that, for every $u\colon A \to B$, every $F\colon A \to \mathcal{C}$ and every $b \in B$, we have isomorphisms

$$\mathrm{Lan}_u F(b) \cong \mathrm{colim}_{u\downarrow b} F \circ \pi^b \quad \text{and} \quad \mathrm{Ran}_u F(b) \cong \lim_{b\downarrow u} F \circ \pi_b,$$

where $\pi^b\colon u \downarrow b \to A$ and $\pi_b\colon b \downarrow u \to A$ denote the projection functors.

More generally, a 2-functor $\mathbb{D}\colon \mathrm{Cat}^{\mathrm{op}} \to \mathrm{CAT}$ is called a *prederivator* and is said to admit Kan extensions if for every functor $u\colon A \to B$ the left and right adjoints of the functor $\mathbb{D}(u)\colon \mathbb{D}(B) \to \mathbb{D}(A)$ exist. We note $u^* = \mathbb{D}(u)$ and denote the left and right adjoints by $u_!$ and u_* respectively.

$$\begin{tikzcd}[column sep=huge]
\mathbb{D}(B) \arrow[r, "u^*" description] & \mathbb{D}(A) \arrow[l, bend right=50, "u_!"'] \arrow[l, bend left=50, "u_*"] \arrow[l, phantom, bend right=25, "\perp"] \arrow[l, phantom, bend left=25, "\perp"]
\end{tikzcd}$$

When considering a functor $p\colon A \to \mathbb{1}$ for a small category A, we note $\mathrm{colim}_A = p_!$ and $\lim_A = p_*$, and speak of colimits and limits of diagrams of shape A with respect to the prederivator $\mathbb{D}$. If a prederivator $\mathbb{D}$ admits Kan extensions, as before, we say that these Kan extensions can be computed pointwise if there are natural isomorphisms

$$b^* u_! \cong \mathrm{colim}_{u\downarrow b}(\pi^b)^* \quad \text{and} \quad b^* u_* \cong \lim_{b\downarrow u}(\pi_b)^*$$

for every $u\colon A \to B$ and every $b \in B$. The two conditions we stated here, nominally the existence and pointwise computation of Kan extensions, are two required axioms for *derivators*. We also require that coproducts map to products and that the families of evaluation functors are conservative. With these axioms, we have a more general notion of colimits and limits associated to each derivator. For example, when a category $\mathcal{C}$ is cocomplete and complete, the represented prederivator associated to this category is a derivator, called the *represented derivator* of $\mathcal{C}$, and the colimits and limits with respect to this derivator correspond to the usual colimits and limits in $\mathcal{C}$.

Initial and final functors are notions related to limits and colimits. A functor $u\colon A \to B$ is said to be *initial* if it preserves limits under precomposition, i.e. if, for every complete category $\mathcal{C}$ and every diagram $F\colon B \to \mathcal{C}$, we have an isomorphism

$$\lim_B F \cong \lim_A F \circ u.$$

Dually, a functor $u\colon A \to B$ is said to be *final* if it preserves colimits under precomposition, i.e. if, for every cocomplete category $\mathcal{C}$ and every diagram $F\colon B \to \mathcal{C}$, we have an isomorphism

$$\operatorname{colim}_A F \circ u \cong \operatorname{colim}_B F.$$

These definitions are also equivalent to saying that there are natural isomorphisms

$$\lim_B \cong \lim_A u^* \quad \text{and} \quad \operatorname{colim}_A u^* \cong \operatorname{colim}_B$$

respectively with respect to every represented derivator. In particular, there exists a surprising characterization of these initial and final functors in terms of comma categories: a functor $u\colon A \to B$ is initial if and only if the comma category $u \downarrow b$ is non-empty and connected for every $b \in B$, and a functor $u\colon A \to B$ is final if and only if the comma category $b \downarrow u$ is non-empty and connected for every $b \in B$.

Define $\mathcal{W}_0$ to be the class of functors $u\colon A \to B$ between small categories such that the map $\pi_0(u)\colon \pi_0(A) \to \pi_0(B)$ is an isomorphism. Then a category A is non-empty and connected if and only if the unique functor $A \to \mathbb{1}$ belongs to $\mathcal{W}_0$. By the characterization of initial functors, this shows that a functor $u\colon A \to B$ is initial if and only if the functor $u \downarrow b \to \mathbb{1}$ belongs to $\mathcal{W}_0$ for every $b \in B$. We focus in the rest of the introduction on initial functors, but one could dualize every statement to final functors.

The class $\mathcal{W}_0$ actually satisfies the axioms of a basic localizer. A *basic localizer* is a class $\mathcal{W}$ of functors between small categories that is weakly saturated, that contains the functor $A \to \mathbb{1}$ for every small category A admitting a terminal object, and that contains all $\mathcal{W}$-local functors, i.e. all functors $u\colon A \to B$ such that, for every commutative triangle

$$\begin{tikzcd} A \arrow[rr, "u"] \arrow[dr, "v"'] & & B \arrow[dl, "w"] \\ & C & \end{tikzcd}$$

the functor $u^c\colon v \downarrow c \to w \downarrow c$ induced by u belongs to $\mathcal{W}$ for every $c \in C$. If $\mathcal{W}$ is a basic localizer, a small category A is called $\mathcal{W}$-aspherical if the unique functor $A \to \mathbb{1}$ belongs to $\mathcal{W}$, and a functor $u\colon A \to B$ is called $\mathcal{W}$-aspherical if the comma category $u \downarrow b$ is

$\mathcal{W}$-aspherical for every $b \in B$. Motivated by the example of $\mathcal{W}_0$, the terminology of $\mathcal{W}$-connected category and $\mathcal{W}$-initial functor would have made sense too. This terminology makes even more sense once we have defined a basic localizer associated to each derivator. In fact, if $\mathbb{D}$ is a derivator, the basic localizer $\mathcal{W}_{\mathbb{D}}$ associated to $\mathbb{D}$ is defined in such a way that the $\mathcal{W}_{\mathbb{D}}$-aspherical functors correspond to the functors that preserve limits under precomposition, i.e. the functors $u\colon A \to B$ such that we have a natural isomorphism

$$\lim_B \cong \lim_A u^*,$$

where $\lim_A$ and $\lim_B$ denote the limits of diagrams of shape A and B respectively with respect to $\mathbb{D}$, and $u^* = \mathbb{D}(u)$. In particular, every basic localizer associated to a represented derivator contains the fundamental basic localizer $\mathcal{W}_0$.

There is also another notion of colimits and limits: the homotopy colimit and the homotopy limit. Let $\mathcal{M}$ be a model category and A be a small category. When the weak equivalences are defined levelwise in $\mathcal{M}^A$, the diagonal functor $\Delta_A\colon \mathcal{M} \to \mathcal{M}^A$ preserves weak equivalences and hence induces a functor $\Delta_A\colon \mathrm{Ho}(\mathcal{M}) \to \mathrm{Ho}(\mathcal{M}^A)$ between their homotopy category. The *homotopy colimit* and the *homotopy limit* of diagrams of shape A are then defined to be the left and right adjoint respectively of this induced functor, when they exist.

$$\begin{tikzcd}
\mathrm{Ho}(\mathcal{M}) \arrow[r, "\Delta_A" description] & \mathrm{Ho}(\mathcal{M}^A) \arrow[l, bend right=50, "\mathrm{hocolim}_A"', "\perp"] \arrow[l, bend left=50, "\mathrm{holim}_A", "\perp"']
\end{tikzcd}$$

The 2-functor $\mathrm{Ho}(\mathcal{M}^{(-)})\colon \mathrm{Cat}^{\mathrm{op}} \to \mathrm{CAT}$, sending a small category A to the homotopy category $\mathrm{Ho}(\mathcal{M}^A)$ with the weak equivalences defined levelwise in $\mathcal{M}^A$, a functor $u\colon A \to B$ to the functor $u^*\colon \mathrm{Ho}(\mathcal{M}^B) \to \mathrm{Ho}(\mathcal{M}^A)$ induced by the precomposition functor $u^*\colon \mathcal{M}^B \to \mathcal{M}^A$, and a natural transformation $\alpha\colon u \Rightarrow v$ to the obvious natural transformation induced by $\alpha^*\colon u^* \Rightarrow v^*$, actually defines a derivator, called the *homotopy derivator* of $\mathcal{M}$. Homotopy initial functors are defined as the functors that preserve homotopy limits under precomposition. In other words, a functor $u\colon A \to B$ is *homotopy initial* if, for every model category $\mathcal{M}$ and every diagram $F \in \mathrm{Ho}(\mathcal{M}^B)$, we have an isomorphism

$$\mathrm{holim}_B F \cong \mathrm{holim}_A F \circ u.$$

This is equivalent to saying that we have a natural isomorphism

$$\mathrm{holim}_B \cong \mathrm{holim}_A u^*$$

with respect to every model category $\mathcal{M}$, or that we have a natural isomorphism

$$\lim_B \cong \lim_A u^*$$

with respect to every homotopy derivator.

Consider the category of simplicial sets sSet equipped with the Quillen model structure. The basic localizer associated to its homotopy derivator is the class $\mathcal{W}_\infty$ of functors between small categories whose nerve is a weak homotopy equivalence of simplicial sets. Then a small category is $\mathcal{W}_\infty$-aspherical if its nerve is homotopy equivalent to a point, and a functor $u\colon A \to B$ is $\mathcal{W}_\infty$-aspherical if the nerve of the comma category $u \downarrow b$ is homotopy equivalent to a point for every $b \in B$. Moreover, Cisinski shows that the basic localizer $\mathcal{W}_\infty$ is the minimal one and this gives us the following characterization of homotopy initial functors: a functor $u\colon A \to B$ is homotopy initial if and only if the nerve of the comma category $u \downarrow b$ is homotopy equivalent to a point for every $b \in B$.

In conclusion, derivators are vehicles to develop the calculus of limits and colimits in classical and more exotic settings and basic localizers allow us to characterize the "initial" and "final" functors with respect to each derivator in terms of comma categories.

Overview

In the first part of this project, we present the necessary background material on Kan extensions and derived functors. This is the content of Section 2. In the second part, we develop the theory of derivators and their basic localizers. The axioms for derivators are presented in Section 3, while the ones for basic localizers are presented in Section 4. In Section 5, we construct a basic localizer associated to each derivator. Our two main examples of derivators are presented in Sections 3 and 6: the represented derivator associated to a cocomplete and complete category and the homotopy derivator associated to a combinatorial model category. We also compute the basic localizer associated to some of these derivators in Sections 5 and 6. Finally, the third part presents some applications of the theory of derivators and basic localizers to the calculus of limits and colimits. Section 7 contains three problems about general pullback squares, and Section 8 contains a characterization for initial and homotopy initial functors, and some general results about colimits and limits.

Let CAT be the 2-category of all categories, functors and natural transformations. Then CAT has a full 2-subcategory Cat whose objects are the small categories. *Derivators* are 2-functors $\mathrm{Cat}^{\mathrm{op}} \to \mathrm{CAT}$, called *prederivators*, satisfying four axioms: coproducts map to products, every family of evalutation functors is conservative, left and right adjoints of every precomposition functor exist, and these adjoints are computed pointwise.

As a first example, for a category $\mathcal{C}$, the 2-functor

$$\mathcal{C}^{(-)}\colon \mathrm{Cat}^{\mathrm{op}} \to \mathrm{CAT}, \quad A \mapsto \mathcal{C}^A,$$

called the represented prederivator of $\mathcal{C}$, could be an example of a derivator, but under which conditions? In this case, the third axiom requires that the left and right adjoints of every precomposition functor exist, while the axiom of pointwise computation requires that these adjoints are computed as colimits or limits over comma categories at each point. In Section 2.1, we introduce the notion of Kan extensions along a functor $u\colon A \to B$ in Cat, which define the left and right adjoints of the precomposition functor $u^*\colon \mathcal{C}^B \to \mathcal{C}^A$ when they exist. Moreover, when the category $\mathcal{C}$ is cocomplete and complete, these Kan extensions always exist and can be computed as the required colimits and limits, as shown in Section 2.2. This allows us to prove that the 2-functor $\mathcal{C}^{(-)}\colon \mathrm{Cat}^{\mathrm{op}} \to \mathrm{CAT}$ is a derivator for a cocomplete and complete category $\mathcal{C}$ (see Theorem 3.3.2).

As a second example, for a model category $\mathcal{M}$, the 2-functor

$$\mathrm{Ho}(\mathcal{M}^{(-)})\colon \mathrm{Cat}^{\mathrm{op}} \to \mathrm{CAT}, \quad A \mapsto \mathrm{Ho}(\mathcal{M}^{A}),$$

where the weak equivalences are defined levelwise, is also a derivator, called the homotopy derivator (see Section 6.1). Since a model category is cocomplete and complete, it admits a represented derivator. Then the image of a functor under the homotopy derivator is the total derived functor of the precomposition functor and the homotopy Kan extensions are constructed as the left and right total derived functors of the left and right adjoints of the precomposition functor respectively, which exist since the represented prederivator is a derivator. These total derived functors are defined in Section 2.4. Moreover, when the considered model category is combinatorial, the adjunctions of the precomposition functor with its Kan extensions are Quillen pairs, while equipping the categories with the projective or injective model structures. Therefore we show in Section 2.5 that, if an adjunction is a Quillen pair, their left and right total derived functors respectively give rise to an adjunction between the homotopy categories.

The axiom about pointwise computations of Kan extensions can also be expressed in terms of Beck-Chevalley squares. In fact, the pointwise computation of the Kan extensions in the case of a represented derivator $\mathcal{C}^{(-)}\colon \mathrm{Cat}^{\mathrm{op}} \to \mathrm{CAT}$ corresponds to the fact that the squares

$$\begin{tikzcd}
\mathcal{C}^{u\downarrow b} & \mathcal{C}^{A} \arrow[l, "(\pi^b)^*"'] \arrow[dl, phantom, "\nu^* \Swarrow"] \\
\mathcal{C} \arrow[u, "\Delta_{u\downarrow b}"] & \mathcal{C}^{B} \arrow[l, "b^*"] \arrow[u, "u^*"']
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{C}^{b\downarrow u} & \mathcal{C}^{A} \arrow[l, "(\pi_b)^*"'] \arrow[dl, phantom, "\mu^* \Nearrow"] \\
\mathcal{C} \arrow[u, "\Delta_{b\downarrow u}"] & \mathcal{C}^{B} \arrow[l, "b^*"] \arrow[u, "u^*"']
\end{tikzcd}$$

are Beck-Chevalley, which means that the left and right mates respectively are natural isomorphisms

$$\mathrm{colim}_{u\downarrow b}(\pi^b)^* \cong b^* \,\mathrm{Lan}_u \quad \text{and} \quad b^* \,\mathrm{Ran}_u \cong \lim_{b\downarrow u}(\pi_b)^*$$

(see Section 2.3). Before stating the axioms for derivators, we first introduce in Section 3.1 the notion of Beck-Chevalley squares and prove that a square is left Beck-Chevalley if and only if it is right Beck-Chevalley. When a prederivator $\mathbb{D}\colon \mathrm{Cat}^{\mathrm{op}} \to \mathrm{CAT}$ is a derivator, all left and right Kan extensions exist and the mates of every square in Cat under $\mathbb{D}$ are well-defined. Hence the notion of left and right Beck-Chevalley squares coincide in this case and gives rise to a notion of $\mathbb{D}$-*Beck-Chevalley square* (Section 5.2). In Section 3.2, we define prederivators and give some previous result about those 2-functors. Finally, in Section 3.3, we state the four axioms of derivators and prove that the represented prederivator of a cocomplete and complete category is a derivator.

A basic localizer is a class $\mathcal{W}$ of functors in Cat that is weakly saturated (in the sense of Grothendieck), that contains the functor $A \to \mathbb{1}$ for all small categories A admitting a terminal or initial object, and that contains the $\mathcal{W}$-local and $\mathcal{W}$-colocal functors (see Section 4.1). To each derivator can be associated such a basic localizer (Section 5.1) and this gives rise to a characterization of the $\mathbb{D}$-Beck-Chevalley squares (Section 5.2). In fact, there exist two properties of squares in Cat coming from a basic localizer $\mathcal{W}$: the $\mathcal{W}$-*exact squares* (Section 4.2) and the *weak* $\mathcal{W}$-*exact squares* (Section 4.3), where the latter requires a stronger condition than the former. If $\mathcal{W}_{\mathbb{D}}$ denotes the basic localizer associated to a derivator $\mathbb{D}$, the notions of weak $\mathcal{W}_{\mathbb{D}}$-exact squares and $\mathbb{D}$-Beck-Chevalley squares coincide and, in most cases, the notions of $\mathcal{W}_{\mathbb{D}}$-exact squares and $\mathbb{D}$-Beck-Chevalley squares are the same. This characterization of the $\mathbb{D}$-Beck-Chevalley squares allows us among others to show that the notions of $\mathcal{W}_{\mathbb{D}}$-aspherical functor and $\mathbb{D}$-initial functors coincide, where a $\mathcal{W}_{\mathbb{D}}$-*aspherical functor* $u\colon A \to B$ is such that the functor $u \downarrow b \to \mathbb{1}$ belongs to $\mathcal{W}_{\mathbb{D}}$ for every $b \in B$, and a $\mathbb{D}$-*initial functor* $u\colon A \to B$ is such that the mate

$$\lim_B \cong \lim_A u^*$$

is a natural isomorphism, where $\lim_A$ and $\lim_B$ denote the limits of diagrams of shapes A and B respectively with respect to the derivator $\mathbb{D}$. In particular, if we compute the basic localizer associated to each represented derivator, we obtain that it contains the fundamental basic localizer $\mathcal{W}_0$, which consists of all functors $u\colon A \to B$ that induces a bijection $\pi_0(u)\colon \pi_0(A) \to \pi_0(B)$, and this gives us the following characterization of initial functors in terms of comma categories: a functor $u\colon A \to B$ is initial if and only if the comma category $u \downarrow b$ is non-empty and connected for every $b \in B$ (see Corollary 8.1.4).

Once initial functors are characterized, we also want to characterize the homotopy initial functors. Since a *homotopy initial functor* $u\colon A \to B$ is such that the mate

$$\operatorname{holim}_B \cong \operatorname{holim}_A u^*$$

is a natural isomorphism, the idea is to compute the basic localizer of each homotopy derivator. It actually suffices to compute the one associated to the homotopy derivator of the category of simplicial sets sSet equipped with the Quillen model structure, since this is the minimal basic localizer, a result due to Cisinski (see [Cis04], Theorem 2.2.11). In fact, the basic localizer associated to this derivator is the class $\mathcal{W}_\infty$ of functors in Cat, whose nerve is a weak homotopy equivalence of simplicial sets. This is not surprizing since the weak equivalences in the Quillen model structure correspond to the weak homotopy equivalences. An idea to go further in this theory would be to compute the basic localizer associated to the homotopy derivator of the category of simplicial sets equipped with another model structure, and this latter would probably correspond to the class of functors whose nerve is a weak equivalence in this model structure. Back to our problem, these results give us the following characterization of homotopy initial functors in terms of comma categories: a functor $u\colon A \to B$ is homotopy initial if and only if the nerve of the comma category $u \downarrow b$ is homotopy equivalent to a point for every $b \in B$ (see Corollary 8.2.5).

In order to compute the basic localizer associated to the homotopy derivator of sSet, we first give, in Section 6.1, a proof of the fact that the homotopy derivator of a combinatorial model category actually defines a derivator. Then, in Section 6.2, we show that a Quillen equivalence gives rise to an equivalence of derivators and hence that they have the same basic localizer. Thomason defines a model structure on Cat such that the weak equivalences correspond to the class $\mathcal{W}_\infty$ and Fritsch and Latch prove that there is a Quillen equivalence between the categories Cat equipped with the Thomason model structure and the category sSet equipped with the Quillen model structure. These results, presented in Section 6.4, imply that the homotopy derivators associated to these two categories have the same basic localizer and, since Maltsiniotis shows that the basic localizer associated to the derivator corresponding to the localization of Cat at a basic localizer $\mathcal{W}$ is $\mathcal{W}$ (see [Mal05], Proposition 3.1.10), this finally shows that the basic localizer associated to the homotopy derivator of sSet equipped with the Quillen model structure is $\mathcal{W}_\infty$.

Section 7 is an application of the theory of derivators to general pullback squares, also called *cartesian squares*. In Section 7.1, we characterize the cartesian squares in terms of *right* $\mathbb{D}$*-Beck-Chevalley square* at a diagram, which are squares in Cat such that the component at this diagram of the mate of the image of the square under $\mathbb{D}$ is an isomorphism, where $\mathbb{D}$ is a derivator. Three problems with cartesian squares are then solved using this characterization in Sections 7.2, 7.3, and 7.4. The first one is the pullback composition and cancellation problem. More explicitly, we show that if we have a diagram

$$\begin{tikzcd} X_{00} \arrow[r] \arrow[d] & X_{10} \arrow[r] \arrow[d] & X_{20} \arrow[d] \\ X_{01} \arrow[r] & X_{11} \arrow[r] & X_{21} \end{tikzcd}$$

such that the right square is cartesian, then the left square is cartesian if and only if the exterior square is cartesian. After that, we adapt the proof of the first problem to show that if we have a diagram

$$\begin{tikzcd} & & X_{110} \arrow[d] \\ & X_{101} \arrow[dr] & \\ X_{011} \arrow[rr] & & X_{111} \end{tikzcd}$$

and we construct each square one after another as a pullback square in order to obtain a diagram

$$\begin{tikzcd}
X_{000} \arrow[r] \arrow[dd] & X_{100} \arrow[dd] \arrow[dr] & \\
& & X_{110} \arrow[dd] \\
X_{001} \arrow[r] \arrow[dr] & X_{101} \arrow[dr] & \\
& X_{011} \arrow[r] & X_{111}
\end{tikzcd}$$

then the object X_{000} corresponds to the limit of the first diagram considered. Finally, for the last problem, we consider a cube

$$\begin{tikzcd}
X_{000} \arrow[r] \arrow[dd] \arrow[dr] & X_{100} \arrow[dd] \arrow[dr] & \\
& X_{010} \arrow[r] \arrow[dd] & X_{110} \arrow[dd] \\
X_{001} \arrow[r] \arrow[dr] & X_{101} \arrow[dr] & \\
& X_{011} \arrow[r] & X_{111}
\end{tikzcd}$$

such that the front and back faces are cartesian and construct the pullback of its left and right face in order to obtain a diagram

$$\begin{tikzcd}
X_{000} \arrow[rrr] \arrow[drr] \arrow[ddr] \arrow[ddd] & & & X_{100} \arrow[drr] \arrow[ddr] \arrow[ddd] & & \\
& & X_{010} \arrow[rrr] \arrow[ddd] & & & X_{110} \arrow[ddd] \\
& X_B \arrow[ur] \arrow[rrr] \arrow[dl] & & & X_A \arrow[ur] \arrow[dl] & \\
X_{001} \arrow[rrr] \arrow[drr] & & & X_{101} \arrow[drr] & & \\
& & X_{011} \arrow[rrr] & & & X_{111}
\end{tikzcd}$$

The aim here is to show that the square

$$\begin{tikzcd}
X_{000} \arrow[r] \arrow[d] & X_{100} \arrow[d] \\
X_B \arrow[r] & X_A
\end{tikzcd}$$

is cartesian.

In Sections 8.1 and 8.2, a proof of the characterizations of initial and homotopy initial functors can be found. In these sections, we define the initial and homotopy initial functors using their comma category characterizations and we show that they correspond to the functors that preserve limits under precomposition with respect to every derivator whose basic localizer contains the fundamental one and with respect to all derivators respectively. Finally, we show three general results about limits and colimits in Section 8.3. The first one is Fubini's theorem for limits and colimits. Then we adapt the computation of limits and colimits of shape the coproduct of two categories to every derivator. Finally, using the characterization of Section 7.1, we show that the right Kan extension of the inclusion $i\colon A \to A^{\triangleleft}$, where $A^{\triangleleft}$ denotes the cone category over A, sends a diagram of shape A to its limit cone, with respect to every derivator.

Most of the definitions and results in Section 2 come from [Rie16]. In Section 2.5, we use results from [Hir03], Sections 7.7 and 8.5, and [DS95], Section 9.

Section 3.1 about Beck-Chevalley squares follows [Gro], Section 8. In Section 3.2, prederivators are defined the way Groth did and we adapt some definitions and results from [Mal01], Section 1.3, to this definition. Maltsiniotis actually defines prederivators as 2-functors $\mathrm{Cat}^{\mathrm{coop}} \to \mathrm{CAT}$, which implies that the 2-arrows are also reversed.

All definitions and results in Sections 4 and 5 come from [Mal11], but we adapted the results of Section 5 to the definition of derivators of Groth.

Section 6.1 about the homotopy derivator associated to a combinatorial model category is inspired by [Gro], Appendix B.3. The fact that the projective and injective model structures are well-defined for a combinatorial model category is a result that can be found in [Lur09], Proposition A.2.8.2. The definition of equivalences of derivator we give in Section 6.2 follows from a result in [Gro13], Proposition 2.9. The results about the basic localizer $\mathcal{W}_\infty$ presented in Section 6.3 are due to Cisinski, see [Cis04]. In Section 6.4, most of the results come from [Tho80], Sections 3 and 4, and [FL79].

The characterization of cartesian squares of Section 7.1 is inspired by results from [Gro], Section 8.2. The dual problem of the one presented in Section 7.2 can be found in [Gro], Section 9.3.

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Part I
Background and Prerequisites

2 Kan Extensions and Derived Functors

In this Section, we introduce the background notions we need, nominally the notions of Kan extensions and of derived functors. In Section 2.1, we first define Kan extensions and show that, if $\mathcal{C}$ is a category and all left and right Kan extensions along a functor $u\colon A \to B$ of Cat in $\mathcal{C}$ exist, they define left and right adjoint functors to the precomposition functor $u^*\colon \mathcal{C}^B \to \mathcal{C}^A$. In Section 2.2, we introduce comma categories in order to show that left and right Kan extensions can be computed "pointwise" as colimits and limits respectively over these comma categories, when all these colimits and limits exist. Corollary 2.2.7 says that, when a category is cocomplete and complete, the left and right Kan extensions along every functor in Cat exist and are computed pointwise. This allows us later to check that the represented prederivator associated to a cocomplete and complete category is a derivator. In Section 2.3, we express limits and colimits of diagrams of shape A as left and right Kan extensions along the unique functor $A \to \mathbb{1}$ and give a characterization of the pointwise computation of Kan extensions in terms of comma squares. This criterion is useful when proving the axiom of pointwise computation for represented derivator.

In Section 2.4, we introduce left and right total derived functors as generalized Kan extensions along the localization functor of a model category. We prove that, if we have an adjunction where the left adjoint is left deformable and the right adjoint is right deformable, then their left and right total derived functors respectively exist and form an adjunction between the homotopy categories. This allows us to show, in Section 2.5, that a Quillen pair gives rise to an adjunction between the homotopy categories. In fact, the left Quillen functor is left deformable with respect to the cofibrant replacement, while the right Quillen functor is right deformable with respect to the fibrant replacement. Finally, the last result says that a Quillen equivalence induces an equivalence between the homotopy categories. This result is useful to show that two homotopy derivators are equivalent when there is a Quillen equivalence between their underlying categories.

2.1 Definition and Adjunctions of Kan Extensions

We first define left and right Kan extensions, which are sort of generalized colimits and limits along functors. The goal here is to show that, when all Kan extensions along a functor in Cat exist, they define functors that are left and right adjoints to the precomposition functor. We consider different sorts of functors: the functors in Cat between small categories and the functors from a small category to a large category, which we call *diagrams*.

Definition 2.1.1. Let $u\colon A \to B$ be a functor in Cat and $F\colon A \to \mathcal{C}$ be a diagram.

(i) A **left Kan extension** of F along u is a diagram $\mathrm{Lan}_u F\colon B \to \mathcal{C}$ together with a natural transformation $\eta\colon F \Rightarrow \mathrm{Lan}_u F \circ u$, called the **unit transformation**, satisfying the universal property: for every diagram $G\colon B \to \mathcal{C}$ and every natural transformation $\gamma\colon F \Rightarrow G \circ u$, there exists a unique natural transformation $\alpha\colon \mathrm{Lan}_u F \Rightarrow G$ such that $\gamma = \alpha u \circ \eta$.

$$
\begin{tikzcd}
A \arrow[rr, "F"] \arrow[dr, "u"'] & \arrow[d, phantom, "\Downarrow\eta"] & \mathcal{C} \\
& B \arrow[ur, "\operatorname{Lan}_u F"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[rr, "F"] \arrow[dr, "u"'] & \arrow[d, phantom, "\Downarrow\gamma"] & \mathcal{C} \\
& B \arrow[ur, "G"'] &
\end{tikzcd}
=
\begin{tikzcd}
A \arrow[rr, "F"] \arrow[dr, "u"'] & \arrow[d, phantom, "\Downarrow\eta"] & \mathcal{C} \\
& B \arrow[ur, "\operatorname{Lan}_u F"] \arrow[ur, bend right=60, "G"'] \arrow[ur, phantom, bend right=30, "\Swarrow\exists!\alpha"] &
\end{tikzcd}
$$

(ii) A **right Kan extension** of F along u is a diagram $\operatorname{Ran}_u F\colon B \to \mathcal{C}$ together with a natural transformation $\epsilon\colon \operatorname{Ran}_u F \circ u \Rightarrow F$, called the **counit transformation**, satisfying the universal property: for every diagram $G\colon B \to \mathcal{C}$ and every natural transformation $\delta\colon G \circ u \Rightarrow F$, there exists a unique natural transformation $\beta\colon G \Rightarrow \operatorname{Ran}_u F$ such that $\delta = \epsilon \circ \beta u$.

$$
\begin{tikzcd}
& B \arrow[dr, "\operatorname{Ran}_u F"] \arrow[d, phantom, "\Downarrow\epsilon"] & \\
A \arrow[ur, "u"] \arrow[rr, "F"'] & & \mathcal{C}
\end{tikzcd}
\qquad
\begin{tikzcd}
& B \arrow[dr, "G"] \arrow[d, phantom, "\Downarrow\delta"] & \\
A \arrow[ur, "u"] \arrow[rr, "F"'] & & \mathcal{C}
\end{tikzcd}
=
\begin{tikzcd}
& B \arrow[dr, "\operatorname{Ran}_u F"'] \arrow[dr, bend left=60, "G"] \arrow[dr, phantom, bend left=30, "\Swarrow\exists!\beta"] \arrow[d, phantom, "\Downarrow\epsilon"] & \\
A \arrow[ur, "u"] \arrow[rr, "F"'] & & \mathcal{C}
\end{tikzcd}
$$

Let $\mathbb{1}$ denote the terminal category. The idea that Kan extensions are generalised colimits and limits comes from the fact that a left (resp. right) Kan extension of a diagram $F\colon A \to \mathcal{C}$ along the unique functor $A \to \mathbb{1}$ is actually the colimit (resp. limit) of F in $\mathcal{C}$, when it exists (see Section 2.3).

Remark 2.1.2. Left and right Kan extensions are dual in the sense of "co", i.e. by reversing the natural transformations, we obtain a left Kan extension from a right Kan extension and conversely.

When all Kan extensions along a functor $u\colon A \to B$ exist, we can define two functors $\operatorname{Lan}_u\colon \mathcal{C}^A \to \mathcal{C}^B$ and $\operatorname{Ran}_u\colon \mathcal{C}^A \to \mathcal{C}^B$. Then we check that they define left and right adjoints to the precomposition functor $u^*\colon \mathcal{C}^B \to \mathcal{C}^A$.

Remark 2.1.3. If the left and right Kan extensions of every diagram $F\colon A \to \mathcal{C}$ along $u\colon A \to B$ exist, we can define two functors $\operatorname{Lan}_u, \operatorname{Ran}_u\colon \mathcal{C}^A \to \mathcal{C}^B$ sending a diagram in $\mathcal{C}^A$ to its left or right Kan extension. For a natural transformation $\alpha\colon F \Rightarrow F'$ in $\mathcal{C}^A$, we define $\operatorname{Lan}_u(\alpha)\colon \operatorname{Lan}_u F \Rightarrow \operatorname{Lan}_u F'$ to be the unique morphism given by the universal property in the following diagram

$$
\begin{tikzcd}
A \arrow[rr, bend left, "F"] \arrow[rr, bend right, "F'"'] \arrow[rr, phantom, "\Downarrow\alpha"] \arrow[dr, "u"'] & \arrow[d, phantom, "\Downarrow\eta'"] & \mathcal{C} \\
& B \arrow[ur, "\operatorname{Lan}_u F'"'] &
\end{tikzcd}
=
\begin{tikzcd}
A \arrow[rr, "F"] \arrow[dr, "u"'] & \arrow[d, phantom, "\Downarrow\eta"] & \mathcal{C} \\
& B \arrow[ur, "\operatorname{Lan}_u F"] \arrow[ur, bend right=60, "\operatorname{Lan}_u F'"'] \arrow[ur, phantom, bend right=30, "\Swarrow\exists!"] &
\end{tikzcd}
$$

where η and η' denote the unit transformation of $\operatorname{Lan}_u F$ and $\operatorname{Lan}_u F'$ respectively. In particular, we have $\eta' \circ \alpha = \operatorname{Lan}_u(\alpha)u \circ \eta$.

Dually, we define $\mathrm{Ran}_u(\alpha)\colon \mathrm{Ran}_u F \Rightarrow \mathrm{Ran}_u F'$ to be the unique morphism given by the universal property in the following diagram

$$
\begin{tikzcd}
& B \arrow[dr, "\mathrm{Ran}_u F"] \arrow[d, phantom, "\Downarrow\epsilon"] & \\
A \arrow[ur, "u"] \arrow[rr, bend left, "F"] \arrow[rr, bend right, "F'"'] \arrow[rr, phantom, "\Downarrow\alpha"] & \, & \mathcal{C}
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
& B \arrow[dr, "\mathrm{Ran}_u F'"'] \arrow[dr, bend left, "\mathrm{Ran}_u F"] \arrow[dr, phantom, bend left=20, "\Swarrow\exists!"] \arrow[d, phantom, "\Downarrow\epsilon'"] & \\
A \arrow[ur, "u"] \arrow[rr, "F'"'] & \, & \mathcal{C}
\end{tikzcd}
$$

where ϵ and ϵ' denote the counit transformation of $\mathrm{Ran}_u F$ and $\mathrm{Ran}_u F'$ respectively. In particular, we have $\alpha \circ \epsilon = \epsilon' \circ \mathrm{Ran}_u(\alpha)u$.

Proposition 2.1.4. *Let $u\colon A \to B$ be a functor in* Cat *and $\mathcal{C}$ be a category. If the left and right Kan extensions of every diagram $F\colon A \to \mathcal{C}$ along u exist, the following adjunctions hold.*

$$
\begin{tikzcd}[column sep=huge]
\mathcal{C}^A \arrow[r, bend left=50, "\mathrm{Lan}_u"] \arrow[r, bend right=50, "\mathrm{Ran}_u"'] \arrow[r, phantom, shift left=3, "\perp"] \arrow[r, phantom, shift right=3, "\perp"] & \mathcal{C}^B \arrow[l, "u^*" description]
\end{tikzcd}
$$

Proof. To show that $\mathrm{Ran}_u \vdash u^*$, for $F\colon A \to \mathcal{C}$ and $G\colon B \to \mathcal{C}$, we define

$$\Psi_{F,G}\colon \mathcal{C}^B(G, \mathrm{Ran}_u F) \to \mathcal{C}^A(G \circ u, F), \quad \beta \mapsto \epsilon \circ \beta u,$$

where $\epsilon\colon \mathrm{Ran}_u F \circ u \Rightarrow F$ denotes the counit transformation of $\mathrm{Ran}_u F$. By the universal property of right Kan extensions, $\Psi_{F,G}$ is an isomorphism. Moreover, it is natural in F and in G. To see this, if we consider $G'\colon B \to \mathcal{C}$, for every $\alpha\colon G \Rightarrow G'$, the following diagram commutes.

$$
\begin{tikzcd}
\mathcal{C}^B(G', \mathrm{Ran}_u F) \arrow[r, "\Psi_{F,G'}"] \arrow[d, "\alpha^*"'] & \mathcal{C}^A(G' \circ u, F) \arrow[d, "(\alpha u)^*"] \\
\mathcal{C}^B(G, \mathrm{Ran}_u F) \arrow[r, "\Psi_{F,G}"'] & \mathcal{C}^A(G \circ u, F)
\end{tikzcd}
$$

To see this

$$
\begin{gathered}
(\alpha u)^* \Psi_{F,G'}(\beta) = (\alpha u)^*(\epsilon \circ \beta u) = \epsilon \circ \beta u \circ \alpha u \\
\Psi_{F,G} \alpha^*(\beta) = \Psi_{F,G}(\beta \circ \alpha) = \epsilon \circ (\beta \circ \alpha) u
\end{gathered}
$$

and the two composites are equal since u^* is a functor. Similarly, if we consider $F'\colon A \to \mathcal{C}$, for every $\gamma\colon F \Rightarrow F'$, the following diagram commutes.

$$\begin{tikzcd}
\mathcal{C}^B(G, \operatorname{Ran}_u F) \arrow[r, "\Psi_{F,G}"] \arrow[d, "(\operatorname{Ran}_u \gamma)_*"'] & \mathcal{C}^A(G \circ u, F) \arrow[d, "\gamma_*"] \\
\mathcal{C}^B(G, \operatorname{Ran}_u F') \arrow[r, "\Psi_{F',G}"'] & \mathcal{C}^A(G \circ u, F')
\end{tikzcd}$$

To see this

$$\begin{gathered}
\gamma_* \Psi_{F,G}(\beta) = \gamma_*(\epsilon \circ \beta u) = \gamma \circ \epsilon \circ \beta u \\
\Psi_{F',G}(\operatorname{Ran}_u \gamma)_*(\beta) = \Psi_{F',G}(\operatorname{Ran}_u \gamma \circ \beta) = \epsilon' \circ (\operatorname{Ran}_u \gamma \circ \beta) u = \epsilon' \circ \operatorname{Ran}_u(\gamma) u \circ \beta u
\end{gathered}$$

and the two composites are equal since $\gamma \circ \epsilon = \epsilon' \circ \operatorname{Ran}_u(\gamma) u$ by construction (see Remark 2.1.3). Hence we have defined an isomorphism $\mathcal{C}^B(G, \operatorname{Ran}_u F) \cong \mathcal{C}^A(G \circ u, F)$ which is natural in both variables. This proves that Ran_u is the right adjoint of u^*.

The proof of the adjunction $\operatorname{Lan}_u F \dashv u^*$ is dual, with the natural isomorphism defined by

$$\Phi_{F,G} \colon \mathcal{C}^B(\operatorname{Lan}_u F, G) \to \mathcal{C}^A(F, G \circ u), \quad \alpha \mapsto \alpha u \circ \eta,$$

for every $F \colon A \to \mathcal{C}$ and $G \colon B \to \mathcal{C}$, where $\eta \colon F \Rightarrow \operatorname{Lan}_u F \circ u$ denotes the unit transformation of $\operatorname{Lan}_u F$. □

Remark 2.1.5. The unit transformations $\eta \colon F \Rightarrow \operatorname{Lan}_u F \circ u$ form the components of the unit of the adjunction $\operatorname{Lan}_u \dashv u^*$ and the counit transformations $\epsilon \colon \operatorname{Ran}_u F \circ u \Rightarrow F$ form the components of the counit of the adjunction $u^* \dashv \operatorname{Ran}_u$.

2.2 Comma Categories and Kan Extensions computed as (Co)limits

Pointwise Kan extensions are diagrams that are computed as colimits or limits over comma categories at each point. To make sense of this, we first introduce comma categories.

Definition 2.2.1. Let $u \colon A \to B$ be a functor in Cat and $b \in B$.

(i) The **comma category** $u \downarrow b$ has

- as objects, pairs $(a, f \colon u(a) \to b)$ where $a \in A$ and f is a morphism in B and
- as morphisms $(a, f) \to (a', f')$, a morphism $h \colon a \to a'$ in A such that the following diagram commutes.

$$\begin{tikzcd}
u(a) \arrow[rr, "u(h)"] \arrow[dr, "f"'] & & u(a') \arrow[dl, "f'"] \\
& b &
\end{tikzcd}$$

(ii) The dual **comma category** $b \downarrow u$ has

- as objects, pairs $(a, f\colon b \to u(a))$ where $a \in A$ and f is a morphism in B and
- as morphisms $(a, f) \to (a', f')$, a morphism $h\colon a \to a'$ in A such that the following diagram commutes.

$$\begin{tikzcd} & b \arrow[dl, "f"'] \arrow[dr, "f'"] & \\ u(a) \arrow[rr, "u(h)"'] & & u(a') \end{tikzcd}$$

Remark 2.2.2. There are **canonical projection functors** from the comma categories $u \downarrow b$ and $b \downarrow u$ to A. We denote them by $\pi^b\colon u \downarrow b \to A$ and $\pi_b\colon b \downarrow u \to A$.

We can generalize the definition of comma categories with respect to two functors in Cat with common codomain. One could also generalize the following to functors between large categories.

Definition 2.2.3. Let $u\colon A \to B$ and $v\colon C \to B$ be two functors in Cat. The **comma category** $u \downarrow v$ has

- as objects, triples $(a, c, f\colon u(a) \to v(c))$ where $a \in A$, $c \in C$ and f is a morphism in B and
- as morphisms $(a, c, f) \to (a', c', f')$, pairs of morphisms $g\colon a \to a'$ in A and $h\colon c \to c'$ in C such that the following diagram commutes.

$$\begin{tikzcd} u(a) \arrow[r, "u(g)"] \arrow[d, "f"'] & u(a') \arrow[d, "f'"] \\ v(c) \arrow[r, "v(h)"'] & v(c') \end{tikzcd}$$

Remark 2.2.4. There are also **canonical projection functors** from $u \downarrow v$ to the categories A and C, denoted by $\pi_u\colon u \downarrow v \to A$ and $\pi_v\colon u \downarrow v \to C$. Moreover, there is a natural transformation $\nu\colon u\pi_u \Rightarrow v\pi_v$ defined by $\nu_{(a,c,f)} = f\colon u(a) \to v(c)$.

$$\begin{tikzcd} u \downarrow v \arrow[r, "\pi_u"] \arrow[d, "\pi_v"'] \arrow[dr, phantom, "\overset{\nu}{\swarrow}"] & A \arrow[d, "u"] \\ C \arrow[r, "v"'] & B \end{tikzcd}$$

Remark 2.2.5. The comma categories $u \downarrow b$ and $b \downarrow u$ are special cases of these general comma categories, if we consider the object b as a functor $b\colon \mathbb{1} \to B$.

$$\begin{tikzcd} u \downarrow b \arrow[r, "\pi^b"] \arrow[d] \arrow[dr, phantom, "\nu \swarrow"] & A \arrow[d, "u"] \\ \mathbb{1} \arrow[r, "b"'] & B \end{tikzcd} \qquad \begin{tikzcd} b \downarrow u \arrow[r] \arrow[d, "\pi_b"'] \arrow[dr, phantom, "\mu \swarrow"] & \mathbb{1} \arrow[d, "b"] \\ A \arrow[r, "u"'] & B \end{tikzcd}$$

When they exist, the colimits (resp. limits) of the projection functor of the comma category $u \downarrow b$ (resp. $b \downarrow u$) composed with a diagram $F\colon A \to \mathcal{C}$ give us a formula to compute the left or right Kan extension of F along u. These formulas are called the **colimit formula** and the **limit formula**.

Theorem 2.2.6. *Let $u\colon A \to B$ be a functor in* Cat *and $F\colon A \to \mathcal{C}$ be a diagram.*

(i) If the colimit

$$\operatorname{Lan}_u F(b) := \operatorname{colim}(u \downarrow b \xrightarrow{\pi^b} A \xrightarrow{F} \mathcal{C})$$

exists for every $b \in B$, the left Kan extension $\operatorname{Lan}_u F\colon B \to \mathcal{C}$ is defined by these colimits and the unit transformation $\eta\colon F \Rightarrow \operatorname{Lan}_u F \circ u$ is extracted from the colimit cone.

(ii) If the limit

$$\operatorname{Ran}_u F(b) := \lim(b \downarrow u \xrightarrow{\pi_b} A \xrightarrow{F} \mathcal{C})$$

exists for every $b \in B$, the right Kan extension $\operatorname{Ran}_u F\colon B \to \mathcal{C}$ is defined by these limits and the counit transformation $\epsilon\colon \operatorname{Ran}_u F \circ u \Rightarrow F$ is extracted from the limit cone.

Proof. (ii) Since all such limits exist, we first construct a diagram $\operatorname{Ran}_u F\colon B \to \mathcal{C}$ such that $\operatorname{Ran}_u F(b) = \lim_{b\downarrow u}(F\pi_b)$ for every $b \in B$. Denote by $\mu^b\colon \operatorname{Ran}_u F(b) \Rightarrow F\pi_b$ the limit cone. For a morphism $g\colon b \to b'$ in B, we have the following commutative diagram.

$$\begin{tikzcd} b' \downarrow u \arrow[rr, "g^*"] \arrow[dr, "\pi_{b'}"'] & & b \downarrow u \arrow[dl, "\pi_b"] \\ & A & \end{tikzcd}$$

Hence $F\pi_{b'} = F\pi_b\, g^*$ and the limit cone of $\operatorname{Ran}_u F(b)$ restricted along g is a cone over $F\pi_{b'}$. Then define $\operatorname{Ran}_u F(g)\colon \operatorname{Ran}_u F(b) \to \operatorname{Ran}_u F(b')$ to be the unique morphism given by the universal property of limits such that the following diagram commutes

$$\begin{tikzcd} \operatorname{Ran}_u F(b) \arrow[rr, dashed, "\exists!\, \operatorname{Ran}_u F(g)"] \arrow[dr, "\mu^b_{f\circ g}"'] & & \operatorname{Ran}_u F(b') \arrow[dl, "\mu^{b'}_f"] \\ & F(a) & \end{tikzcd}$$

for every object $(a, f\colon b' \to u(a)) \in b' \downarrow u$. Uniqueness implies that $\mathrm{Ran}_u F$ is functorial. We now define the counit transformation $\epsilon\colon \mathrm{Ran}_u F \circ u \Rightarrow F$ to be

$$\epsilon_a = \mu^{u(a)}_{1_{u(a)}}\colon \mathrm{Ran}_u F(u(a)) \to F(a)$$

for every $a \in A$. Consider a morphism $h\colon a \to a'$ in A. Then we have the following commutative diagram.

$$\begin{tikzcd}
\mathrm{Ran}_u F(u(a)) \arrow[r, "\epsilon_a = \mu^{u(a)}_{1_{u(a)}}"] \arrow[d, "\mathrm{Ran}_u F(u(h))"'] \arrow[dr, "\mu^{u(a)}_{u(h)}"] & F(a) \arrow[d, "F(h)"] \\
\mathrm{Ran}_u F(u(a')) \arrow[r, "\epsilon_{a'} = \mu^{u(a')}_{1_{u(a')}}"'] & F(a')
\end{tikzcd}$$

The upper triangle actually commutes by naturality of $\mu^{u(a)}\colon \mathrm{Ran}_u F(u(a)) \Rightarrow F\pi_{u(a)}$ while the lower triangle commutes by construction of $\mathrm{Ran}_u F(u(h))$. This shows that ϵ is a natural transformation.

It remains to show that $(\mathrm{Ran}_u F, \epsilon)$ satisfies the universal property of right Kan extensions. For $(G\colon B \to \mathcal{C}, \delta\colon G \circ u \Rightarrow F)$, we must construct a natural transformation $\beta\colon G \Rightarrow \mathrm{Ran}_u F$ which is unique with the following property.

$$\begin{tikzcd}
& B \arrow[dr, "G"] \arrow[d, phantom, "\Downarrow\delta"] & \\
A \arrow[ur, "u"] \arrow[rr, "F"'] & {} & \mathcal{C}
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
& B \arrow[dr, "\mathrm{Ran}_u F"'] \arrow[dr, bend left, "G"] \arrow[d, phantom, "\Downarrow\epsilon"] \arrow[dr, phantom, bend left=20, "\Leftarrow \exists!\beta"] & \\
A \arrow[ur, "u"] \arrow[rr, "F"'] & {} & \mathcal{C}
\end{tikzcd}$$

Consider a morphism $h\colon (a, f) \to (a', f')$ in $b \downarrow u$. This gives us a cone over $F\pi_b$ with summit $G(b)$ given by the following right-hand commutative diagram.

$$\begin{tikzcd}
& u(a) \arrow[d, "u(h)"] \\
b \arrow[ur, "f"] \arrow[dr, "f'"'] & \\
& u(a')
\end{tikzcd}
\qquad
\begin{tikzcd}
& Gu(a) \arrow[r, "\delta_a"] \arrow[d, "Gu(h)"] & F(a) \arrow[d, "Fh"] \\
G(b) \arrow[ur, "Gf"] \arrow[dr, "Gf'"'] & & \\
& Gu(a') \arrow[r, "\delta_{a'}"'] & F(a')
\end{tikzcd}$$

The right-hand triangle commutes since G is a functor and the square commutes by naturality of δ. We define $\beta_b\colon G(b) \to \mathrm{Ran}_u F(b)$ to be the unique morphism given by the universal property of limits.

$$
\begin{tikzcd}
 & G(b) \arrow[dl, "Gf"'] \arrow[dr, "Gf'"] \arrow[d, dashed, "\exists!\, \beta_b"] & \\
Gu(a) \arrow[d, "\delta_a"'] & \operatorname{Ran}_u F(b) \arrow[dl, "\mu^b_f"'] \arrow[dr, "\mu^b_{f'}"] & Gu(a') \arrow[d, "\delta_{a'}"] \\
F(a) \arrow[rr, "Fh"'] & & F(a')
\end{tikzcd}
$$

To show that β is natural, consider a morphism $g\colon b \to b'$ in B. Then the following diagram commutes

$$
\begin{tikzcd}
G(b) \arrow[rr, "Gg"] \arrow[d, "\beta_b"'] & & G(b') \arrow[d, "\beta_{b'}"] \\
\operatorname{Ran}_u F(b) \arrow[rr, "\operatorname{Ran}_u F(g)"] \arrow[dr, "\mu^b_{f\circ g}"'] & & \operatorname{Ran}_u F(b') \arrow[dl, "\mu^{b'}_f"] \\
 & F(a) &
\end{tikzcd}
$$

for every $(a, f\colon b' \to u(a)) \in b' \downarrow u$. The lower triangle commutes by construction of $\operatorname{Ran}_u F(g)$ and the exterior of the diagram commutes by construction of β_b and $\beta_{b'}$. Since this holds for every $(a, f\colon b' \to u(a)) \in b' \downarrow u(a)$ and $\operatorname{Ran}_u F(b')$ is a limit, the upper diagram commutes and hence β is natural. Moreover, the natural transformation β is such that $\delta = \epsilon \circ \beta u$ since, for every $a \in A$, the following diagram commutes by construction of β.

$$
\begin{tikzcd}
Gu(a) \arrow[r, equal] \arrow[d, "\beta_{u(a)}"'] & Gu(a) \arrow[d, "\delta_a"] \\
\operatorname{Ran}_u F(u(a)) \arrow[r, "\epsilon_a = \mu^{u(a)}_{1_{u(a)}}"'] & F(a)
\end{tikzcd}
$$

Finally, the natural transformation β is unique with this property. To see this, for every $(a, f\colon b \to u(a)) \in b \downarrow u$, the following diagram must commute in order for β to be natural.

$$\begin{tikzcd}
G(b) \arrow[rr, "Gf"] \arrow[d, "\beta_b"'] & & Gu(a) \arrow[d, "\beta_{u(a)}"] \arrow[dl, bend right, "\delta_a"'] \\
\operatorname{Ran}_u F(b) \arrow[rr, "\operatorname{Ran}_u F(f)"] \arrow[dr, "\mu^b_f"'] & {} & \operatorname{Ran}_u F(u(a)) \arrow[dl, "\epsilon_a = \mu^{u(a)}_{1_{u(a)}}"] \\
& F(a) &
\end{tikzcd}$$

This forces β to be defined as we did.
(i) The proof is dual to (ii). $\square$

Finally, we check that, when $\mathcal{C}$ is a cocomplete (resp. complete) category, the left (resp. right) Kan extension along a functor in Cat always exist and can be computed by the colimit (resp. limit) formula.

Corollary 2.2.7. *Let $u\colon A \to B$ be a functor in* Cat *and $\mathcal{C}$ be a category.*

(i) If $\mathcal{C}$ is cocomplete, then the left Kan extension functor

$$\begin{tikzcd}
\mathcal{C}^A \arrow[r, bend left, "\operatorname{Lan}_u"] \arrow[r, phantom, "\perp"] & \mathcal{C}^B \arrow[l, bend left, "u^*"]
\end{tikzcd}$$

exists and is given by the colimit formula.

(ii) If $\mathcal{C}$ is complete, then the right Kan extension functor

$$\begin{tikzcd}
\mathcal{C}^A \arrow[r, bend left, "u^*"] \arrow[r, phantom, "\perp"] & \mathcal{C}^B \arrow[l, bend left, "\operatorname{Ran}_u"]
\end{tikzcd}$$

exists and is given by the limit formula.

Proof. (ii) Since A and B are small categories, the category $b \downarrow u$ is also small, for every $b \in B$. If $\mathcal{C}$ is complete, the limit $\lim(b \downarrow u \xrightarrow{\pi_b} A \xrightarrow{F} \mathcal{C})$ exists, for every diagram $F\colon A \to \mathcal{C}$ and every $b \in B$, and these limits define the right Kan extension $\operatorname{Ran}_u F\colon B \to \mathcal{C}$ by Theorem 2.2.6 (ii).
(i) The proof is dual to (ii). $\square$

2.3 Colimits and Limits as Kan Extensions

In this section, we verify that Kan extensions along the unique functor $A \to \mathbb{1}$ correspond to colimits and limits of diagrams of shape A, for every small category A. We also introduce another way to describe the colimit and limit formulas, which will be useful to check one of the axioms for derivators for the 2-functor $\mathcal{C}^{(-)}\colon \mathrm{Cat}^{\mathrm{op}} \to \mathrm{CAT}$.

Theorem 2.3.1. *Let $F\colon A \to \mathcal{C}$ be a diagram.*

(i) The left Kan extension of F along the unique functor $p\colon A \to \mathbb{1}$ defines the colimit of F in $\mathcal{C}$, each existing if and only if the other does.

$$\begin{tikzcd} A \arrow[rr, "F"] \arrow[dr, "p"'] & \arrow[d, phantom, "\Downarrow\eta"] & \mathcal{C} \\ & \mathbb{1} \arrow[ur, "\mathrm{Lan}_p F \cong \mathrm{colim}_A F"'] & \end{tikzcd}$$

(ii) The right Kan extension of F along the unique functor $p\colon A \to \mathbb{1}$ defines the limit of F in $\mathcal{C}$, each existing if and only if the other does.

$$\begin{tikzcd} & \mathbb{1} \arrow[dr, "\mathrm{Ran}_p F \cong \lim_A F"] \arrow[d, phantom, "\Downarrow\epsilon"] & \\ A \arrow[ur, "p"] \arrow[rr, "F"'] & {} & \mathcal{C} \end{tikzcd}$$

Proof. (ii) The composition $\mathrm{Ran}_p F \circ p\colon A \to \mathcal{C}$ corresponds to the constant diagram $\Delta \mathrm{Ran}_p F\colon A \to \mathcal{C}$. Hence $\epsilon\colon \Delta \mathrm{Ran}_p F \Rightarrow F$ defines a cone over F. Moreover, the universal property of the right Kan extension is: for every $a\colon \mathbb{1} \to A$ and every $\delta\colon \Delta a \Rightarrow F$, there exists a unique morphism $s\colon a \to \mathrm{Ran}_p F$ such that $\delta = \epsilon \circ \Delta s$, which is exactly the universal property of the limit of F. By uniqueness of limits, we obtain $\mathrm{Ran}_p F \cong \lim_A F$.
(i) The proof is dual to (ii). $\square$

Corollary 2.3.2. *Let $u\colon A \to B$ be a functor in* Cat *and $F\colon A \to \mathcal{C}$ be a diagram.*

(i) The left Kan extension $(\mathrm{Lan}_u F\colon B \to \mathcal{C}, \eta\colon F \Rightarrow \mathrm{Lan}_u F \circ u)$ *can be computed by the colimit formula (Theorem 2.2.6 (i)) if and only if*

$$\begin{tikzcd} u \downarrow b \arrow[r, "\pi^b"] \arrow[dr, "r"'] & A \arrow[rr, "F"] \arrow[dr, "u"] \arrow[d, phantom, "\Downarrow\nu"] & \arrow[d, phantom, "\Downarrow\eta"] & \mathcal{C} \\ & \mathbb{1} \arrow[r, "b"'] & B \arrow[ur, "\mathrm{Lan}_u F"'] & \end{tikzcd}$$

defines the left Kan extension of $F\pi^b$ along r for every $b \in B$.

(ii) The right Kan extension $(\mathrm{Ran}_u F\colon B \to \mathcal{C}, \epsilon\colon \mathrm{Ran}_u F \circ u \Rightarrow F)$ *can be computed by the limit formula (Theorem 2.2.6 (ii)) if and only if*

$$\begin{tikzcd}
 & \mathbb{1} \arrow[r, "b"] \arrow[d, phantom, "\Downarrow\mu"] & B \arrow[dr, "\mathrm{Ran}_u F"] \arrow[d, phantom, "\Downarrow\epsilon"] & \\
b\downarrow u \arrow[ur, "s"] \arrow[r, "\pi_b"'] & A \arrow[ur, "u"] \arrow[rr, "F"'] & & \mathcal{C}
\end{tikzcd}$$

defines the right Kan extension of $F\pi_b$ along s for every $b \in B$.

Proof. This is immediate from Theorem 2.3.1. □

Remark 2.3.3. We also write $u_! = \mathrm{Lan}_u\colon \mathcal{C}^A \to \mathcal{C}^B$ for the left Kan extension functor along $u\colon A \to B$ and $u_* = \mathrm{Ran}_u\colon \mathcal{C}^A \to \mathcal{C}^B$ for the right Kan extension functor along u.

2.4 Derived Functors

We now introduce the notion of total derived functors. A total derived functor is a Kan extension along the localization functor of the homotopy category of a model category. We first give the construction of the homotopy category of a model category.

Theorem 2.4.1. *For every model category $\mathcal{M}$, there exists a homotopy category* $\mathrm{Ho}(\mathcal{M})$ *that is the localization at the weak equivalences of $\mathcal{M}$. In other words, there exists a category* $\mathrm{Ho}(\mathcal{M})$ *and a functor $\gamma\colon \mathcal{M} \to \mathrm{Ho}(\mathcal{M})$ carrying weak equivalences to isomorphisms such that the pair $(\mathrm{Ho}(\mathcal{M}), \gamma)$ satisfies the universal property: for every pair $(\mathcal{E}, F\colon \mathcal{M} \to \mathcal{E})$, where $\mathcal{E}$ is a category and F is a functor carrying weak equivalences to isomorphisms, there exists a unique functor $\overline{F}\colon \mathrm{Ho}(\mathcal{M}) \to \mathcal{E}$ such that $F = \overline{F} \circ \gamma$.*

$$\begin{tikzcd}
\mathcal{M} \arrow[rr, "F"] \arrow[dr, "\gamma"'] & & \mathcal{E} \\
 & \mathrm{Ho}(\mathcal{M}) \arrow[ur, dashed, "\exists!\,\overline{F}"'] &
\end{tikzcd}$$

*The functor $\gamma\colon \mathcal{M} \to \mathrm{Ho}(\mathcal{M})$ is called the **localization functor** of $\mathcal{M}$.*

Proof. We construct the homotopy category $\mathrm{Ho}(\mathcal{M})$ to be the category that admits

- the same objects as the category $\mathcal{M}$;
- as morphisms, equivalence classes of finite zig-zag of morphisms in $\mathcal{M}$, where only the weak equivalences are allowed to go backward, modulo the following rules:

– Adjacent arrows pointing in the same direction may be composed;

– Adjacent pairs $\xrightarrow{w}\xleftarrow{w}$ or $\xleftarrow{w}\xrightarrow{w}$, with w a weak equivalence in $\mathcal{M}$, may be removed;

– Identities may be removed.

Then, we define $\gamma\colon \mathcal{M} \to \mathrm{Ho}(\mathcal{M})$ to be the functor

- that is the identity on the objects;
- that carries a morphism $f\colon X \to Y$ in $\mathcal{M}$ to the unary zig-zag $f\colon X \to Y$ in $\mathrm{Ho}(\mathcal{M})$.

This defines a localization at the weak equivalences of $\mathcal{M}$, as it is proved in [GZ67], Chapter 1. □

As a consequence of this theorem, we can define the notion of total derived functors of a functor between model categories. One could also generalize this notion to every category that admits a homotopy category.

Definition 2.4.2. Let $F\colon \mathcal{M} \to \mathcal{N}$ be a functor between model categories and let $\gamma\colon \mathcal{M} \to \mathrm{Ho}(\mathcal{M})$ and $\delta\colon \mathcal{N} \to \mathrm{Ho}(\mathcal{N})$ be the localization functors.

(i) If the right Kan extension of δF along γ exists, it defines the **total left derived functor** $\mathbb{L}F$ of F.

$$\begin{tikzcd} \mathrm{Ho}(\mathcal{M}) \arrow[r, dashed, "\mathbb{L}F"] & \mathrm{Ho}(\mathcal{N}) \\ \mathcal{M} \arrow[u, "\gamma"] \arrow[r, "F"'] \arrow[ur, phantom, "\Downarrow"] & \mathcal{N} \arrow[u, "\delta"'] \end{tikzcd}$$

(ii) If the left Kan extension of δF along γ exists, it defines the **total right derived functor** $\mathbb{R}F$ of F.

$$\begin{tikzcd} \mathcal{M} \arrow[r, "F"] \arrow[d, "\gamma"'] \arrow[dr, phantom, "\Downarrow"] & \mathcal{N} \arrow[d, "\delta"] \\ \mathrm{Ho}(\mathcal{M}) \arrow[r, dashed, "\mathbb{R}F"'] & \mathrm{Ho}(\mathcal{N}) \end{tikzcd}$$

Remark 2.4.3. Kan extensions along functors between large categories are defined in the same way as the ones along functors in Cat (Definition 2.1.1).

The question is now: when does a functor admit a left or right total derived functor? This is, in particular, the case of a left or right deformable functor. This kind of functors interest us since an adjunction between model categories that is a Quillen pair gives rise to left and right deformable functors with respect to the cofibrant and fibrant replacement functors respectively. Here are first the definitions of a deformation and of a deformable functor.

Definition 2.4.4. Let $\mathcal{M}$ be a model category.

(i) A **left deformation** of $\mathcal{M}$ consists of a pair $(Q\colon \mathcal{M} \to \mathcal{M}, q\colon Q \Rightarrow 1_{\mathcal{M}})$, where Q is an endofunctor of $\mathcal{M}$ and q is a natural weak equivalence, i.e. q is a natural transformation such that $q_X\colon QX \to X$ is weak equivalence for every $X \in \mathcal{M}$.

(ii) A **right deformation** of $\mathcal{M}$ consists of a pair $(R\colon \mathcal{M} \to \mathcal{M}, r\colon 1_{\mathcal{M}} \Rightarrow R)$, where R is an endofunctor of $\mathcal{M}$ and r is a natural weak equivalence.

Definition 2.4.5. Let $F\colon \mathcal{M} \to \mathcal{N}$ be a functor between model categories.

(i) The functor F is **left deformable** if there exists a left deformation (Q, q) of $\mathcal{M}$ such that F preserves all weak equivalences between objects in a full subcategory containing the image of Q. We say that (Q, q) is a **left deformation** of F.

(ii) The functor F is **right deformable** if there exists a right deformation (R, r) of $\mathcal{M}$ such that F preserves all weak equivalences between objects in a full subcategory containing the image of R. We say that (R, r) is a **right deformation** of F.

The next proposition shows that the left or right total derived functor of a left or right deformable functor exists. Moreover, it is an absolute Kan extension. We first define the notion of absolute total derived functors.

Definition 2.4.6. Let $F\colon \mathcal{M} \to \mathcal{N}$ be a functor between model categories and let $\gamma\colon \mathcal{M} \to \mathrm{Ho}(\mathcal{M})$ and $\delta\colon \mathcal{N} \to \mathrm{Ho}(\mathcal{N})$ be the localization functors.

(i) The total left derived functor $\mathbb{L}F\colon \mathrm{Ho}(\mathcal{M}) \to \mathrm{Ho}(\mathcal{N})$ is **absolute** if, for every functor $R\colon \mathrm{Ho}(\mathcal{N}) \to \mathcal{E}$, the functor $R\mathbb{L}F\colon \mathrm{Ho}(\mathcal{M}) \to \mathcal{E}$ is the right Kan extension of $R\delta F$ along γ.

$$\begin{tikzcd}
\mathrm{Ho}(\mathcal{M}) \arrow[r, "\mathbb{L}F"] & \mathrm{Ho}(\mathcal{N}) \arrow[r, "R"] & \mathcal{E} \\
\mathcal{M} \arrow[u, "\gamma"] \arrow[r, "F"'] \arrow[ur, phantom, "\Downarrow"] & \mathcal{N} \arrow[u, "\delta"'] &
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
& \mathrm{Ho}(\mathcal{M}) \arrow[dr, "\mathrm{Ran}_\gamma R\delta F"] \arrow[d, phantom, "\Downarrow"] & \\
\mathcal{M} \arrow[ur, "\gamma"] \arrow[rr, "R\delta F"'] & {} & \mathcal{E}
\end{tikzcd}$$

(ii) The total right derived functor $\mathbb{R}F\colon \mathrm{Ho}(\mathcal{M}) \to \mathrm{Ho}(\mathcal{N})$ is **absolute** if, for every functor $L\colon \mathrm{Ho}(\mathcal{N}) \to \mathcal{E}$, the functor $L\mathbb{R}F\colon \mathrm{Ho}(\mathcal{M}) \to \mathcal{E}$ is the left Kan extension of $L\delta F$ along γ

$$\begin{tikzcd} \mathcal{M} \arrow[r, "F"] \arrow[d, "\gamma"'] \arrow[dr, phantom, "\Downarrow"] & \mathcal{N} \arrow[d, "\delta"] & \\ \mathrm{Ho}(\mathcal{M}) \arrow[r, "\mathbb{R}F"'] & \mathrm{Ho}(\mathcal{N}) \arrow[r, "L"'] & \mathcal{E} \end{tikzcd} \quad = \quad \begin{tikzcd} \mathcal{M} \arrow[rr, "L\delta F"] \arrow[dr, "\gamma"'] & \Downarrow & \mathcal{E} \\ & \mathrm{Ho}(\mathcal{M}) \arrow[ur, "\mathrm{Lan}_\gamma L\delta F"'] & \end{tikzcd}$$

Proposition 2.4.7.

(i) The total left derived functor of a left deformable functor exists and is absolute.

(ii) The total right derived functor of a right deformable functor exists and is absolute.

Proof. See Proposition 6.4.12 in [Rie16]. □

Finally, if an adjunction between model categories is such that the left adjoint admits an absolute total left derived functor and the right adjoint admits an absolute total right derived functor, these form an adjunction between the homotopy categories. In particular, this happens when the left adjoint is left deformable and the right adjoint is right deformable.

Theorem 2.4.8. *Let $F\colon \mathcal{M} \to \mathcal{N}$ and $G\colon \mathcal{N} \to \mathcal{M}$ be functors between model categories such that $F \dashv G$ is an adjunction. If the total left derived functor $\mathbb{L}F$ of F and the total right derived functor $\mathbb{R}G$ of G exist and are absolute, then they form an adjunction*

$$\begin{tikzcd} \mathrm{Ho}(\mathcal{M}) \arrow[r, bend left, "\mathbb{L}F"] \arrow[r, phantom, "\perp"] & \mathrm{Ho}(\mathcal{N}) \arrow[l, bend left, "\mathbb{R}G"] \end{tikzcd}$$

between the homotopy categories of $\mathcal{M}$ and $\mathcal{N}$.

Proof. Let $\eta\colon 1_{\mathcal{M}} \Rightarrow GF$ denote the unit and $\epsilon\colon FG \Rightarrow 1_{\mathcal{N}}$ denote the counit of the adjunction $F \dashv G$. We construct the unit and counit for the adjunction $\mathbb{L}F \dashv \mathbb{R}G$. Since the total left derived functor $\mathbb{L}F$ is absolute, the functor $\mathbb{R}G \circ \mathbb{L}F\colon \mathrm{Ho}(\mathcal{M}) \to \mathrm{Ho}(\mathcal{M})$ is the right Kan extension of $\mathbb{R}G\delta F$ along γ.

$$\begin{tikzcd} \mathrm{Ho}(\mathcal{M}) \arrow[r, "\mathbb{L}F"] & \mathrm{Ho}(\mathcal{N}) \arrow[r, "\mathbb{R}G"] & \mathrm{Ho}(\mathcal{M}) \\ \mathcal{M} \arrow[u, "\gamma"] \arrow[r, "F"'] \arrow[ur, phantom, "\Downarrow"] & \mathcal{N} \arrow[u, "\delta"'] & \end{tikzcd}$$

By the universal property of this right Kan extension, there exists a unique natural transformation $\eta'\colon 1_{\mathrm{Ho}(\mathcal{M})} \Rightarrow \mathbb{R}G \circ \mathbb{L}F$ in the following diagram

$$
\begin{tikzcd}
 & & & & & \mathrm{Ho}(\mathcal{M}) \\
 & \mathcal{M} \arrow[r, "\gamma"] \arrow[dr, phantom, "\Downarrow"] & \mathrm{Ho}(\mathcal{M}) & & \mathrm{Ho}(\mathcal{M}) \arrow[ur, equal] \arrow[ur, phantom, "\exists!\eta'\Downarrow"'] \arrow[r, "\mathbb{L}F"'] \arrow[dr, phantom, "\Downarrow"] & \mathrm{Ho}(\mathcal{N}) \arrow[u, "\mathbb{R}G"'] \\
\mathcal{M} \arrow[ur, equal] \arrow[ur, phantom, "\eta\Downarrow"'] \arrow[r, "F"'] & \mathcal{N} \arrow[u, "G"] \arrow[r, "\delta"'] & \mathrm{Ho}(\mathcal{N}) \arrow[u, "\mathbb{R}G"'] & = & \mathcal{M} \arrow[u, "\gamma"] \arrow[r, "F"'] & \mathcal{N} \arrow[u, "\delta"']
\end{tikzcd}
$$

and we define η' as the unit of $\mathbb{L}F \dashv \mathbb{R}G$. Similarly, since the total right derived functor $\mathbb{R}G$ is absolute, the functor $\mathbb{L}F \circ \mathbb{R}G\colon \mathrm{Ho}(\mathcal{N}) \to \mathrm{Ho}(\mathcal{N})$ is the left Kan extension of $\mathbb{L}F\gamma G$ along δ.

$$
\begin{tikzcd}
\mathcal{N} \arrow[r, "G"] \arrow[d, "\delta"'] \arrow[dr, phantom, "\Downarrow"] & \mathcal{M} \arrow[d, "\gamma"] & \\
\mathrm{Ho}(\mathcal{N}) \arrow[r, "\mathbb{R}G"'] & \mathrm{Ho}(\mathcal{M}) \arrow[r, "\mathbb{L}F"'] & \mathrm{Ho}(\mathcal{N})
\end{tikzcd}
$$

By the universal property of this left Kan extension, there exists a unique natural transformation $\epsilon'\colon \mathbb{L}F \circ \mathbb{R}G \Rightarrow 1_{\mathrm{Ho}(\mathcal{N})}$ in the following diagram

$$
\begin{tikzcd}
\mathcal{N} \arrow[r, "G"] \arrow[dr, equal] \arrow[dr, phantom, "\epsilon\Downarrow"] & \mathcal{M} \arrow[r, "\gamma"] \arrow[d, "F"] \arrow[dr, phantom, "\Downarrow"] & \mathrm{Ho}(\mathcal{M}) \arrow[d, "\mathbb{L}F"] & & \mathcal{N} \arrow[r, "G"] \arrow[d, "\delta"'] \arrow[dr, phantom, "\Downarrow"] & \mathcal{M} \arrow[d, "\gamma"] \\
 & \mathcal{N} \arrow[r, "\delta"'] & \mathrm{Ho}(\mathcal{N}) & = & \mathrm{Ho}(\mathcal{N}) \arrow[r, "\mathbb{R}G"] \arrow[dr, equal] \arrow[dr, phantom, "\exists!\epsilon'\Downarrow"] & \mathrm{Ho}(\mathcal{M}) \arrow[d, "\mathbb{L}F"] \\
 & & & & & \mathrm{Ho}(\mathcal{N})
\end{tikzcd}
$$

and we define ϵ' as the counit of $\mathbb{L}F \dashv \mathbb{R}G$. The fact that η' and ϵ' satisfy the triangle identities follows from the triangle identities of η and ϵ, and from the universal properties of Kan extensions. □

Corollary 2.4.9. *Let $F\colon \mathcal{M} \to \mathcal{N}$ and $G\colon \mathcal{N} \to \mathcal{M}$ be functors between model categories such that $F \dashv G$ is an adjunction. If F is left deformable and G is right deformable, then the total left derived functor $\mathbb{L}F$ of F and the total right derived functor $\mathbb{R}G$ of G form an adjunction*

$$\begin{tikzcd} \mathrm{Ho}(\mathcal{M}) \arrow[r, bend left, "\mathbb{L}F"] \arrow[r, phantom, "\perp"] & \mathrm{Ho}(\mathcal{N}) \arrow[l, bend left, "\mathbb{R}G"] \end{tikzcd}$$

between the homotopy categories of $\mathcal{M}$ and $\mathcal{N}$.

Proof. This follows immediately from Proposition 2.4.7 and Theorem 2.4.8. □

2.5 Quillen Pairs and Derived Functors

In this section, we introduce the notion of Quillen pairs and show that the left Quillen functor is left deformable with respect to the cofibrant replacement functor and the right Quillen functor is right deformable with respect to the fibrant replacement functor. The results of last section imply that the appropriate total derived functors of the Quillen functors form an adjunction between the homotopy categories, which will allow us to construct the Kan extensions for the homotopy derivator of a combinatorial model category. In order that the cofibrant and fibrant replacements are functorial, we suppose here that all model categories admit functorial factorizations, which is in particular the case of the combinatorial ones.

Definition 2.5.1. Let $F\colon \mathcal{M} \to \mathcal{N}$ and $G\colon \mathcal{N} \to \mathcal{M}$ be functors between model categories such that $F \dashv G$ is an adjunction. The pair $F \dashv G$ is a **Quillen pair** if one of the following equivalent holds:

(i) The left adjoint F preserves cofibrations and trivial cofibrations.

(ii) The right adjoint G preserves fibrations and trivial fibrations.

We now define the notions of cofibrant and fibrant replacements and show that they induce left and right deformations for a model category.

Definition 2.5.2. Let $\mathcal{M}$ be a model category and let $X \in \mathcal{M}$.

(i) By the factorization axiom for model categories, the morphism $\varnothing \to X$ factors through

$$\begin{tikzcd} \varnothing \arrow[rr] \arrow[dr, hook] & & X \\ & QX \arrow[ur, two heads, "\sim", "q_X"'] & \end{tikzcd}$$

where $\varnothing \hookrightarrow QX$ is a cofibration and q_X is a trivial fibration, and we call QX a **cofibrant replacement** of X.

(ii) By the factorization axiom for model categories, the morphism $X \to *$ factors through

$$\begin{tikzcd} X \arrow[rr] \arrow[dr, hook, "\sim", "r_X"'] & & * \\ & RX \arrow[ur, two heads] & \end{tikzcd}$$

where r_X is a trivial fibration and $RX \twoheadrightarrow *$ is a fibration, and we call RX a **fibrant replacement** of X.

Lemma 2.5.3. *Let $\mathcal{M}$ be a model category.*

(i) The pair $(Q\colon \mathcal{M} \to \mathcal{M}, q\colon Q \Rightarrow 1_{\mathcal{M}})$ defines a left deformation of $\mathcal{M}$, where, for every $X \in \mathcal{M}$, QX is a cofibrant replacement of X and $q_X\colon QX \overset{\sim}{\twoheadrightarrow} X$ is the trivial fibration of Definition 2.5.2 (i).

(ii) The pair $(R\colon \mathcal{M} \to \mathcal{M}, r\colon 1_{\mathcal{M}} \Rightarrow R)$ defines a right deformation of $\mathcal{M}$, where, for every $X \in \mathcal{M}$, RX is a fibrant replacement of X and $r_X\colon X \overset{\sim}{\hookrightarrow} RX$ is the trivial cofibration of Definition 2.5.2 (ii).

Proof. (i) We first define how $Q\colon \mathcal{M} \to \mathcal{M}$ acts on the morphisms of $\mathcal{M}$. Let $f\colon X \to Y$ be a morphism in $\mathcal{M}$. By functorial factorizations, there exists a lift in the following diagram

$$\begin{tikzcd} \varnothing \arrow[r, equal] \arrow[d, hook] & \varnothing \arrow[d, hook] \\ QX \arrow[r, dashed, "\exists Qf"] \arrow[d, two heads, "q_X"', "\sim"] & QY \arrow[d, two heads, "q_Y", "\sim"'] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

which we define as $Qf\colon QX \to QY$, and this implies that $Q\colon \mathcal{M} \to \mathcal{M}$ is a functor and $q\colon Q \Rightarrow 1_{\mathcal{M}}$ is a natural transformation. Since $q_X\colon QX \overset{\sim}{\twoheadrightarrow} X$ is a weak equivalence for every $X \in \mathcal{M}$, this shows that (Q, q) is a left deformation of $\mathcal{M}$.
(ii) The proof is dual to (i). $\square$

To show that the left adjoint functor of a Quillen pair is left deformable and that the right one is right deformable, we use Ken Brown's Lemma, stated below.

Lemma 2.5.4 (Ken Brown's Lemma)**.** *Let $F\colon \mathcal{M} \to \mathcal{N}$ be a functor between model categories.*

(i) If the functor F carries trivial cofibrations between cofibrant objects in $\mathcal{M}$ to weak equivalences in $\mathcal{N}$, then it carries all weak equivalences between cofibrant objects in $\mathcal{M}$ to weak equivalences in $\mathcal{N}$.

(ii) If the functor F carries trivial fibrations between fibrant objects in $\mathcal{M}$ to weak equivalences in $\mathcal{N}$, then it carries all weak equivalences between fibrant objects in $\mathcal{M}$ to weak equivalences in $\mathcal{N}$.

Proof. See Corollary 7.7.2 in [Hir03]. $\square$

Lemma 2.5.5. *Let $F\colon \mathcal{M} \to \mathcal{N}$ and $G\colon \mathcal{N} \to \mathcal{M}$ be functors between model categories such that $F \dashv G$ is a Quillen pair.*

(i) The functor F is left deformable.

(ii) The functor G is right deformable.

Proof. (i) Since F preserves all trivial cofibrations, it carries in particular trivial cofibrations between cofibrant objects in $\mathcal{M}$ to weak equivalences in $\mathcal{N}$. By Ken Brown's Lemma, this implies that F preserves all weak equivalences between cofibrant objects. Hence, if we consider the cofibrant replacement left deformation (Q, q) of $\mathcal{M}$, then this defines a left deformation for F.
(ii) The proof is dual to (i). $\square$

It follows that a Quillen pair between model categories gives an adjunction between the homotopy categories.

Theorem 2.5.6. *Let $F\colon \mathcal{M} \to \mathcal{N}$ and $G\colon \mathcal{N} \to \mathcal{M}$ be functors between model categories such that $F \dashv G$ is a Quillen pair. Then the total left derived functor $\mathbb{L}F$ of F and the total right derived functor $\mathbb{R}G$ of G exist and are absolute. Moreover, they form an adjunction*

$$\begin{tikzcd} \mathrm{Ho}(\mathcal{M}) \arrow[r, bend left, "\mathbb{L}F"] \arrow[r, phantom, "\perp"] & \mathrm{Ho}(\mathcal{N}) \arrow[l, bend left, "\mathbb{R}G"] \end{tikzcd}$$

between the homotopy categories of $\mathcal{M}$ and $\mathcal{N}$.

Proof. This follows immediately from Lemma 2.5.5 and Corollary 2.4.9. $\square$

Finally, when the Quillen pair is a Quillen equivalence, the adjunction above is an equivalence. This will be useful to show that a Quillen equivalence between combinatorial model categories induces an equivalence between their homotopy derivators.

Definition 2.5.7. Let $F\colon \mathcal{M} \to \mathcal{N}$ and $G\colon \mathcal{N} \to \mathcal{M}$ be functors between model categories such that $F \dashv G$ is a Quillen pair. The pair $F \dashv G$ is a **Quillen equivalence** if, for every cofibrant object $X \in \mathcal{M}$ and every fibrant object $Y \in \mathcal{N}$, a morphism $f\colon X \to GY$ is a weak equivalence in $\mathcal{M}$ if and only if its adjunct $f^{\#}\colon FX \to Y$ is a weak equivalence in $\mathcal{N}$.

Theorem 2.5.8. *Let $F\colon \mathcal{M} \to \mathcal{N}$ and $G\colon \mathcal{N} \to \mathcal{M}$ be functors between model categories such that $F \dashv G$ is a Quillen equivalence. Then the total left derived functor $\mathbb{L}F$ of F and the total right derived functor $\mathbb{R}G$ of G are equivalences*

$$\begin{tikzcd} \mathrm{Ho}(\mathcal{M}) \arrow[rr, bend left, "\mathbb{L}F"] \arrow[rr, phantom, "\simeq"] & & \mathrm{Ho}(\mathcal{N}) \arrow[ll, bend left, "\mathbb{R}G"] \end{tikzcd}$$

between the homotopy categories of $\mathcal{M}$ and $\mathcal{N}$.

Proof. See Theorem 8.5.23 in [Hir03]. $\square$

Part II
Basic Localizers and Derivators

3 Derivators

The aim of this section is to state the four axioms that define a derivator and to show that the 2-functor $\mathcal{C}^{(-)}\colon \mathrm{Cat}^{\mathrm{op}} \to \mathrm{CAT}$ satisfies these four axioms, when $\mathcal{C}$ is a cocomplete and complete category. In order to do this, we introduce, in Section 3.1, the notion of left and right Beck-Chevalley squares, which are 2-squares in CAT such that their left or right mate respectively is a natural isomorphism. We show that, when the left and right mates are well-defined, the notion of left and right Beck-Chevalley squares coincide. Since an axiom for derivators requires that the left and right Kan extensions exist, this implies that the mates of the image under $\mathbb{D}$ of every square in Cat are well-defined, and this gives rise to a notion of $\mathbb{D}$-Beck-Chevalley square (Section 2.5), where $\mathbb{D}$ is a derivator. Corollary 2.3.2 gives an example of Beck-Chevalley squares: if $u\colon A \to B$ is a functor in Cat, $b \in B$ and $\mathcal{C}$ is a cocomplete and complete category, the squares

$$\begin{tikzcd}
\mathcal{C}^{u\downarrow b} & \mathcal{C}^{A} \arrow[l, "(\pi^b)^*"'] \\
\mathcal{C} \arrow[u, "\Delta_{u\downarrow b}"] \arrow[ur, phantom, "\nu^* \swarrow"] & \mathcal{C}^{B} \arrow[l, "b^*"] \arrow[u, "u^*"']
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
\mathcal{C}^{b\downarrow u} & \mathcal{C}^{A} \arrow[l, "(\pi_b)^*"'] \\
\mathcal{C} \arrow[u, "\Delta_{b\downarrow u}"] \arrow[ur, phantom, "\mu^* \nearrow"] & \mathcal{C}^{B} \arrow[l, "b^*"] \arrow[u, "u^*"']
\end{tikzcd}$$

are such that their mates are natural isomorphisms

$$\mathrm{colim}_{u\downarrow b}(\pi^b)^* \cong b^* \operatorname{Lan}_u \quad \text{ and } \quad b^* \operatorname{Ran}_u \cong \lim_{b\downarrow u}(\pi_b)^*.$$

This is exactly the axiom for derivators about pointwise computation of Kan extensions applied to the 2-functor $\mathcal{C}^{(-)}\colon \mathrm{Cat}^{\mathrm{op}} \to \mathrm{CAT}$, and the notion of Beck-Chevalley squares allows us to state this axiom in the general case. In Section 3.2, we give the definition of a prederivator, which is just a 2-functor of the form $\mathrm{Cat}^{\mathrm{op}} \to \mathrm{CAT}$, and introduce related notions, as for example limits and colimits with respect to a prederivator. We also examine how a prederivator acts on adjunctions and on fully faithful functors that are part of an adjunction. These results are useful to describe the aspherical categories with respect to a basic localizer associated to a derivator. Finally, in Section 3.3, we state the four axioms of a derivator and show that the represented derivator associated to a cocomplete and complete category defines a derivator.

3.1 Beck-Chevalley Squares

A *square* in CAT is a diagram of the form

$$\begin{tikzcd}
\mathcal{C} & \mathcal{A} \arrow[l, "p^*"'] \\
\mathcal{D} \arrow[u, "D = v^*"] \arrow[ur, phantom, "\alpha \swarrow"] & \mathcal{B} \arrow[l, "q^*"] \arrow[u, "u^*"']
\end{tikzcd}$$

i.e. the data of four categories $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{D}$, of four functors $u^*\colon \mathcal{B} \to \mathcal{A}$, $v^*\colon \mathcal{D} \to \mathcal{C}$, $p^*\colon \mathcal{A} \to \mathcal{C}$ and $q^*\colon \mathcal{B} \to \mathcal{D}$ and a natural transformation $\alpha\colon p^*u^* \Rightarrow v^*q^*$. We consider in this section a class of squares in CAT, called the Beck-Chevalley squares. This notion of squares is useful to state the axiom for derivators about pointwise computation of Kan extensions.

Definition 3.1.1. Consider a square D in CAT.

$$D = \begin{tikzcd} \mathcal{C} & \mathcal{A} \arrow[l, "p^*"'] \\ \mathcal{D} \arrow[u, "v^*"] \arrow[ur, phantom, "\alpha \swArrow"] & \mathcal{B} \arrow[l, "q^*"] \arrow[u, "u^*"'] \end{tikzcd}$$

(i) The square D is **left Beck-Chevalley** if u^* and v^* admit left adjoints $u_!$ and $v_!$ respectively and the left mate $\alpha_!\colon v_!p^* \Rightarrow q^*u_!$ defined by

$$\begin{tikzcd} \mathcal{D} \arrow[dr, equal] & \mathcal{C} \arrow[l, "v_!"'] \arrow[dl, phantom, "\epsilon \swArrow"] & \mathcal{A} \arrow[l, "p^*"'] \arrow[dr, equal] & \\ & \mathcal{D} \arrow[u, "v^*"'] \arrow[ur, phantom, "\alpha \swArrow"] & \mathcal{B} \arrow[l, "q^*"] \arrow[u, "u^*"] \arrow[ur, phantom, "\eta \swArrow"] & \mathcal{A} \arrow[l, "u_!"] \end{tikzcd}$$

is a natural isomorphism, where $\epsilon\colon v_!v^* \Rightarrow 1_D$ is the counit and $\eta\colon\ 1_A \Rightarrow u^*u_!$ is the unit of the adjunctions.

(ii) Dually, the square D is **right Beck-Chevalley** if p^* and q^* admit right adjoints p_* and q_* respectively and the right mate $\alpha_*\colon u^*q_* \Rightarrow p_*v^*$ defined by

$$\begin{tikzcd} \mathcal{A} \arrow[dr, equal] \arrow[dr, phantom, "\eta \swArrow"'] & \\ \mathcal{C} \arrow[u, "p_*"] & \mathcal{A} \arrow[l, "p^*"] \\ \mathcal{D} \arrow[u, "v^*"] \arrow[ur, phantom, "\alpha \swArrow"] \arrow[dr, equal] & \mathcal{B} \arrow[l, "q^*"'] \arrow[u, "u^*"'] \arrow[dl, phantom, "\epsilon \swArrow"] \\ & \mathcal{D} \arrow[u, "q_*"'] \end{tikzcd}$$

is a natural isomorphism, where $\epsilon\colon q^*q_* \Rightarrow 1_D$ is the counit and $\eta\colon 1_A \Rightarrow p_*p^*$ is the unit of the adjunctions.

Remark 3.1.2. A square

$$D = \begin{tikzcd} \mathcal{C} & \mathcal{A} \arrow[l, "p^*"'] \\ \mathcal{D} \arrow[u, "v^*"] \arrow[ur, phantom, "\overset{\alpha}{\Swarrow}"] & \mathcal{B} \arrow[l, "q^*"] \arrow[u, "u^*"'] \end{tikzcd}$$

is left Beck-Chevalley if and only if its opposite square

$$D^{\mathrm{op}} = \begin{tikzcd} \mathcal{C}^{\mathrm{op}} & \mathcal{A}^{\mathrm{op}} \arrow[l, "p^*"'] \\ \mathcal{D}^{\mathrm{op}} \arrow[u, "v^*"] \arrow[ur, phantom, "\overset{\alpha^{\mathrm{op}}}{\Nearrow}"] & \mathcal{B}^{\mathrm{op}} \arrow[l, "q^*"] \arrow[u, "u^*"'] \end{tikzcd}$$

is right Beck-Chevalley.

We want to show that, when both of the appropriate pairs of adjoints in a square in CAT exist, the notions of left and right Beck-Chevalley squares coincide. Since one of the axioms for derivators requires that all adjoints exist, these two notions will always be the same while considering derivators. We first prove two properties of mates.

Proposition 3.1.3. *The passages to mates are compatible with horizontal pastings, i.e. if we consider the following horizontal pasting in* CAT

$$\begin{tikzcd} \mathcal{E} & \mathcal{C} \arrow[l, "r^*"'] & \mathcal{A} \arrow[l, "p^*"'] \\ \mathcal{F} \arrow[u, "w^*"] \arrow[ur, phantom, "\overset{\beta}{\Swarrow}"] & \mathcal{D} \arrow[l, "s^*"] \arrow[u, "v^*"] \arrow[ur, phantom, "\overset{\alpha}{\Swarrow}"] & \mathcal{B} \arrow[l, "q^*"] \arrow[u, "u^*"'] \end{tikzcd} = \begin{tikzcd} \mathcal{E} & \mathcal{A} \arrow[l, "r^*p^*"'] \\ \mathcal{F} \arrow[u, "w^*"] \arrow[ur, phantom, "\overset{\gamma}{\Swarrow}"] & \mathcal{B} \arrow[l, "s^*q^*"] \arrow[u, "u^*"'] \end{tikzcd}$$

we have the following diagrams

$$
\begin{tikzcd}
\mathcal{E} \arrow[d, "w_!"'] \arrow[dr, phantom, "\Searrow\beta_!"] & \mathcal{C} \arrow[l, "r^*"'] \arrow[d, "v_!"] \arrow[dr, phantom, "\Searrow\alpha_!"] & \mathcal{A} \arrow[l, "p^*"'] \arrow[d, "u_!"] \\
\mathcal{F} & \mathcal{D} \arrow[l, "s^*"] & \mathcal{B} \arrow[l, "q^*"]
\end{tikzcd}
=
\begin{tikzcd}
\mathcal{E} \arrow[d, "w_!"'] \arrow[dr, phantom, "\Searrow\gamma_!"] & \mathcal{A} \arrow[l, "r^*p^*"'] \arrow[d, "u_!"] \\
\mathcal{F} & \mathcal{B} \arrow[l, "s^*q^*"]
\end{tikzcd}
$$

$$
\begin{tikzcd}
\mathcal{E} \arrow[r, "r_*"] \arrow[dr, phantom, "\Nwarrow\beta_*"] & \mathcal{C} \arrow[r, "p_*"] \arrow[dr, phantom, "\Nwarrow\alpha_*"] & \mathcal{A} \\
\mathcal{F} \arrow[u, "w^*"] \arrow[r, "s_*"'] & \mathcal{D} \arrow[u, "v^*"'] \arrow[r, "q_*"'] & \mathcal{B} \arrow[u, "u^*"']
\end{tikzcd}
=
\begin{tikzcd}
\mathcal{E} \arrow[r, "p_*r_*"] \arrow[dr, phantom, "\Nwarrow\gamma_*"] & \mathcal{A} \\
\mathcal{F} \arrow[u, "w^*"] \arrow[r, "q_*s_*"'] & \mathcal{B} \arrow[u, "u^*"']
\end{tikzcd}
$$

i.e. $\gamma_! = s^*\alpha_! \circ \beta_! p^*$ *and* $\gamma_* = p_*\beta_* \circ \alpha_* s_*$.

Proof. We first show that $\gamma_! = s^*\alpha_! \circ \beta_! p^*$. Explicitly, the pasting of $\alpha_!$ and $\beta_!$ is given by

$$
\begin{tikzcd}
\mathcal{D} \arrow[dr, equal] \arrow[dr, phantom, shift left=3, "\epsilon\Swarrow"] & \mathcal{E} \arrow[l, "w_!"'] \arrow[dr, phantom, "\beta\Swarrow"] & \mathcal{C} \arrow[l, "r^*"'] \arrow[dr, equal] \arrow[dr, phantom, shift right=3, "\eta\Swarrow"'] & & & \\
& \mathcal{F} \arrow[u, "w^*"] & \mathcal{D} \arrow[l, "s^*"] \arrow[u, "v^*"'] \arrow[dr, equal] \arrow[dr, phantom, shift left=3, "\epsilon\Swarrow"] & \mathcal{C} \arrow[l, "v_!"'] \arrow[dr, phantom, "\alpha\Swarrow"] & \mathcal{A} \arrow[l, "p^*"'] \arrow[dr, equal] \arrow[dr, phantom, shift right=3, "\eta\Swarrow"'] & \\
& & & \mathcal{D} \arrow[u, "v^*"] & \mathcal{B} \arrow[l, "q^*"] \arrow[u, "u^*"'] & \mathcal{A} \arrow[l, "u_!"]
\end{tikzcd}
=
\begin{tikzcd}
\mathcal{F} \arrow[dr, equal] \arrow[dr, phantom, shift left=3, "\epsilon\Swarrow"] & \mathcal{E} \arrow[l, "w_!"'] \arrow[dr, phantom, "\gamma\Swarrow"] & \mathcal{A} \arrow[l, "r^*p^*"'] \arrow[dr, equal] \arrow[dr, phantom, shift right=3, "\eta\Swarrow"'] & \\
& \mathcal{F} \arrow[u, "w^*"] & \mathcal{B} \arrow[l, "s^*q^*"] \arrow[u, "u^*"'] & \mathcal{A} \arrow[l, "u_!"]
\end{tikzcd}
$$

where the equality comes from the triangle identities and the definition of γ and where ϵ and η denote the appropriate counits and units of adjunctions. This shows the first equality. The second equality $\gamma_* = p_*\beta_* \circ \alpha_* s_*$ comes from the fact that the unit of the adjunction $r^*p^* \dashv p_*r_*$ is the pasting of the units of the adjunctions $p^* \dashv p_*$ and $r^* \dashv r_*$ and the counit of the adjunction $s^*q^* \dashv q_*s_*$ is the pasting of the counits of the adjunctions $q^* \dashv q_*$ and $s^* \dashv s_*$. Hence the passages to mates is compatible with horizontal pasting. $\square$

Proposition 3.1.4. *The passages to mates are compatible with vertical pastings.*

Proof. The proof is dual to the proof of Proposition 3.1.3. $\square$

Remark 3.1.5. In particular, the class of Beck-Chevalley squares is stable under horizontal and vertical pasting.

Proposition 3.1.6. *The two passages to mates are inverse to each other, i.e. if we consider a square D in* CAT*,*

$$D = \begin{tikzcd} \mathcal{C} & \mathcal{A} \arrow[l, "p^*"'] \\ \mathcal{D} \arrow[u, "v^*"] & \mathcal{B} \arrow[l, "q^*"] \arrow[u, "u^*"'] \arrow[ul, phantom, "\overset{\alpha}{\Swarrow}"] \end{tikzcd}$$

then $\alpha = (\alpha_!)_*$ and $\alpha = (\alpha_*)_!$.

Proof. Explicitly, the natural transformation $(\alpha_!)_*$ is given by

$$\begin{tikzcd} \mathcal{C} & \\ \mathcal{D} \arrow[u, "v^*"] & \mathcal{C} \arrow[ul, equal] \arrow[l, "v_!"] \arrow[ul, phantom, "\overset{\eta}{\Swarrow}"'] \\ \mathcal{B} \arrow[u, "q^*"] & \mathcal{A} \arrow[l, "u_!"'] \arrow[u, "p^*"'] \arrow[ul, phantom, "\overset{\alpha_!}{\Swarrow}"] \\ & \mathcal{B} \arrow[ul, equal] \arrow[u, "u^*"'] \arrow[ul, phantom, "\overset{\epsilon}{\Swarrow}"] \end{tikzcd} = \begin{tikzcd} \mathcal{C} & & & \\ \mathcal{D} \arrow[u, "v^*"] & \mathcal{C} \arrow[ul, equal] \arrow[l, "v_!"'] \arrow[ul, phantom, "\overset{\eta}{\Swarrow}"'] & \mathcal{A} \arrow[l, "p^*"'] & \\ & \mathcal{D} \arrow[ul, equal] \arrow[u, "v^*"] \arrow[ul, phantom, "\overset{\epsilon}{\Swarrow}"] & \mathcal{B} \arrow[l, "q^*"] \arrow[u, "u^*"'] \arrow[ul, phantom, "\overset{\alpha}{\Swarrow}"] & \mathcal{A} \arrow[ul, equal] \arrow[l, "u_!"'] \arrow[ul, phantom, "\overset{\eta}{\Swarrow}"'] \\ & & & \mathcal{B} \arrow[ul, equal] \arrow[u, "u^*"'] \arrow[ul, phantom, "\overset{\epsilon}{\Swarrow}"] \end{tikzcd} = \begin{tikzcd} \mathcal{C} & \mathcal{A} \arrow[l, "p^*"'] \\ \mathcal{D} \arrow[u, "v^*"] & \mathcal{B} \arrow[l, "q^*"] \arrow[u, "u^*"'] \arrow[ul, phantom, "\overset{\alpha}{\Swarrow}"] \end{tikzcd}$$

where the equalities come from the triangle identities and where ϵ and η denote the appropriate counits and units of adjunctions. Hence $(\alpha_!)_* = \alpha$. Similarly, we can show that $(\alpha_*)_! = \alpha$. $\square$

A natural transformation $\alpha\colon u^* \Rightarrow v^*$ corresponds to squares in CAT

$$\begin{tikzcd} \mathcal{A} \arrow[r, equal] & \mathcal{A} \\ \mathcal{B} \arrow[u, "v^*"] \arrow[r, equal] & \mathcal{B} \arrow[u, "u^*"'] \arrow[ul, phantom, "\overset{\alpha}{\Swarrow}"] \end{tikzcd} \qquad \begin{tikzcd} \mathcal{A} & \mathcal{B} \arrow[l, "u^*"'] \\ \mathcal{A} \arrow[u, equal] & \mathcal{B} \arrow[l, "v^*"] \arrow[u, equal] \arrow[ul, phantom, "\overset{\alpha}{\Swarrow}"] \end{tikzcd}$$

and the left and right mates respectively of these squares define two natural transformations $\alpha_!\colon v_! \Rightarrow u_!$ and $\alpha_*\colon v_* \Rightarrow u_*$, which are said to be conjugate to α. We prove that a natural transformation is a natural isomorphism if and only if its conjugate is. For a square D in CAT,

$$\begin{tikzcd}
\mathcal{C} \arrow[dr, phantom, "\alpha \swarrow"] & \mathcal{A} \arrow[l, "p^*"'] \\
\mathcal{D} \arrow[u, "D = v^*"] & \mathcal{B} \arrow[l, "q^*"] \arrow[u, "u^*"']
\end{tikzcd}$$

we show that the mates $\alpha_!$ and α_* are conjugate to each other, which implies that a square is left Beck-Chevalley if and only if it is right Beck-Chevalley, when the mates are well-defined.

Definition 3.1.7. Let $\alpha\colon u^* \Rightarrow v^*$ be a natural transformation in CAT.

(i) Suppose u^* and v^* admit left adjoints $u_!$ and $v_!$ respectively. The natural transformation $\alpha_!\colon v_! \Rightarrow u_!$ defined by

$$v_! \xRightarrow{\eta} v_! u^* u_! \xRightarrow{\alpha} v_! v^* u_! \xRightarrow{\epsilon} u_!$$

is called the **left conjugate** to α, where $\eta\colon\ 1 \Rightarrow u^* u_!$ is the unit and $\epsilon\colon v_! v^* \Rightarrow 1$ is the counit of the adjunctions.

(ii) Dually, suppose u^* and v^* admit right adjoints u_* and v_* respectively. The natural transformation $\alpha_*\colon v_* \Rightarrow u_*$ defined by

$$v_* \xRightarrow{\eta} u_* u^* v_* \xRightarrow{\alpha} u_* v^* v_* \xRightarrow{\epsilon} u_*$$

is called the **right conjugate** to α, where $\eta\colon\ 1 \Rightarrow u_* u^*$ is the unit and $\epsilon\colon v^* v_* \Rightarrow 1$ is the counit of the adjunctions.

Remark 3.1.8. By Propositions 3.1.3 and 3.1.4, if $\alpha\colon u^* \Rightarrow v^*$ and $\beta\colon v^* \Rightarrow w^*$ are two natural transformations in CAT such that the appropriate adjoints of u^*, v^*, and w^* exist, then $(\beta \circ \alpha)_! = \alpha_! \circ \beta_!$ and $(\beta \circ \alpha)_* = \alpha_* \circ \beta_*$. Moreover, by Proposition 3.1.6, if $\alpha\colon u^* \Rightarrow v^*$ is a natural transformation in CAT such that the appropriate adjoints of u^* and v^* exist, we have that $\alpha = (\alpha_!)_*$ and $\alpha = (\alpha_*)_!$. This implies that the natural transformation α is the right conjugate to $\alpha_!$ and the left conjugate to α_* and we say that α and $\alpha_!$ (resp. α_*) are conjugate to each other. Finally, if $u^*\colon \mathcal{B} \to \mathcal{A}$ is a functor in CAT admitting the appropriate adjoint, $(1_{u^*})_! = 1_{u_!}$ and $(1_{u^*})_* = 1_{u_*}$.

It follows from this remark that a natural transformation is a natural isomorphism if and only if its conjugate is.

Lemma 3.1.9. *Let $\alpha\colon u^* \Rightarrow v^*$ be a natural transformation in* CAT.

(i) Suppose u^* and v^* admit left adjoints $u_!$ and $v_!$ respectively. Then $\alpha\colon u^* \Rightarrow v^*$ is a natural isomorphism if and only if $\alpha_!\colon v_! \Rightarrow u_!$ is a natural isomorphism.

(ii) Suppose u^* and v^* admit right adjoints u_* and v_* respectively. Then $\alpha\colon u^* \Rightarrow v^*$ is a natural isomorphism if and only if $\alpha_*\colon v_* \Rightarrow u_*$ is a natural isomorphism.

Proof. (i) Suppose α is a natural isomorphism and let $\beta\colon v^* \Rightarrow u^*$ be the inverse of α. By Remark 3.1.8, if $\beta_!\colon u_! \Rightarrow v_!$ is the left conjugate to β, we have

$$1_{u_!} = (1_{u^*})_! = (\beta \circ \alpha)_! = \alpha_! \circ \beta_! \quad \text{and} \quad 1_{v_!} = (1_{v^*})_! = (\alpha \circ \beta)_! = \beta_! \circ \alpha_!.$$

Hence $\alpha_!$ is a natural isomorphism with inverse $\beta_!$. The other implication is dual.
(ii) The proof is dual to (i). $\square$

From the definition of mates, we have the following lemma, stating that a natural transformation α in a square in CAT is conjugate to the mates $(\alpha_!)_!$ and $(\alpha_*)_*$. This allows us to show that the left and right mates of a square in CAT are conjugate.

Lemma 3.1.10. *Consider a square D in* CAT*.*

$$D = \begin{tikzcd} \mathcal{C} \arrow[r, leftarrow, "p^*"] & \mathcal{A} \\ \mathcal{D} \arrow[u, "v^*"] \arrow[r, leftarrow, "q^*"'] \arrow[ur, phantom, "\alpha \swarrow"] & \mathcal{B} \arrow[u, "u^*"'] \end{tikzcd}$$

(i) If the four functors in the square D admit a left adjoint, the mate $(\alpha_!)_!\colon q_!v_! \Rightarrow u_!p_!$ is left conjugate to $\alpha\colon p^*u^* \Rightarrow v^*q^*$.

(ii) If the four functors in the square D admit a right adjoint, the mate $(\alpha_*)_*\colon q_*v_* \Rightarrow u_*p_*$ is right conjugate to $\alpha\colon p^*u^* \Rightarrow v^*q^*$.

Proof. Immediate from the definition of the mates. $\square$

Proposition 3.1.11. *Consider a square D in* CAT*.*

$$D = \begin{tikzcd} \mathcal{C} \arrow[r, leftarrow, "p^*"] & \mathcal{A} \\ \mathcal{D} \arrow[u, "v^*"] \arrow[r, leftarrow, "q^*"'] \arrow[ur, phantom, "\alpha \swarrow"] & \mathcal{B} \arrow[u, "u^*"'] \end{tikzcd}$$

If u^*, v^* admit a left adjoint and p^*, q^* admit a right adjoint, the square D is left Beck-Chevalley if and only if it is right Beck-Chevalley.

Proof. We have to show that the mate $\alpha_!\colon v_!p^* \Rightarrow q^*u_!$ is a natural isomorphism if and only if the mate $\alpha_*\colon u^*q_* \Rightarrow p_*v^*$ is a natural isomorphism. By Proposition 3.1.6 and Lemma 3.1.10, we have that $\alpha_* = ((\alpha_!)_*)_*$ and hence that α_* and $\alpha_!$ are conjugate. It follows from Lemma 3.1.9 that one of these mates is a natural isomorphism if and only if the other is. $\square$

Hence, when the appropriate adjoints in a square in CAT exist, the notions of left and right Beck-Chevalley squares coincide and this gives rise to the following terminology.

Definition 3.1.12. Consider a square D in CAT.

$$\begin{tikzcd} & \mathcal{C} \arrow[dr, phantom, "\alpha \Swarrow"] & \mathcal{A} \arrow[l, "p^*"'] \\ D = & \mathcal{D} \arrow[u, "v^*"] & \mathcal{B} \arrow[l, "q^*"] \arrow[u, "u^*"'] \end{tikzcd}$$

When u^*, v^* admit a left adjoint and p^*, q^* admit a right adjoint, we say that the square is **Beck-Chevalley** if it is left or right Beck-Chevalley.

3.2 Prederivators

Before defining derivators, we give some notations, definitions and results related to the 2-functors $\mathrm{Cat}^{\mathrm{op}} \to \mathrm{CAT}$, called prederivators. We also define more explicitly the prederivator $\mathcal{C}^{(-)} \colon \mathrm{Cat}^{\mathrm{op}} \to \mathrm{CAT}$ for a category $\mathcal{C}$.

Definition 3.2.1. A **prederivator** $\mathbb{D}$ is a strict 2-functor $\mathbb{D} \colon \mathrm{Cat}^{\mathrm{op}} \to \mathrm{CAT}$ carrying

- a small category A to a category $\mathbb{D}(A)$;
- a functor $u \colon A \to B$ in Cat to a functor $u^* \colon \mathbb{D}(B) \to \mathbb{D}(A)$;
- a natural transformation $\alpha \colon u \Rightarrow v$ in Cat to a natural transformation $\alpha^* \colon u^* \Rightarrow v^*$.

Let $\mathbb{D}$ be a prederivator. The category $\mathbb{D}(\mathbb{1})$ has a special role, containing information about the prederivator. For example, in the case of the prederivator $\mathcal{C}^{(-)} \colon \mathrm{Cat}^{\mathrm{op}} \to \mathrm{CAT}$, it happens that this category corresponds to the category $\mathcal{C}$. We also introduce the notions of evaluation functors and constant coefficients, which correspond to the usual evalutation functors and diagonal functors respectively while considering the prederivator $\mathcal{C}^{(-)} \colon \mathrm{Cat}^{\mathrm{op}} \to \mathrm{CAT}$.

Definition 3.2.2. We call $\mathbb{D}(\mathbb{1})$ the **underlying category** of $\mathbb{D}$. The objects of $\mathbb{D}(\mathbb{1})$ are called the **absolute coefficients**.

Definition 3.2.3. Let A be a small category and $a \in A$. Then a induces a functor $a \colon \mathbb{1} \to A$ and hence a functor $a^* \colon \mathbb{D}(A) \to \mathbb{D}(\mathbb{1})$, called **evaluation functor**.

Definition 3.2.4. Let A be a small category and $p \colon A \to \mathbb{1}$ be the unique functor to the terminal category. A **constant coefficient** of A is an object of $\mathbb{D}(A)$ of the form p^*X, where $X \in \mathbb{D}(\mathbb{1})$ is an absolute coefficient.

Example 3.2.5. Let $\mathcal{C}$ be a category in CAT. There is a **represented prederivator** $\mathbb{D}_{\mathcal{C}} = \mathcal{C}^{(-)} \colon \mathrm{Cat}^{\mathrm{op}} \to \mathrm{CAT}$ carrying

- a small category A to the category of diagrams $\mathcal{C}^A$;
- a functor $u\colon A \to B$ in Cat to the precomposition functor $u^*\colon \mathcal{C}^B \to \mathcal{C}^A$ sending an object $F \in \mathcal{C}^B$ to the object $F \circ u \in \mathcal{C}^A$;
- a natural transformation $\alpha\colon u \Rightarrow v$ to the natural transformation $\alpha^*\colon u^* \Rightarrow v^*$ defined by $(\alpha^*)_F = F\alpha\colon F \circ u \Rightarrow F \circ v$, for every $F \in \mathcal{C}^B$.

The underlying category of $\mathbb{D}_{\mathcal{C}}$ is $\mathcal{C}^{\mathbb{1}} \cong \mathcal{C}$. Moreover, if A is a small category and $a \in A$, then the evaluation functor $a^*\colon \mathcal{C}^A \to \mathcal{C}$ corresponds to the usual evaluation functor sending a diagram $F\colon A \to \mathcal{C}$ to $F(a) \in \mathcal{C}$. Finally, if $p\colon A \to \mathbb{1}$, then $p^*\colon \mathcal{C} \to \mathcal{C}^A$ corresponds to the diagonal functor Δ_A sending an element $c \in \mathcal{C}$ to the constant functor $\Delta c\colon A \to \mathcal{C}$, which justifies the terminology of constant coefficient.

Since the left and right adjoints of the diagonal functor are the colimit and limit functors respectively, this example justifies the following notations.

Definition 3.2.6. Let A be a small category and $p\colon A \to \mathbb{1}$.

(i) We denote the image of p under $\mathbb{D}$ by

$$\Delta_A = p^*\colon \mathbb{D}(\mathbb{1}) \to \mathbb{D}(A).$$

(ii) If the left adjoint of Δ_A exists, we denote it by

$$\mathrm{colim}_A = p_!\colon \mathbb{D}(A) \to \mathbb{D}(\mathbb{1})$$

and call it the **colimit functor** of diagrams of shape A.

(iii) If the right adjoint of Δ_A exists, we denote it by

$$\lim_A = p_*\colon \mathbb{D}(A) \to \mathbb{D}(\mathbb{1})$$

and call it the **limit functor** of diagrams of shape A.

In the same way we defined squares in CAT, we can define squares in Cat to be diagrams

$$\mathcal{D} = \begin{tikzcd} C \arrow[r, "p"] \arrow[d, "v"'] \arrow[dr, phantom, "\alpha\swarrow"] & A \arrow[d, "u"] \\ D \arrow[r, "q"'] & B \end{tikzcd}$$

where A, B, C and D are small categories, $u\colon A \to B$, $v\colon C \to D$, $p\colon C \to A$ and $q\colon D \to B$ are functors in Cat and $\alpha\colon up \Rightarrow qv$ is a natural transformation. Since a prederivator is a 2-functor $\mathbb{D}\colon \mathrm{Cat}^{\mathrm{op}} \to \mathrm{CAT}$, from a square in Cat, we obtain a square in CAT.

Definition 3.2.7. Let

$$
\begin{tikzcd}
\mathcal{D} = & C \arrow[r, "p"] \arrow[d, "v"'] \arrow[dr, phantom, "\alpha \swarrow"] & A \arrow[d, "u"] \\
& D \arrow[r, "q"'] & B
\end{tikzcd}
$$

be a square in Cat. The **image under** $\mathbb{D}$ of the square $\mathcal{D}$ is the square $\mathbb{D}(\mathcal{D})$.

$$
\begin{tikzcd}
\mathbb{D}(\mathcal{D}) = & \mathbb{D}(C) \arrow[dr, phantom, "\alpha^* \swarrow"] & \mathbb{D}(A) \arrow[l, "p^*"'] \\
& \mathbb{D}(D) \arrow[u, "v^*"] & \mathbb{D}(B) \arrow[l, "q^*"] \arrow[u, "u^*"']
\end{tikzcd}
$$

Finally, we give two results about prederivators. The first one says that the image of an adjunction in Cat under $\mathbb{D}$ gives rise to an adjunction in CAT, but reversed, and the second one says that, if a fully faithful functor in Cat admits an adjoint, then the image of this adjoint under $\mathbb{D}$ is a fully faithful functor. These results are useful while considering the basic localizer of a derivator (see Section 5).

Proposition 3.2.8. *Let $u\colon A \to B$ and $v\colon B \to A$ be two functors in* Cat *such that $u \dashv v$ is an adjunction. Then $v^* \dashv u^*$ is an adjunction in* CAT.

Proof. Let $\eta\colon 1_A \Rightarrow vu$ and $\epsilon\colon uv \Rightarrow 1_B$ be the unit and counit of the adjunction. We have the triangle identities.

$$
\begin{tikzcd}
B \arrow[r, "v"] \arrow[dr, equal] \arrow[dr, phantom, "\epsilon \swarrow", near start] & A \arrow[d, "u"] \arrow[dr, equal] \arrow[dr, phantom, "\eta \swarrow"] & \\
& B \arrow[r, "v"'] & A
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[r, "u"] \arrow[dr, equal] \arrow[dr, phantom, "\eta \nearrow", near start] & B \arrow[d, "v"] \arrow[dr, equal] \arrow[dr, phantom, "\epsilon \nearrow"] & \\
& A \arrow[r, "u"'] & B
\end{tikzcd}
$$

Applying $\mathbb{D}$ to these diagrams, we obtain

$$
\begin{tikzcd}
\mathbb{D}(A) \arrow[r, "v^*"] \arrow[dr, equal] \arrow[dr, phantom, "\eta^* \nearrow", near start] & \mathbb{D}(B) \arrow[d, "u^*"] \arrow[dr, equal] \arrow[dr, phantom, "\epsilon^* \nearrow"] & \\
& \mathbb{D}(A) \arrow[r, "v^*"'] & \mathbb{D}(B)
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{D}(B) \arrow[r, "u^*"] \arrow[dr, equal] \arrow[dr, phantom, "\epsilon^* \swarrow", near start] & \mathbb{D}(A) \arrow[d, "v^*"] \arrow[dr, equal] \arrow[dr, phantom, "\eta^* \swarrow"] & \\
& \mathbb{D}(B) \arrow[r, "u^*"'] & \mathbb{D}(A)
\end{tikzcd}
$$

It follows from this that $v^* \dashv u^*$ is an adjunction with unit $\eta^*\colon 1_{\mathbb{D}(A)} \Rightarrow u^*v^*$ and counit $\epsilon^*\colon v^*u^* \Rightarrow 1_{\mathbb{D}(B)}$. $\square$

The second result we want to show follows directly from this proposition and the following lemma, giving a connection between the fully faithfulness of functors in an adjunction and the unit and counit of the adjunction.

Lemma 3.2.9. *Let $F\colon \mathcal{C} \to \mathcal{D}$ and $G\colon \mathcal{D} \to \mathcal{C}$ be two functors such that $F \dashv G$ is an adjunction with unit $\eta\colon 1_{\mathcal{C}} \Rightarrow GF$ and counit $\epsilon\colon FG \Rightarrow 1_{\mathcal{D}}$.*

(i) The functor F is fully faithful if and only if the unit η is a natural isomorphism.

(ii) The functor G is fully faithful if and only if the counit ϵ is a natural isomorphism.

Proof. (i) Suppose first that η is a natural isomorphism. We show that F is full and faithful. Let $a, b \in \mathcal{C}$ and $g\colon Fa \to Fb$ in $\mathcal{D}$. Define $h\colon a \to b$ to be the composite

$$a \xrightarrow{\eta_a} GFa \xrightarrow{Gg} GFb \xrightarrow{\eta_b^{-1}} b.$$

Then $Fh = g$ since the diagram

$$\begin{tikzcd}
Fa \arrow[r, "F\eta_a"] \arrow[dr, equal] & FGFa \arrow[rr, "FGg"] \arrow[d, "\epsilon_{Fa}"] & & FGFb \arrow[r, "F\eta_b^{-1}"] \arrow[d, "\epsilon_{Fb}"'] & Fb \\
& Fa \arrow[rr, "g"'] & & Fb \arrow[ur, equal] &
\end{tikzcd}$$

commutes by the triangle identities and the naturality of ϵ. Hence the functor F is full. Now consider $f, g\colon a \to b$ such that $Ff = Fg$. Then $GFf = GFg$ and, since the following diagrams commute by naturality of η,

$$\begin{tikzcd}
a \arrow[r, "f"] \arrow[d, "\eta_a"'] & b \\
GFa \arrow[r, "GFf"'] & GFb \arrow[u, "\eta_b^{-1}"']
\end{tikzcd}
\qquad
\begin{tikzcd}
a \arrow[r, "g"] \arrow[d, "\eta_a"'] & b \\
GFa \arrow[r, "GFg"'] & GFb \arrow[u, "\eta_b^{-1}"']
\end{tikzcd}$$

we have that $f = g$. Hence the functor F is faithful.

Now suppose that the functor F is fully faithul. We show that η is a natural isomorphism. Since F is fully faithful, it suffices to show that $F\eta\colon F \Rightarrow FGF$ is a natural isomorphism. Let $a \in \mathcal{C}$. By the triangle identities, we have $\epsilon_{Fa} \circ F\eta_a = 1_{Fa}$. Moreover, there exists a morphism $f\colon GFa \to a$ in $\mathcal{C}$ such that $Ff = \epsilon_{Fa}\colon FGFa \to Fa$, by fully faithfulness of F. Then the diagram

$$\begin{tikzcd}
GFa \arrow[r, "f"] \arrow[d, "\eta_{GFa}"'] \arrow[dr, equal] & a \arrow[d, "\eta_a"] \\
GFGFa \arrow[r, "G\epsilon_{Fa} = GFf"'] & GFa
\end{tikzcd}$$

commutes by the triangle identities and the naturality of η. Hence $\eta_a \circ f = 1_{GFa}$ and then $F\eta_a \circ Ff = F\eta_a \circ \epsilon_{Fa} = 1_{FGFa}$. Thus $\epsilon F\colon FGF \Rightarrow F$ is an inverse to $F\eta$ and finally $F\eta$ is a natural isomorphism.
(ii) The proof is dual to (i). □

Corollary 3.2.10. *Let $u\colon A \to B$ and $v\colon B \to A$ be two functors in* Cat *such that $u \dashv v$ is an adjunction.*

(i) If the functor u is fully faithful, then the functor v^* is fully faithful.

(ii) If the functor v is fully faithful, then the functor u^* is fully faithful.

Proof. (i) Let $\eta\colon 1_A \Rightarrow vu$ be the unit of the adjunction $u \dashv v$. By Lemma 3.2.9, if the functor u is fully faithful, then η is a natural isomorphism. By functoriality of $\mathbb{D}$, this implies that $\eta^*\colon 1_{\mathbb{D}(A)} \Rightarrow u^*v^*$ is a natural isomorphism. By Proposition 3.2.8, the natural transformation η^* defines the unit of the adjunction $v^* \dashv u^*$ and hence the functor v^* is fully faithful by Lemma 3.2.9.
(ii) The proof is dual to (i). □

3.3 Axioms for Derivators

Finally, we state the four axioms for derivators.

Definition 3.3.1. A prederivator $\mathbb{D}\colon \mathrm{Cat}^{\mathrm{op}} \to \mathrm{CAT}$ is a **derivator** if it satisfies the following axioms.

[D1] The prederivator $\mathbb{D}$ takes coproducts to products, i.e. the canonical morphism

$$\mathbb{D}(\coprod_{i\in I} A_i) \longrightarrow \prod_{i\in I} \mathbb{D}(A_i)$$

is an equivalence of categories for every family of small categories $\{A_i\}_{i\in I}$.

[D2] The family of evalutation functors $\{a^*\colon \mathbb{D}(A) \to \mathbb{D}(\mathbb{1})\}_{a\in A}$ is conservative for every small category A, i.e. if ϕ is a morphism in $\mathbb{D}(A)$ such that $a^*(\phi)$ is an isomorphism in $\mathbb{D}(\mathbb{1})$ for every $a \in A$, then ϕ is an isomorphism.

[D3] The prederivator $\mathbb{D}$ admits Kan extensions, i.e. for every $u\colon A \to B$ in Cat, the functor $u^*\colon \mathbb{D}(B) \to \mathbb{D}(A)$ admits a left adjoint $u_!\colon \mathbb{D}(A) \to \mathbb{D}(B)$ and a right adjoint $u_*\colon \mathbb{D}(A) \to \mathbb{D}(B)$.

[D4] The prederivator $\mathbb{D}$ admits pointwise Kan extensions, i.e. for every $u\colon A \to B$ in Cat and every $b \in B$, the image under $\mathbb{D}$ of the following comma squares

$$\begin{tikzcd} u\downarrow b \arrow[r, "\pi^b"] \arrow[d] \arrow[dr, phantom, "\nu \swArrow"] & A \arrow[d, "u"] \\ \mathbb{1} \arrow[r, "b"'] & B \end{tikzcd} \qquad \begin{tikzcd} b\downarrow u \arrow[r] \arrow[d, "\pi_b"'] \arrow[dr, phantom, "\mu \swArrow"] & \mathbb{1} \arrow[d, "b"] \\ A \arrow[r, "u"'] & B \end{tikzcd}$$

are Beck-Chevalley; in other words, there are natural isomorphisms

$$\operatorname{colim}_{u\downarrow b}(\pi^b)^* \cong b^*u_! \quad \text{and} \quad b^*u_* \cong \lim_{b\downarrow u}(\pi_b)^*.$$

The next result shows that the represented prederivator $\mathbb{D}_{\mathcal{C}}$ is a derivator when $\mathcal{C}$ is a cocomplete and complete category. This follows from the results about Kan extensions of Section 2.

Theorem 3.3.2. *Let $\mathcal{C}$ be a cocomplete and complete category. The represented prederivator*

$$\mathbb{D}_{\mathcal{C}} \colon \mathrm{Cat}^{\mathrm{op}} \to \mathrm{CAT}, \quad A \mapsto \mathcal{C}^A$$

*is a derivator, called the **represented derivator** of $\mathcal{C}$.*

Proof. Let $\{A_i\}_{i\in I}$ be a family of small categories and let $\iota_j \colon A_j \to \coprod_{i\in I} A_i$ denotes the inclusion for every $j \in I$. Then the morphisms ι_j induce functors

$$\iota_j^* \colon \mathcal{C}^{\amalg_{i\in I} A_i} \to \mathcal{C}^{A_j}$$

that satisfy the universal property of products. By uniqueness of products, we have that $\mathcal{C}^{\amalg_{i\in I} A_i} \cong \prod_{i\in I} \mathcal{C}^{A_i}$ and hence the prederivator $\mathbb{D}_{\mathcal{C}}$ satisfies axiom **[D1]**.

Axiom **[D2]** is clear since a natural transformation is a natural isomorphism if and only if each of its components is an isomorphism in $\mathcal{C}$.

By Corollary 2.2.7, the prederivator $\mathbb{D}_{\mathcal{C}}$ admits Kan extensions (axiom **[D3]**) and these Kan extensions are computed pointwise, i.e.

$$\operatorname{colim}_{u\downarrow b}(\pi^b)^* \cong b^*u_! \quad \text{and} \quad b^*u_* \cong \lim_{b\downarrow u}(\pi_b)^*.$$

Moreover, Corollary 2.3.2 says that the natural isomorphisms of the limit and colimit formulas are exactly the mates of axiom **[D4]**. $\square$

4 Basic Localizers

In Section 4.1, we give the definition of a basic localizer $\mathcal{W}$, which is a weakly saturated class of functors in Cat that contains the functors $A \to \mathbb{1}$ for small categories A admitting a terminal or initial object and that contains the $\mathcal{W}$-local and $\mathcal{W}$-colocal functors (Definition 4.1.16). For a basic localizer $\mathcal{W}$, we say that a category A is $\mathcal{W}$-aspherical category, if the unique functor $A \to \mathbb{1}$ belongs to $\mathcal{W}$. Then a functor $u\colon A \to B$ is said to be $\mathcal{W}$-aspherical (resp. $\mathcal{W}$-coaspherical) if the comma category $u \downarrow b$ (resp. $b \downarrow u$) is $\mathcal{W}$-aspherical for every $b \in B$. In particular, Proposition 4.1.18 implies that a $\mathcal{W}$-aspherical (resp. $\mathcal{W}$-coaspherical) functor is $\mathcal{W}$-local (resp. $\mathcal{W}$-colocal) with respect to every commutative triangle and hence the third axiom for basic localizers implies that it belongs to $\mathcal{W}$. Examples of basic localizers are the fundamental basic localizer $\mathcal{W}_0$, which consists of all functor $u\colon A \to B$ in Cat that induce isomorphisms $\pi_0(u)\colon \pi_0(A) \to \pi_0(B)$, and the minimal basic localizer $\mathcal{W}_\infty$, which consists of all functors in Cat whose nerves are weak homotopy equivalences of simplicial sets. Considering the first example, a category is $\mathcal{W}_0$-aspherical if it is non-empty and connected, and hence a functor is $\mathcal{W}_0$-aspherical (resp. $\mathcal{W}_0$-coaspherical) if it is initial (resp. final). Considering the second example, a category is $\mathcal{W}_\infty$-aspherical if its nerve is homotopy equivalent to a point, and hence a functor is $\mathcal{W}_\infty$-aspherical (resp. $\mathcal{W}_\infty$-coaspherical) if it is homotopy initial (resp. homotopy final). In Section 5, we construct a basic localizer $\mathcal{W}_\mathbb{D}$ associated to a derivator $\mathbb{D}$ and, keeping in mind these two examples, we can show that $\mathcal{W}_\mathbb{D}$-aspherical functors correspond to $\mathbb{D}$-initial functors.

In Section 4.2, we introduce the notion of $\mathcal{W}$-exact squares, which are squares in Cat such that induced functors between comma categories is $\mathcal{W}$-aspherical or $\mathcal{W}$-coaspherical. A stronger condition is to require that these induced functors are $\mathcal{W}$-local or $\mathcal{W}$-colocal over a certain category. This is the definition of a weak $\mathcal{W}$-exact square, which is given in Section 4.3. These two notions of squares actually coincide, except when considering the indiscrete basic localizer $\mathcal{W}_{\text{ind}}$, which consists of all functors between small categories that are both empty or both non-empty.

4.1 Definition and First Results

In this section, we give the definition of a basic localizer $\mathcal{W}$ and introduce two classes of functors in Cat induced by $\mathcal{W}$: the $\mathcal{W}$-aspherical and $\mathcal{W}$-local functors. In particular, these classes of functors are contained in $\mathcal{W}$. The first axiom for basic localizers requires that it is weakly saturated, as defined here.

Definition 4.1.1. A class $\mathcal{W}$ of functors in Cat is **weakly saturated** if it satisfies the following axioms.

[WS1] (Identities) The class $\mathcal{W}$ contains the identities.

[WS2] (Two of three) If two of three functors in a commutative triangle in Cat belong to $\mathcal{W}$, then so does the third one.

[WS3] (Retracts) If we have a commutative diagram in Cat of the form

$$
\begin{tikzcd}
A \arrow[r] \arrow[d, "u"'] \arrow[rr, bend left, "1_A"] & C \arrow[r] \arrow[d, "v"] & A \arrow[d, "u"] \\
B \arrow[r] \arrow[rr, bend right, "1_B"'] & D \arrow[r] & B
\end{tikzcd}
$$

i.e. the functor u is a retract of the functor v, and the functor v belongs to $\mathcal{W}$, then so does the functor u.

[WS4] (Opposite) If a functor $u\colon A \to B$ in Cat belongs to $\mathcal{W}$, then so does its opposite functor $u\colon A^{\mathrm{op}} \to B^{\mathrm{op}}$.

Remark 4.1.2. The axiom **[WS4]** actually follows from the three other ones. A proof can be found in [Mal05], Proposition 1.1.22.

We now state the three axioms for basic localizers.

Definition 4.1.3. A **basic localizer** is a class $\mathcal{W}$ of functors in Cat satisfying the following axioms.

[BL1] The class $\mathcal{W}$ is weakly saturated.

[BL2] If A is a small category admitting a terminal object, then $A \to \mathbb{1}$ belongs to $\mathcal{W}$.

[BL3] If

$$
\begin{tikzcd}
A \arrow[rr, "u"] \arrow[dr, "v"'] & & B \arrow[dl, "w"] \\
& C &
\end{tikzcd}
$$

is a commutative triangle in Cat and the functor

$$u^c\colon v \downarrow c \to w \downarrow c, \quad (a, v(a) \xrightarrow{f} c) \mapsto (u(a), wu(a) = v(a) \xrightarrow{f} c)$$

over u belongs to $\mathcal{W}$ for every $c \in C$, then the functor u belongs to $\mathcal{W}$.

Remark 4.1.4. By axiom **[WS4]**, we could have replaced axioms **[BL2]** and **[BL3]** by their following dual axioms.

[BL2'] If A is a small category admitting an initial object, then $A \to \mathbb{1}$ belongs to $\mathcal{W}$.

[BL3'] If

$$\begin{tikzcd} A \arrow[rr, "u"] \arrow[dr, "v"'] & & B \arrow[dl, "w"] \\ & C & \end{tikzcd}$$

is a commutative triangle in Cat and the functor

$$u_c \colon c \downarrow v \to c \downarrow w, \quad (a, c \xrightarrow{f} v(a)) \mapsto (u(a), c \xrightarrow{f} wu(a) = v(a))$$

over u belongs to $\mathcal{W}$ for every $c \in C$, then the functor u belongs to $\mathcal{W}$.

Let $\mathcal{W}$ be a basic localizer. We introduce the notions of $\mathcal{W}$-aspherical categories, as for example the small categories admitting a terminal object, and of $\mathcal{W}$-aspherical functors, which belong in particular to $\mathcal{W}$ by axiom [**BL3**].

Definition 4.1.5. An element of $\mathcal{W}$ is called a **$\mathcal{W}$-equivalence**.

Definition 4.1.6. A small category A is **$\mathcal{W}$-aspherical** if the functor $A \to \mathbb{1}$ belongs to $\mathcal{W}$.

Remark 4.1.7. By axiom [**WS4**], a small category A is $\mathcal{W}$-aspherical if and only if its opposite category A^{op} is $\mathcal{W}$-aspherical. The axiom [**BL2**] says that if A admits a terminal object, it is $\mathcal{W}$-aspherical. Dually, if A admits an initial object, it is also $\mathcal{W}$-aspherical.

Definition 4.1.8. Let $u \colon A \to B$ be a functor in Cat.

(i) The functor u is **$\mathcal{W}$-aspherical** if the functor

$$u^b \colon u \downarrow b \to B \downarrow b, \quad (a, u(a) \xrightarrow{f} b) \mapsto (u(a), u(a) \xrightarrow{f} b)$$

over u is a $\mathcal{W}$-equivalence for every $b \in B$.

(ii) Dually, the functor u is **$\mathcal{W}$-coaspherical** if the functor

$$u_b \colon b \downarrow u \to b \downarrow B, \quad (a, b \xrightarrow{f} u(a)) \mapsto (u(a), b \xrightarrow{f} u(a))$$

over u is a $\mathcal{W}$-equivalence for every $b \in B$.

Remark 4.1.9. The class of $\mathcal{W}$-aspherical functors contains the identities, satisfy the two-of-three axiom and is stable under retracts, according to axioms [**WS1**], [**WS2**] and [**WS3**]. Moreover, by axiom [**WS4**], a functor $u \colon A \to B$ in Cat is $\mathcal{W}$-aspherical if and only if its opposite functor $u \colon A^{\mathrm{op}} \to B^{\mathrm{op}}$ is $\mathcal{W}$-coaspherical, since the opposite functor of $u^b \colon u \downarrow b \to B \downarrow b$ is $u_b \colon b \downarrow u \to b \downarrow B$ for every $b \in B$. Furthermore, since the functors $u^b \colon u \downarrow b \to B \downarrow b$, for $b \in B$, correspond to the functors over u of axiom [**BL3**] applied to the commutative triangle

$$\begin{tikzcd} A \arrow[rr, "u"] \arrow[dr, "u"'] & & B \arrow[dl, "1_B"] \\ & B & \end{tikzcd}$$

if follows from this that if $u\colon A \to B$ is $\mathcal{W}$-aspherical, it is a $\mathcal{W}$-equivalence. Dually, if $u\colon A \to B$ is $\mathcal{W}$-coaspherical, it is also a $\mathcal{W}$-equivalence.

The next proposition and its corollary give relations between $\mathcal{W}$-aspherical (resp. $\mathcal{W}$-coaspherical) functors and $\mathcal{W}$-aspherical categories. Note that the notion of $\mathcal{W}$-aspherical categories is self-dual, while the notion of $\mathcal{W}$-aspherical functors is dual to the notion of $\mathcal{W}$-coaspherical functors.

Proposition 4.1.10. *Let $u\colon A \to B$ be a functor in* Cat*.*

(i) The functor u is $\mathcal{W}$-aspherical if and only if the category $u \downarrow b$ is $\mathcal{W}$-aspherical for every $b \in B$.

(ii) The functor u is $\mathcal{W}$-coaspherical if and only if the category $b \downarrow u$ is $\mathcal{W}$-aspherical for every $b \in B$.

Proof. (i) Let $b \in B$ and consider the following commutative diagram.

$$\begin{tikzcd} u \downarrow b \arrow[rr, "u^b"] \arrow[dr] & & B \downarrow b \arrow[dl] \\ & \mathbb{1} & \end{tikzcd}$$

Since $B \downarrow b$ admits a terminal object $(b, 1_b)$, it is $\mathcal{W}$-aspherical by axiom **[BL2]** and $B \downarrow b \to \mathbb{1}$ is a $\mathcal{W}$-equivalence. By axiom **[WS2]**, it follows that $u^b\colon u \downarrow b \to B \downarrow b$ is a $\mathcal{W}$-equivalence if and only if $u \downarrow b \to \mathbb{1}$ is a $\mathcal{W}$-equivalence, i.e. the category $u \downarrow b$ is $\mathcal{W}$-aspherical. Hence the functor u is $\mathcal{W}$-aspherical if and only if the category $u \downarrow b$ is $\mathcal{W}$-aspherical for every $b \in B$.
(ii) The proof is dual to (i). $\square$

Corollary 4.1.11. *Let A be a small category.*

(i) The category A is $\mathcal{W}$-aspherical if and only if the functor $A \to \mathbb{1}$ is $\mathcal{W}$-aspherical.

(ii) The category A is $\mathcal{W}$-aspherical if and only if the functor $A \to \mathbb{1}$ is $\mathcal{W}$-coaspherical.

Proof. (i) Denote by $p\colon A \to \mathbb{1}$ the unique functor to the terminal category. Then, by Proposition 4.1.10, the functor p is $\mathcal{W}$-aspherical if and only if the category $p \downarrow * \cong A$ is $\mathcal{W}$-aspherical.
(ii) The proof is dual to (i). $\square$

Remark 4.1.12. In particular, it follows from this corollary that, for every small category A, the unique functor $A \to \mathbb{1}$ is $\mathcal{W}$-aspherical if and only if it is $\mathcal{W}$-coaspherical.

In particular, the functors in Cat admitting a right (resp. left) adjoint are examples of $\mathcal{W}$-aspherical (resp. $\mathcal{W}$-coaspherical) functors. This results immediately from the following lemma and the axiom [**BL2**]. Moreover, since $\mathcal{W}$-aspherical (resp. $\mathcal{W}$-coaspherical) functors are $\mathcal{W}$-equivalences, this implies that functors that are part of an adjunction are $\mathcal{W}$-equivalences and that basic localizers contain all isomorphisms and equivalences of small categories.

Lemma 4.1.13. *Let $u\colon A \to B$ be a functor in* Cat*.*

(i) The functor u admits a right adjoint if and only if the category $u \downarrow b$ admits a terminal object for every $b \in B$.

(ii) The functor u admits a left adjoint if and only if the category $b \downarrow u$ admits an initial object for every $b \in B$.

Proof. (i) If $u\colon A \to B$ admits a right adjoint $v\colon B \to A$, then $(v(b), uv(b) \xrightarrow{\epsilon_b} b)$ is a terminal object of $u \downarrow b$ for every $b \in B$. Conversely, if the category $u \downarrow b$ admits a terminal object $(a_b, u(a_b) \xrightarrow{f_b} b)$ for every $b \in B$, define $v\colon B \to A$ to be the functor sending

- an object $b \in B$ to $v(b) = a_b$,
- a morphism $g\colon b \to b'$ in B to the unique morphism $v(g)\colon a_b \to a_{b'}$ given by the universal property of terminal objects such that the diagram

$$\begin{tikzcd} u(a_b) \arrow[r, "uv(g)"] \arrow[d, "f_b"'] & u(a_{b'}) \arrow[d, "f_{b'}"] \\ b \arrow[r, "g"'] & b' \end{tikzcd}$$

commutes.

Then v is functorial and defines a right adjoint to u.
(ii) The proof is dual to (i). $\square$

Proposition 4.1.14. *Let $u\colon A \to B$ be a functor in* Cat*.*

(i) If the functor u admits a right adjoint, it is $\mathcal{W}$-aspherical.

(ii) If the functor u admits a left adjoint, it is $\mathcal{W}$-coaspherical.

Proof. (i) By Lemma 4.1.13, if the functor u admits a right adjoint, the category $u \downarrow b$ admits a terminal object for every $b \in B$. Hence, by axiom [**BL2**], the category $u \downarrow b$ is $\mathcal{W}$-aspherical for every $b \in B$, which implies that the functor u is $\mathcal{W}$-aspherical by Proposition 4.1.10.
(ii) The proof is dual to (i). $\square$

Remark 4.1.15. It follows from Proposition 4.1.14 that all equivalences and isomorphisms are $\mathcal{W}$-(co)aspherical and, in particular, belong to $\mathcal{W}$.

We introduce a stronger notion of functors, called the $\mathcal{W}$-local functors over a small category, which are functors satisfying the hypotheses of axiom [**BL3**]. For example, $\mathcal{W}$-aspherical functors are $\mathcal{W}$-local functors over their target.

Definition 4.1.16. Consider the following commutative triangle in Cat.

$$\begin{tikzcd} A \arrow[rr, "u"] \arrow[dr, "v"'] & & B \arrow[dl, "w"] \\ & C & \end{tikzcd}$$

(i) The functor u is **$\mathcal{W}$-local** over C if the functor

$$u^c\colon v\downarrow c \to w\downarrow c, \quad (a, v(a) \xrightarrow{f} c) \mapsto (u(a), wu(a) = v(a) \xrightarrow{f} c)$$

over u is a $\mathcal{W}$-equivalence for every $c \in C$.

(ii) The functor u is **$\mathcal{W}$-colocal** over C if the functor

$$u_c\colon c\downarrow v \to c\downarrow w, \quad (a, c \xrightarrow{f} v(a)) \mapsto (u(a), c \xrightarrow{f} wu(a) = v(a))$$

over u is a $\mathcal{W}$-equivalence for every $c \in C$.

Remark 4.1.17. The class of $\mathcal{W}$-local functors contains the identities, satisfy the two-of-three axiom and is stable under retracts, according to axioms [**WS1**], [**WS2**] and [**WS3**]. Moreover, by axiom [**WS4**], a functor $u\colon A \to B$ in Cat is $\mathcal{W}$-local over C if and only if its opposite functor $u^{\mathrm{op}}\colon A^{\mathrm{op}} \to B^{\mathrm{op}}$ is $\mathcal{W}$-colocal over C^{op}, since the opposite functor $u^c\colon v\downarrow c \to w\downarrow c$ is $u_c\colon c\downarrow v \to c\downarrow w$ for every $c \in C$. Furthermore, the axiom [**BL3**] says that if $u\colon A \to B$ is $\mathcal{W}$-local over C, it is a $\mathcal{W}$-equivalence. Dually, if $u\colon A \to B$ is $\mathcal{W}$-colocal over C, it is also a $\mathcal{W}$-equivalence.

The next proposition says that if there is a functor between two small categories, then a $\mathcal{W}$-local functor over the source of this functor is a $\mathcal{W}$-local functor over the target of this functor. In particular, this implies that a $\mathcal{W}$-aspherical functor is $\mathcal{W}$-local.

Proposition 4.1.18. *For every functor $f\colon C \to C'$ in* Cat*, a $\mathcal{W}$-local functor over C is also $\mathcal{W}$-local over C'.*

Proof. Consider the following commutative diagram in Cat

$$\begin{tikzcd} A \arrow[rr, "u"] \arrow[dr, "v"'] & & B \arrow[dl, "w"] \\ & C \arrow[d, "f"] & \\ & C' & \end{tikzcd}$$

and suppose the functor u is $\mathcal{W}$-local over C. Then the diagram

$$\begin{tikzcd} fv \downarrow c' \arrow[rr, "u^{c'}"] \arrow[dr, "v^{c'}"'] & & fw \downarrow c' \arrow[dl, "w^{c'}"] \\ & f \downarrow c' & \end{tikzcd}$$

commutes for every $c' \in C'$. For $c' \in C'$ and an object $(c, f(c) \xrightarrow{h} c') \in f \downarrow c'$, we have the isomorphisms of categories

$$v^{c'} \downarrow (c,h) \cong v \downarrow c \quad \text{and} \quad w^{c'} \downarrow (c,h) \cong w \downarrow c$$

since their objects and morphisms are in correspondence. Then the functor $u^{c'} \downarrow (c,h)$ corresponds to the functor u^c, which is a $\mathcal{W}$-equivalence by hypothesis. This means that the functor $u^{c'}$ is $\mathcal{W}$-local over $f \downarrow c'$ and it follows from axiom [**BL3**] that it is a $\mathcal{W}$-equivalence. Since this holds for every $c' \in C'$, the functor u is $\mathcal{W}$-local over C'. $\square$

We defined in this section three classes of morphisms satisfying axioms [**WS1**], [**WS2**] and [**WS3**]: the class $\mathcal{W}$ of $\mathcal{W}$-equivalences, the class $\mathcal{W}_{\text{asph}}$ of $\mathcal{W}$-aspherical functors and the class $\mathcal{W}_{\text{loc}}$ of $\mathcal{W}$-local functors. In particular, following from Proposition 4.1.18, a $\mathcal{W}$-aspherical functor is a $\mathcal{W}$-local functor with respect to every commutative triangle in Cat as in 4.1.16 and hence we have the following inclusions

$$\mathcal{W}_{\text{asph}} \subseteq \mathcal{W}_{\text{loc}} \subseteq \mathcal{W}.$$

We finally give four examples of basic localizers.

Example 4.1.19. The class $\mathcal{W}_{\text{tr}}$ of all functors in Cat is a basic localizer, called the **trivial basic localizer**.

Example 4.1.20. The class $\mathcal{W}_{\text{ind}}$ of all functors between small categories that are both empty or both non-empty is a basic localizer, called the **indiscrete basic localizer**.

Example 4.1.21. Let A be a small category. There is an equivalence relation on the objects of A defined by $a \sim a'$ if and only if there exists a finite zig-zag of morphisms from a to a', where $a, a' \in A$. Let $\pi_0(A)$ be the discrete category of equivalence classes of A under this relation and let $\mathcal{W}_0$ be the class of functors $u\colon A \to B$ in Cat such that the map $\pi_0(u)\colon \pi_0(A) \to \pi_0(B)$ induced by u is a bijection. Then the class $\mathcal{W}_0$ forms a basic localizer (see [Mal11], Section 1.2), called the **fundamental basic localizer**. From this definition, a small category A is $\mathcal{W}_0$-aspherical if it is non-empty and connected and a functor $u\colon A \to B$ in Cat is $\mathcal{W}_0$-aspherical if the comma category $u \downarrow b$ is non-empty and connected for every $b \in B$, i.e. if u is initial. Dually, the $\mathcal{W}_0$-coaspherical functors correspond to the final functors.

Example 4.1.22. The class $\mathcal{W}_\infty$ of all functors whose nerve is a weak homotopy equivalence of simplicial sets is a basic localizer, which is the minimal one. More details can be found in Section 6.3.

Note that we have the inclusions $\mathcal{W}_0 \subset \mathcal{W}_{\text{ind}} \subset \mathcal{W}_{\text{tr}}$. In fact, the three basic localizers $\mathcal{W}_0$, $\mathcal{W}_{\text{tr}}$ and $\mathcal{W}_{\text{ind}}$ are the three largest basic localizers, while the basic localizer $\mathcal{W}_\infty$ is the minimal one (Theorem 6.3.5).

Theorem 4.1.23. *If $\mathcal{W}$ is a basic localizer that is not trivial or indiscrete, we have the following inclusions*

$$\mathcal{W} \subseteq \mathcal{W}_0 \subset \mathcal{W}_{\text{ind}} \subset \mathcal{W}_{\text{tr}}.$$

Proof. See [Mal11], Section 1.2. □

4.2 Exact Squares

Let $\mathcal{W}$ be a basic localizer. The basic localizer $\mathcal{W}$ induces a class of squares in Cat, called $\mathcal{W}$-exact squares, such that a functor between comma categories induced by the square is $\mathcal{W}$-aspherical. We show that this class of squares is the smallest one stable under horizontal pasting and horizontal descent and containing the comma squares and squares of the form

$$\begin{tikzcd} A \arrow[r, "u"] \arrow[d] \arrow[dr, phantom, "\overset{=}{\swarrow}"] & B \arrow[d] \\ \mathbb{1} \arrow[r, equal] & \mathbb{1} \end{tikzcd}$$

where the functor u is $\mathcal{W}$-coaspherical.

We first define the functors between comma categories induced by a square.

Definition 4.2.1. Consider a square $\mathcal{D}$ in Cat.

$$\mathcal{D} = \begin{tikzcd} C \arrow[r, "p"] \arrow[d, "v"'] \arrow[dr, phantom, "\overset{\alpha}{\swarrow}"] & A \arrow[d, "u"] \\ D \arrow[r, "q"'] & B \end{tikzcd}$$

(i) The functor p induces a functor $p^d : v \downarrow d \to u \downarrow q(d)$, for every $d \in D$, carrying an object $(c, v(c) \xrightarrow{h} d) \in v \downarrow d$ to the object $(p(c), up(c) \xrightarrow{\alpha_c} qv(c) \xrightarrow{q(h)} q(d))$ and a morphism $g : c \to c'$ in $v \downarrow d$ to the morphism $p(g) : p(c) \to p(c')$.

(ii) The functor v induces a functor $v_a : a \downarrow p \to u(a) \downarrow q$, for every $a \in A$, carrying an object $(c, a \xrightarrow{f} p(c)) \in a \downarrow p$ to the object $(v(c), u(a) \xrightarrow{u(f)} up(c) \xrightarrow{\alpha_c} qv(c))$ and a morphism $g : c \to c'$ in $a \downarrow p$ to the morphism $v(g) : v(c) \to v(c')$.

Remark 4.2.2. We say that $p^d : v \downarrow d \to u \downarrow q(d)$, for $d \in D$, is a functor over $p : C \to A$ and $v_a : a \downarrow p \to u(a) \downarrow q$, for $a \in A$, is a functor over $v : C \to D$ since we have the following commutative diagrams.

$$\begin{tikzcd} v \downarrow d \arrow[r, "p^d"] \arrow[d, "\pi^d"'] & u \downarrow q(d) \arrow[d, "\pi^{q(d)}"] \\ C \arrow[r, "p"'] & A \end{tikzcd} \qquad \begin{tikzcd} a \downarrow p \arrow[r, "v_a"] \arrow[d, "\pi_a"'] & u(a) \downarrow q \arrow[d, "\pi_{u(a)}"] \\ C \arrow[r, "v"'] & D \end{tikzcd}$$

Since the notions of $\mathcal{W}$-aspherical and $\mathcal{W}$-coaspherical functors are dual, we can define $\mathcal{W}$-exact squares in two ways: either the functors over p are $\mathcal{W}$-coaspherical or the functors over v are $\mathcal{W}$-aspherical.

Proposition 4.2.3. *Consider a square $\mathcal{D}$ in* Cat*.*

$$\mathcal{D} = \begin{tikzcd} C \arrow[r, "p"] \arrow[d, "v"'] \arrow[dr, phantom, "\alpha \swarrow"] & A \arrow[d, "u"] \\ D \arrow[r, "q"'] & B \end{tikzcd}$$

The following are equivalent:

(i) The functor $p^d\colon v \downarrow d \to u \downarrow q(d)$ over p is $\mathcal{W}$-coaspherical for every $d \in D$.

(ii) The functor $v_a\colon a \downarrow p \to u(a) \downarrow q$ over v is $\mathcal{W}$-aspherical for every $a \in A$.

Proof. Let $a \in A$, $d \in D$ and $g\colon u(a) \to q(d)$ in B. An object of the category $(a, g) \downarrow p^d$ is a triplet $(c, v(c) \xrightarrow{h} d, a \xrightarrow{f} p(c))$ such that the diagram

$$\begin{tikzcd} u(a) \arrow[r, "g"] \arrow[d, "u(f)"'] & q(d) \\ up(c) \arrow[r, "\alpha_c"'] & qv(c) \arrow[u, "q(h)"'] \end{tikzcd}$$

commutes and an object of the category $v_a \downarrow (d, g)$ is a triplet $(c, a \xrightarrow{f} p(c), v(c) \xrightarrow{h} d)$ such that the diagram above commutes. Moreover, morphisms in these categories are morphisms $l\colon c \to c'$ in C such that the following diagrams commute.

$$\begin{tikzcd} & p(c) \arrow[dd, "p(l)"] \\ a \arrow[ur, "f"] \arrow[dr, "f'"'] & \\ & p(c') \end{tikzcd} \qquad \begin{tikzcd} v(c) \arrow[dd, "v(l)"'] \arrow[dr, "h"] & \\ & d \\ v(c') \arrow[ur, "h'"'] & \end{tikzcd}$$

Hence the categories $(a, g) \downarrow p^d$ and $v_a \downarrow (d, g)$ are isomorphic. Since the functor p^d is $\mathcal{W}$-coaspherical if and only if the category $(a, g) \downarrow p^d$ is $\mathcal{W}$-aspherical for every $(a, g) \in u \downarrow q(d)$ and the functor v_a is $\mathcal{W}$-aspherical if and only if the category $v_a \downarrow (d, g)$

is $\mathcal{W}$-aspherical for every $(d, g) \in u(a) \downarrow q$ (Proposition 4.1.10), it follows that the functor p^d is $\mathcal{W}$-coaspherical for every $d \in D$ if and only if the functor v_a is $\mathcal{W}$-aspherical for every $a \in A$. $\square$

Definition 4.2.4. A square is **$\mathcal{W}$-exact** if it satisfies one of the two equivalent conditions (i) and (ii) of Proposition 4.2.3.

In particular, the next proposition gives a relation between $\mathcal{W}$-aspherical functors and $\mathcal{W}$-exact squares.

Proposition 4.2.5. *Let $u\colon A \to B$ be a functor in* Cat*.*

(i) The square

$$\mathcal{D} = \begin{tikzcd} A \arrow[r] \arrow[d, "u"'] \arrow[dr, phantom, "\overset{=}{\swArrow}"] & \mathbb{1} \arrow[d, equal] \\ B \arrow[r] & \mathbb{1} \end{tikzcd}$$

is $\mathcal{W}$-exact if and only if the functor u is $\mathcal{W}$-aspherical.

(ii) The square

$$\mathcal{D} = \begin{tikzcd} A \arrow[r, "u"] \arrow[d] \arrow[dr, phantom, "\overset{=}{\swArrow}"] & B \arrow[d] \\ \mathbb{1} \arrow[r, equal] & \mathbb{1} \end{tikzcd}$$

is $\mathcal{W}$-exact if and only if the functor u is $\mathcal{W}$-coaspherical.

Proof. (i) The functor of Definition 4.2.1 (ii) over u is exactly u. Hence the square $\mathcal{D}$ is $\mathcal{W}$-exact if and only if the functor u is $\mathcal{W}$-aspherical. Moreover, the functor of Definition 4.2.1 (i) over $A \to \mathbb{1}$ is $u \downarrow b \to \mathbb{1}$ for every $b \in B$. Since the square $\mathcal{D}$ is $\mathcal{W}$-exact if and only if $u \downarrow b \to \mathbb{1}$ is $\mathcal{W}$-coaspherical for every $b \in B$, it follows from Corollary 4.1.11 and Proposition 4.1.10 that the square $\mathcal{D}$ is $\mathcal{W}$-exact if and only if the category $u \downarrow b$ is $\mathcal{W}$-aspherical for every $b \in B$ if and only if the functor u is $\mathcal{W}$-aspherical.
(ii) The proof is dual to (i). $\square$

This proposition says that the class of $\mathcal{W}$-exact squares already satisfies one of the properties we required at the beginning of the section. In particular, considering the fundamental basic localizer $\mathcal{W}_0$, this proposition gives us a criteria for final and initial functors (see Theorem 8.1.3).

Remark 4.2.6. If a functor $u\colon A \to B$ in Cat admits a right adjoint, it is $\mathcal{W}$-aspherical (Proposition 4.1.14) and the square

$$\mathcal{D} = \begin{tikzcd} A \arrow[r] \arrow[d, "u"'] \arrow[dr, phantom, "\overset{=}{\swArrow}"] & \mathbb{1} \arrow[d, equal] \\ B \arrow[r] & \mathbb{1} \end{tikzcd}$$

is $\mathcal{W}$-exact. And, if a functor $u\colon A \to B$ in Cat admits a left adjoint, it is $\mathcal{W}$-coaspherical (Proposition 4.1.14) and the square

$$\mathcal{D} = \begin{tikzcd} A \arrow[r, "u"] \arrow[d] \arrow[dr, phantom, "\overset{=}{\swArrow}"] & B \arrow[d] \\ \mathbb{1} \arrow[r, equal] & \mathbb{1} \end{tikzcd}$$

is $\mathcal{W}$-exact.

There is also a relation between $\mathcal{W}$-aspherical categories and $\mathcal{W}$-exact squares.

Corollary 4.2.7. *Let A be a small category. The square*

$$\begin{tikzcd} A \arrow[r] \arrow[d] \arrow[dr, phantom, "\overset{=}{\swArrow}"] & \mathbb{1} \arrow[d, equal] \\ \mathbb{1} \arrow[r, equal] & \mathbb{1} \end{tikzcd}$$

is $\mathcal{W}$-exact if and only if the category A is $\mathcal{W}$-aspherical.

Proof. Immediate from Proposition 4.2.5 and Corollary 4.1.11. $\square$

We now verify the rest of the properties we required for the class of $\mathcal{W}$-exact squares, i.e. the stability under horizontal pasting and horizontal descent and the fact that it contains the comma squares.

Proposition 4.2.8. *The class of $\mathcal{W}$-exact squares*

(i) is stable under horizontal pasting;

(ii) is stable under horizontal descent, i.e. if J is a set and

$$\mathcal{D} = \begin{tikzcd} C \arrow[r, "p"] \arrow[d, "v"'] \arrow[dr, phantom, "\overset{\alpha}{\swArrow}"] & A \arrow[d, "u"] \\ D \arrow[r, "q"'] & B \end{tikzcd} \qquad \mathcal{D}_j = \begin{tikzcd} C_j \arrow[r, "p_j"] \arrow[d, "v_j"'] \arrow[dr, phantom, "\overset{\alpha_j}{\swArrow}"] & C \arrow[d, "v"] \\ D_j \arrow[r, "q_j"'] & D \end{tikzcd}$$

are squares in Cat *such that* $\mathrm{Ob}(D) = \bigcup_{j\in J} q_j(\mathrm{Ob}(D_j))$ *and the squares* $\mathcal{D}_j$ *and the horizontal pasting* $\mathcal{D}_j \circ \mathcal{D}$ *are* $\mathcal{W}$*-exact for every* $j \in J$*, then the square* $\mathcal{D}$ *is also* $\mathcal{W}$*-exact.*

(iii) contains all comma squares of the form

$$\begin{tikzcd} u\downarrow b \arrow[r, "\pi^b"] \arrow[d, "s"'] \arrow[dr, phantom, "\nu \swarrow"] & A \arrow[d, "u"] \\ \mathbb{1} \arrow[r, "b"'] & B \end{tikzcd}$$

(iv) contains all comma squares.

Proof. (i) Immediate from the fact that the class of $\mathcal{W}$-aspherical functors is stable under composition (see Remark 4.1.9).
(ii) Let $d \in D$. By hypothesis, there exists some $j \in J$ and $d_j \in D_j$ such that $d = q_j(d_j)$. Since the squares $\mathcal{D}_j$ and $\mathcal{D}_j \circ \mathcal{D}$ are $\mathcal{W}$-exact, the functor $v_j \downarrow d_j \to v \downarrow d$ over p_j and the functor $v_j \downarrow d_j \to v \downarrow d \to u \downarrow q(d)$ over $p \circ p_j$ are $\mathcal{W}$-coaspherical. By the two-of-three axiom (see Remark 4.1.9), the functor $v \downarrow d \to u \downarrow q(d)$ over p is also $\mathcal{W}$-coaspherical and hence the square $\mathcal{D}$ is $\mathcal{W}$-exact.
(iii) It suffices to see that the morphism $s \downarrow * \to u \downarrow b$ over π^b is the identity of $u \downarrow b$ and hence is $\mathcal{W}$-coaspherical (see Remark 4.1.9).
(iv) Let $\mathcal{D}$ be a comma square in Cat

$$\mathcal{D} = \begin{tikzcd} u\downarrow v \arrow[r, "\pi_u"] \arrow[d, "\pi_v"'] \arrow[dr, phantom, "\nu \swarrow"] & A \arrow[d, "u"] \\ C \arrow[r, "v"'] & B \end{tikzcd}$$

and let $c \in C$. Then the functor $r\colon u \downarrow v(c) \to \pi_v \downarrow c$ which carries an object $(a, u(a) \xrightarrow{f} v(c)) \in u \downarrow v(c)$ to $(a, c, u(a) \xrightarrow{f} v(c), c \xrightarrow{1_c} c)$ has left adjoint the functor $\pi_u^c\colon \pi_v \downarrow c \to u \downarrow v(c)$ over π_u. Consider the following diagram.

$$\begin{tikzcd} u\downarrow v(c) \arrow[r, "r"] \arrow[d] \arrow[dr, phantom, "= \swarrow"] & \pi_v\downarrow c \arrow[r, "\pi^c"] \arrow[d] \arrow[dr, phantom, "\swarrow"] & u\downarrow v \arrow[r, "\pi_u"] \arrow[d, "\pi_v"'] \arrow[dr, phantom, "\nu \swarrow"] & A \arrow[d, "u"] \\ \mathbb{1} \arrow[r, equal] & \mathbb{1} \arrow[r, "c"'] & C \arrow[r, "v"'] & B \end{tikzcd} = \begin{tikzcd} u\downarrow v(c) \arrow[r, "\pi^{v(c)}"] \arrow[d] \arrow[dr, phantom, "\swarrow"] & A \arrow[d, "u"] \\ \mathbb{1} \arrow[r, "v(c)"'] & B \end{tikzcd}$$

$\mathcal{D}''$ $\quad$ $\mathcal{D}'$ $\quad$ $\mathcal{D}$ $\quad\quad$ $\mathcal{D}'''$

By Remark 4.2.6, the square $\mathcal{D}''$ is $\mathcal{W}$-exact since the functor r admits a left adjoint. By (iii), the squares $\mathcal{D}'$ and $\mathcal{D}'''$ are $\mathcal{W}$-exact and, by (i), the horizontal pasting $\mathcal{D}'' \circ \mathcal{D}'$ is also $\mathcal{W}$-exact. Since $\mathrm{Ob}(C) = \bigcup_{c \in C} \{c\}$, it follows from (ii) that the square $\mathcal{D}$ is $\mathcal{W}$-exact. Hence all comma squares are $\mathcal{W}$-exact. □

Dually, we also have the stability under vertical pasting and vertical descent.

Proposition 4.2.9. *The class of $\mathcal{W}$-exact squares*

(i) is stable under vertical pasting;

(ii) is stable under vertical descent, i.e. if J is a set and

$$\mathcal{D} = \begin{tikzcd} C \arrow[r, "p"] \arrow[d, "v"'] \arrow[dr, phantom, "\alpha \swarrow"] & A \arrow[d, "u"] \\ D \arrow[r, "q"'] & B \end{tikzcd} \qquad \mathcal{D}_j = \begin{tikzcd} C_j \arrow[r, "p_j"] \arrow[d, "v_j"'] \arrow[dr, phantom, "\alpha_j \swarrow"] & A_j \arrow[d, "u_j"] \\ C \arrow[r, "p"'] & A \end{tikzcd}$$

are squares in Cat *such that* $\mathrm{Ob}(A) = \bigcup_{j \in J} u_j(\mathrm{Ob}(A_j))$ *and the square* $\mathcal{D}_j$ *and the vertical pasting* $\mathcal{D} \circ \mathcal{D}_j$ *are* $\mathcal{W}$*-exact for every* $j \in J$*, then the square* $\mathcal{D}$ *is also* $\mathcal{W}$*-exact.*

Proof. The proof is dual to the proof of Proposition 4.2.8 (i) and (ii). □

Finally, we can prove that the class of $\mathcal{W}$-exact squares is the smallest one with the required properties. This results will be useful to show that the class of $\mathbb{D}$-Beck-Chevalley squares (Section 5.2) contains the $\mathcal{W}$-exact squares, where $\mathcal{W}$ is the basic localizer associated to a derivator $\mathbb{D}$.

Theorem 4.2.10. *The class of* $\mathcal{W}$*-exact squares is the smallest class of squares in* Cat *stable under horizontal pasting and horizontal descent and that contains the comma squares and the squares of the form*

$$\begin{tikzcd} A \arrow[r, "u"] \arrow[d] \arrow[dr, phantom, "= \swarrow"] & B \arrow[d] \\ \mathbb{1} \arrow[r, equal] & \mathbb{1} \end{tikzcd}$$

where the functor u is $\mathcal{W}$*-coaspherical.*

Proof. By Propositions 4.2.8 and 4.2.5, the class of $\mathcal{W}$-exact squares satisfies all conditions of the theorem. Let $\mathcal{Q}$ be a class of squares in Cat satisfying all these conditions. We show that $\mathcal{Q}$ contains the $\mathcal{W}$-exact squares. Let $\mathcal{D}$ be a $\mathcal{W}$-exact square in Cat.

$$\mathcal{D} = \begin{tikzcd} C \arrow[r, "p"] \arrow[d, "v"'] & A \arrow[d, "u"] \\ D \arrow[r, "q"'] \arrow[ur, phantom, "\overset{\alpha}{\swArrow}"] & B \end{tikzcd}$$

Then, by definition, the functor $p^d \colon v \downarrow d \to u \downarrow q(d)$ over p is $\mathcal{W}$-coaspherical for every $d \in D$. For $d \in D$, we have the following diagram.

$$\begin{tikzcd} v \downarrow d \arrow[r, "\pi^d"] \arrow[d] & C \arrow[r, "p"] \arrow[d, "v"] & A \arrow[d, "u"] \\ \mathbb{1} \arrow[r, "d"'] \arrow[ur, phantom, "\swArrow"] & D \arrow[r, "q"'] \arrow[ur, phantom, "\overset{\alpha}{\swArrow}"] & B \\ & \mathcal{D}' \quad\quad \mathcal{D} & \end{tikzcd} = \begin{tikzcd} v \downarrow d \arrow[r, "p^d"] \arrow[d] & u \downarrow q(d) \arrow[r, "\pi^{q(d)}"] \arrow[d] & A \arrow[d, "u"] \\ \mathbb{1} \arrow[r, equal] \arrow[ur, phantom, "\overset{=}{\swArrow}"] & \mathbb{1} \arrow[r, "q(d)"'] \arrow[ur, phantom, "\swArrow"] & B \\ & \mathcal{D}''' \quad\quad \mathcal{D}'' & \end{tikzcd}$$

By hypothesis, the comma squares $\mathcal{D}'$ and $\mathcal{D}''$ belong to $\mathcal{Q}$ and the square $\mathcal{D}'''$ belongs to $\mathcal{Q}$ since p^d is $\mathcal{W}$-coaspherical. Then the pasting $\mathcal{D}''' \circ \mathcal{D}''$ also belongs to $\mathcal{Q}$ by stability under horizontal pasting. Finally, since $\mathcal{Q}$ is stable under horizontal descent and $\mathrm{Ob}(D) = \bigcup_{d \in D}\{d\}$, the square $\mathcal{D}$ belongs to $\mathcal{Q}$. Hence the class $\mathcal{Q}$ contains all $\mathcal{W}$-exact squares and the class of $\mathcal{W}$-exact squares is the smallest class of squares satisfying the conditions of the theorem. $\square$

Remark 4.2.11. Dually, the class of $\mathcal{W}$-exact squares is the smallest class of squares in Cat stable under vertical pasting and vertical descent and that contains the comma squares and the squares of the form

$$\begin{tikzcd} A \arrow[r] \arrow[d, "u"'] & \mathbb{1} \arrow[d, equal] \\ B \arrow[r] \arrow[ur, phantom, "\overset{=}{\swArrow}"] & \mathbb{1} \end{tikzcd}$$

where the functor u is $\mathcal{W}$-aspherical.

4.3 Weak Exact Squares

Let $\mathcal{W}$ be a basic localizer. There is another class of squares, whose defining property is similar to the one of $\mathcal{W}$-exact squares, but using $\mathcal{W}$-local or $\mathcal{W}$-colocal functors in place of $\mathcal{W}$-aspherical and $\mathcal{W}$-coaspherical functors. This class of squares is generally the same as the class of $\mathcal{W}$-exact squares, except when $\mathcal{W} = \mathcal{W}_{\mathrm{ind}}$. But this class of squares is important in the case of a basic localizer associated to a derivator $\mathbb{D}$, since then it coincides with the class of $\mathbb{D}$-Beck-Chevalley squares (Section 5.2).

Proposition 4.3.1. *Consider a square $\mathcal{D}$ in* Cat*.*

$$\mathcal{D} = \begin{tikzcd} C \arrow[r, "p"] \arrow[d, "v"'] \arrow[dr, phantom, "\overset{\alpha}{\swarrow}"] & A \arrow[d, "u"] \\ D \arrow[r, "q"'] & B \end{tikzcd}$$

The following are equivalent:

(i) The functor $p^d\colon v \downarrow d \to u \downarrow q(d)$ over p is $\mathcal{W}$-colocal over A for every $d \in D$.

(ii) The functor $v_a\colon a \downarrow p \to u(a) \downarrow q$ over v is $\mathcal{W}$-local over D for every $a \in A$.

Moreover, both conditions are implied by

(iii) The square is $\mathcal{W}$-exact,

and (iii) is equivalent to (i) and (ii) when $\mathcal{W} \neq \mathcal{W}_{\text{ind}}$.

Proof. Suppose first that the square $\mathcal{D}$ is $\mathcal{W}$-exact, i.e. the functor $p^d\colon v \downarrow d \to u \downarrow q(d)$ over p is $\mathcal{W}$-coaspherical. In particular, the functor p^d is $\mathcal{W}$-colocal over $u \downarrow q(d)$ and, since there is a functor $\pi^{q(d)}\colon u \downarrow q(d) \to A$, it follows from Proposition 4.1.18 that the functor p^d is $\mathcal{W}$-colocal over A. Hence (iii) implies (i) and, dually, we can show that (iii) implies (ii).

We describe conditions (i) and (ii). For $a \in A$ and $d \in D$, consider the following diagrams.

$$\begin{tikzcd} v \downarrow d \arrow[r, "p^d"] \arrow[d, "\pi^d"'] & u \downarrow q(d) \arrow[d, "\pi^{q(d)}"] \\ C \arrow[r, "p"'] & A \end{tikzcd} \qquad \begin{tikzcd} a \downarrow p \arrow[r, "v_a"] \arrow[d, "\pi_a"'] & u(a) \downarrow q \arrow[d, "\pi_{u(a)}"] \\ C \arrow[r, "v"'] & D \end{tikzcd}$$

Condition (i) says that $(p^d)_a\colon a \downarrow (p\pi^d) \to a \downarrow \pi^{q(d)}$ is a $\mathcal{W}$-equivalence and condition (ii) says that $(v_a)^d\colon (v\pi_a) \downarrow d \to \pi_{u(a)} \downarrow d$ is a $\mathcal{W}$-equivalence.

Suppose first that $\mathcal{W} = \mathcal{W}_{\text{ind}}$ and check that (i) is equivalent to (ii). Note that there is an isomorphism of categories $a \downarrow (p\pi^d) \cong (v\pi_a) \downarrow d$. Moreover, the category $a \downarrow \pi^{q(d)}$ is non-empty if and only if the set $B(u(a), q(d))$ is non-empty if and only if the category $\pi_{u(a)} \downarrow d$ is non-empty. Hence $(p^d)_a$ is a $\mathcal{W}_{\text{ind}}$-equivalence if and only if $(v_a)^d$ is a $\mathcal{W}_{\text{ind}}$-equivalence, for every $a \in A$ and $d \in D$, and (i) is equivalent to (ii).

Now suppose that $\mathcal{W} \neq \mathcal{W}_{\text{ind}}$. If $\mathcal{W} = \mathcal{W}_{\text{tr}}$, it is clear that all equivalences hold. Hence suppose also that $\mathcal{W} \neq \mathcal{W}_{\text{tr}}$. By Theorem 4.1.23, this implies that $\mathcal{W} \subseteq \mathcal{W}_0$.

We show that in this case (i) implies (iii). The functor $(p^d)_a\colon a \downarrow (p\pi^d) \to a \downarrow \pi^{q(d)}$ can be identified with the coproduct functor

$$\coprod_{g\colon u(a)\to q(d)} (a,g) \downarrow p^d \longrightarrow \coprod_{g\colon u(a)\to q(d)} (a,g) \downarrow (u \downarrow q(d)). \tag{1}$$

By (i), this coproduct functor is a $\mathcal{W}$-equivalence and it is in particular a $\mathcal{W}_0$-equivalence since $\mathcal{W} \subseteq \mathcal{W}_0$. This implies that the category $(a,g) \downarrow p^d$ is non-empty for every $g\colon u(a) \to q(d)$ in B. Therefore the functor $(a,g) \downarrow p^d \to (a,g) \downarrow (u \downarrow q(d))$ is a retract of (1) for every morphism $g\colon u(a) \to q(d)$ in B and, by axiom **[WS3]**, it is a $\mathcal{W}$-equivalence. This shows that the functor p^d is $\mathcal{W}$-coaspherical and thus that the square $\mathcal{D}$ is $\mathcal{W}$-exact. Dually, we can show that in this case (ii) implies (iii). □

Definition 4.3.2. A square is **weak $\mathcal{W}$-exact** if it satisfies one of the two equivalent conditions (i) and (ii) of Proposition 4.3.1.

Remark 4.3.3. A $\mathcal{W}$-exact square in Cat is weak $\mathcal{W}$-exact. Moreover, if $\mathcal{W} \neq \mathcal{W}_{\text{ind}}$, a square in Cat is $\mathcal{W}$-exact if and only if it is weak $\mathcal{W}$-exact.

As for $\mathcal{W}$-exact squares, the class of weak $\mathcal{W}$-exact squares is stable under horizontal and vertical pasting and horizontal and vertical descent and contains the comma squares.

Proposition 4.3.4. *The class of weak $\mathcal{W}$-exact squares*

(i) is stable under horizontal and vertical pasting;

(ii) is stable under horizontal and vertical descent;

(iii) contains all comma squares.

Proof. The proof is essentially the same as the one for $\mathcal{W}$-exact squares (see Propositions 4.2.8 and 4.3.4). □

The following example shows that, for the indiscrete basic localizer $\mathcal{W}_{\text{ind}}$, there exist weak $\mathcal{W}_{\text{ind}}$-exact squares that are not $\mathcal{W}_{\text{ind}}$-exact, which indicates that the distinction is meaningful.

Example 4.3.5. Consider the indiscrete basic localizer $\mathcal{W}_{\text{ind}}$. Then there exist weak $\mathcal{W}_{\text{ind}}$-exact squares that are not $\mathcal{W}_{\text{ind}}$-exact. Here is an example. Let B be the small category

$$B = \quad \begin{tikzcd} 0 \arrow[r, bend left, "\alpha"] \arrow[r, bend right, "\beta"'] & 1 \end{tikzcd}$$

and let $\mathcal{D}$ be the following square in Cat.

$$\mathcal{D} = \begin{tikzcd} \mathbb{1} \arrow[r, equal] \arrow[d, equal] \arrow[dr, phantom, "\alpha \swarrow"] & \mathbb{1} \arrow[d, "0"] \\ \mathbb{1} \arrow[r, "1"'] & B \end{tikzcd}$$

Then the functor $\mathbb{1} \downarrow * \to 0 \downarrow 1$ over the upper functor can be identified with the functor $\mathbb{1} \xrightarrow{\alpha} \{\alpha, \beta\}$, where $\{\alpha, \beta\}$ is the discrete category whose objects are α and β. Then this functor is $\mathcal{W}_{\mathrm{ind}}$-colocal over $\mathbb{1}$, i.e. it is a $\mathcal{W}_{\mathrm{ind}}$-equivalence, since its source and target categories are both non-empty. But it is not $\mathcal{W}_{\mathrm{ind}}$-coaspherical, since the functor $\beta \downarrow \alpha \to \beta \downarrow \{\alpha, \beta\}$ is not a $\mathcal{W}_{\mathrm{ind}}$-equivalence. The category $\beta \downarrow \alpha$ is actually empty, while the category $\beta \downarrow \{\alpha, \beta\}$ is not. Hence the square $\mathcal{D}$ is weak $\mathcal{W}_{\mathrm{ind}}$-exact, but not $\mathcal{W}_{\mathrm{ind}}$-exact.

5 Basic Localizer of a Derivator

Given a derivator $\mathbb{D}$, we define, in Section 5.1, a class $\mathcal{W}_{\mathbb{D}}$ of functors in Cat that forms a basic localizer. For example, if $\mathcal{C}$ is a cocomplete and complete category, we can compute the basic localizer associated to the derivator $\mathbb{D}_{\mathcal{C}}$, which is either $\mathcal{W}_0$, $\mathcal{W}_{\text{ind}}$ or $\mathcal{W}_{\text{tr}}$. After verifying the axioms for basic localizers for $\mathcal{W}_{\mathbb{D}}$, we introduce, in Section 5.2, the notion of $\mathbb{D}$-Beck-Chevalley squares, which are squares in Cat whose images under $\mathbb{D}$ are Beck-Chevalley squares. Finally, we show that this notion of squares coincides with the notion of weak $\mathcal{W}_{\mathbb{D}}$-exact squares and hence, in most cases, with the notion of $\mathcal{W}_{\mathbb{D}}$-exact squares.

5.1 Definition and Verification of the Axioms

Let $\mathbb{D}$ be a derivator. We construct here a class of functors in Cat associated to the derivator $\mathbb{D}$ and check that this class satisfies all axioms of a basic localizer.

Definition 5.1.1. A functor $u\colon A \to B$ in Cat is a **$\mathbb{D}$-equivalence** if the functor $u^*\colon \mathbb{D}(B) \to \mathbb{D}(A)$ induces a fully faithful functor on the full subcategory of constant coefficients of $\mathbb{D}(B)$, i.e. for every $X, Y \in \mathbb{D}(\mathbb{1})$, the map

$$\mathbb{D}(B)(\Delta_B X, \Delta_B Y) \to \mathbb{D}(A)(\Delta_A X, \Delta_A Y), \quad \phi \mapsto u^*(\phi)$$

is a bijection. We define $\mathcal{W}_{\mathbb{D}}$ to be the class of $\mathbb{D}$-equivalences.

We verify each axiom for basic localizers for $\mathcal{W}_{\mathbb{D}}$. Here is the first one.

Proposition 5.1.2. *The class $\mathcal{W}_{\mathbb{D}}$ of $\mathbb{D}$-equivalences is weakly saturated.*

Proof. Since the class of bijections in Set satisfies the axioms [**WS1**], [**WS2**] and [**WS3**], it follows from the definition of $\mathbb{D}$-equivalences that the class $\mathcal{W}_{\mathbb{D}}$ satisfies the axioms [**WS1**], [**WS2**] and [**WS3**]. Moreover, if a functor $u\colon A \to B$ in Cat is a $\mathbb{D}$-equivalence, then its opposite functor $u\colon A^{\text{op}} \to B^{\text{op}}$ is also a $\mathbb{D}$-equivalence since the corresponding map of Definition 5.1.1 is the composite

$$\begin{tikzcd}
\mathbb{D}(B^{\text{op}})(\Delta_{B^{\text{op}}}X, \Delta_{B^{\text{op}}}Y) \arrow[r] \arrow[d, "\cong"'] & \mathbb{D}(A^{\text{op}})(\Delta_{A^{\text{op}}}X, \Delta_{A^{\text{op}}}Y) \\
\mathbb{D}(B)(\Delta_B X, \Delta_B Y) \arrow[r, "u^*"'] & \mathbb{D}(A)(\Delta_A X, \Delta_A Y) \arrow[u, "\cong"']
\end{tikzcd}$$

where $X, Y \in \mathbb{D}(\mathbb{1})$. This shows axiom [**WS4**]. □

As for basic localizers, there are notions of $\mathbb{D}$-aspherical categories, $\mathbb{D}$-(co)aspherical functors, $\mathbb{D}$-(co)local functors, and (weak) $\mathbb{D}$-exact squares.

Definition 5.1.3. As in Section 4.1, we have the following terminology.

(i) A small category A is **$\mathbb{D}$-aspherical** if it is $\mathcal{W}_{\mathbb{D}}$-aspherical, i.e. if the functor $\Delta_A\colon \mathbb{D}(\mathbb{1}) \to \mathbb{D}(A)$ is fully faithful.

(ii) A functor $u\colon A \to B$ is **$\mathbb{D}$-aspherical** (resp. **$\mathbb{D}$-coaspherical**) if it is $\mathcal{W}_{\mathbb{D}}$-aspherical (resp. $\mathcal{W}_{\mathbb{D}}$-coaspherical).

(iii) Considering a commutative triangle in Cat,

$$\begin{tikzcd} A \arrow[rr, "u"] \arrow[dr, "v"'] & & B \arrow[dl, "w"] \\ & C & \end{tikzcd}$$

the functor u is **$\mathbb{D}$-local** (resp. **$\mathbb{D}$-colocal**) over C if it is $\mathcal{W}_{\mathbb{D}}$-local (resp. $\mathcal{W}_{\mathbb{D}}$-colocal) over C.

(iv) A square $\mathcal{D}$ in Cat

$$\mathcal{D} = \begin{tikzcd} C \arrow[r, "p"] \arrow[d, "v"'] \arrow[dr, phantom, "\overset{\alpha}{\swarrow}"] & A \arrow[d, "u"] \\ D \arrow[r, "q"'] & B \end{tikzcd}$$

is **$\mathbb{D}$-exact** if it is $\mathcal{W}_{\mathbb{D}}$-exact and it is **weak $\mathbb{D}$-exact** if it is weak $\mathcal{W}_{\mathbb{D}}$-exact.

The second axiom of a basic localizer follows immediately from the definition of $\mathbb{D}$-aspherical category.

Proposition 5.1.4.

(i) A small category A admitting a terminal object is $\mathbb{D}$-aspherical.

(ii) A small category A admitting an initial object is $\mathbb{D}$-aspherical.

Proof. (i) Let $t \in A$ denote the terminal object. Then $t\colon \mathbb{1} \to A$ is a fully faithful functor with left adjoint $p\colon A \to \mathbb{1}$. By Corollary 3.2.10, this implies that $p^* = \Delta_A$ is fully faithful and, by definition, that the category A is $\mathbb{D}$-aspherical.
(ii) The proof is dual to (i). $\square$

The next proposition gives a useful criterion for $\mathbb{D}$-equivalences in terms of natural isomorphisms induced by the unit and counit of the adjunctions formed by the Kan extensions.

Proposition 5.1.5. *Let $u\colon A \to B$ be a functor in* Cat. *The following are equivalent:*

(i) The functor u is a $\mathbb{D}$-equivalence.

(ii) The natural transformation

$$\lim_B \Delta_B \Rightarrow \lim_B u_* u^* \Delta_B \cong \lim_A \Delta_A$$

induced by the unit $\eta\colon 1_{\mathbb{D}(B)} \Rightarrow u_* u^*$ is a natural isomorphism.

(iii) The natural transformation

$$\operatorname{colim}_A \Delta_A \cong \operatorname{colim}_B u_! u^* \Delta_B \Rightarrow \operatorname{colim}_B \Delta_B$$

induced by the counit $\epsilon\colon u_! u^* \Rightarrow 1_{\mathbb{D}(B)}$ is a natural isomorphism.

Proof. Condition (i) means that the map $\mathbb{D}(B)(\Delta_B X, \Delta_B Y) \to \mathbb{D}(A)(\Delta_A X, \Delta_A Y)$ induced by u is bijective, for every $X, Y \in \mathbb{D}(\mathbb{1})$. By adjunction, this means that the map

$$\mathbb{D}(\mathbb{1})(X, \lim_B \Delta_B Y) \to \mathbb{D}(\mathbb{1})(X, \lim_A \Delta_A Y)$$

induced by the unit $\eta\colon 1_{\mathbb{D}(B)} \Rightarrow u_* u^*$ is bijective, for every $X, Y \in \mathbb{D}(\mathbb{1})$, which is equivalent to saying that the morphism $\lim_B \Delta_B Y \to \lim_A \Delta_A Y$ induced by the unit η is an isomorphism, for every $Y \in \mathbb{D}(\mathbb{1})$. In other words, condition (i) holds if and only if the natural transformation $\lim_B \Delta_B \Rightarrow \lim_A \Delta_A$ induced by the unit η is a natural isomorphism. Since the natural transformations in (ii) and (iii) are conjugate, the equivalence of (ii) and (iii) follows from Lemma 3.1.9. □

We also have a criterion in terms of natural isomorphisms induced by the unit and counit of the adjunctions formed by the Kan extensions for $\mathcal{W}$-local and $\mathcal{W}$-colocal functors.

Proposition 5.1.6. *Consider a commutative triangle in* Cat.

$$\begin{tikzcd} A \arrow[rr, "u"] \arrow[dr, "v"'] & & B \arrow[dl, "w"] \\ & C & \end{tikzcd}$$

The following are equivalent:

(i) The functor u is $\mathbb{D}$-local over C.

(ii) The natural transformation

$$v_! \Delta_A \cong w_! u_! u^* \Delta_B \Rightarrow w_! \Delta_B$$

induced by the counit $\epsilon\colon u_! u^* \Rightarrow 1_{\mathbb{D}(B)}$ is a natural isomorphism.

(iii) The natural transformation

$$\lim_B w^* \Rightarrow \lim_B u_* u^* w^* \cong \lim_A v^*$$

induced by the unit $\eta\colon 1_{\mathbb{D}(B)} \Rightarrow u_* u^*$ *is a natural isomorphism.*

And dually, the following are equivalent:

(iv) The functor u *is* $\mathbb{D}$*-colocal over* C*.*

(v) The natural transformation

$$w_* \Delta_B \Rightarrow w_* u_* u^* \Delta_B \cong v_* \Delta_A$$

induced by the unit $\eta\colon 1_{\mathbb{D}(B)} \Rightarrow u_* u^*$ *is a natural isomorphism.*

(vi) The natural transformation

$$\operatorname{colim}_A v^* \cong \operatorname{colim}_B u_! u^* w^* \Rightarrow \operatorname{colim}_B w^*$$

induced by the counit $\epsilon\colon u_! u^* \Rightarrow 1_{\mathbb{D}(B)}$ *is a natural isomorphism.*

Proof. By axiom **[D2]**, condition (ii) is equivalent to saying that $c^* v_! \Delta_A \Rightarrow c^* w_! \Delta_B$ is a natural isomorphism for every $c \in C$. By axiom **[D4]**, for $c \in C$, the images under $\mathbb{D}$ of the comma squares

$$\begin{tikzcd} v \downarrow c \arrow[r, "\pi_v^c"] \arrow[d] \arrow[dr, phantom, "\swarrow"] & A \arrow[d, "v"] \\ \mathbb{1} \arrow[r, "c"'] & C \end{tikzcd} \qquad \begin{tikzcd} w \downarrow c \arrow[r, "\pi_w^c"] \arrow[d] \arrow[dr, phantom, "\swarrow"] & B \arrow[d, "w"] \\ \mathbb{1} \arrow[r, "c"'] & C \end{tikzcd}$$

are Beck-Chevalley, i.e. we have natural isomorphisms

$$\operatorname{colim}_{v\downarrow c} (\pi_v^c)^* \cong c^* v_! \quad \text{and} \quad \operatorname{colim}_{w\downarrow c} (\pi_w^c)^* \cong c^* w_!.$$

Then the natural transformation $c^* v_! \Delta_A \Rightarrow c^* w_! \Delta_B$ corresponds to the natural transformation $\operatorname{colim}_{v\downarrow c} \Delta_{v\downarrow c} = \operatorname{colim}_{v\downarrow c} (\pi_v^c)^* \Delta_A \Rightarrow \operatorname{colim}_{w\downarrow c} (\pi_w^c)^* \Delta_B = \operatorname{colim}_{w\downarrow c} \Delta_{w\downarrow c}$ induced by the counit $\epsilon\colon (u^c)_!(u^c)^* \Rightarrow 1_{\mathbb{D}(w\downarrow c)}$. Hence it is a natural isomorphism if and only if $u^c\colon v \downarrow c \to w \downarrow c$ is a $\mathbb{D}$-equivalence, by Proposition 5.1.5. This shows $v_! \Delta_A \Rightarrow w_! \Delta_B$ is a natural isomorphism if and only if the functor u is $\mathbb{D}$-local over C and hence that (i) and (ii) are equivalent. Since the natural transformations in (ii) and (iii) are conjugate, the equivalence of (ii) and (iii) follows from Lemma 3.1.9.

The second part of the proposition is dual. □

This proposition allows us to prove the last axiom for basic localizers for $\mathcal{W}_{\mathbb{D}}$ and hence conclude that $\mathcal{W}_{\mathbb{D}}$ is a basic localizer.

Proposition 5.1.7. *Consider a commutative triangle in* Cat*.*

$$\begin{tikzcd} A \arrow[rr, "u"] \arrow[dr, "v"'] & & B \arrow[dl, "w"] \\ & C & \end{tikzcd}$$

If the functor u is $\mathbb{D}$-local over C, then it is a $\mathbb{D}$-equivalence.

Proof. By Proposition 5.1.6, the natural transformation $v_!\Delta_A \Rightarrow w_!\Delta_B$ induced by the counit $\epsilon\colon u_!u^* \Rightarrow 1_{\mathbb{D}(B)}$ is a natural isomorphism. Then the natural transformation

$$\operatorname{colim}_A \Delta_A \cong \operatorname{colim}_C v_!\Delta_A \Rightarrow \operatorname{colim}_C w_!\Delta_B \cong \operatorname{colim}_B \Delta_B$$

is also a natural isomorphism. By Proposition 5.1.5, this means that the functor u is a $\mathbb{D}$-equivalence. □

Theorem 5.1.8. *The class $\mathcal{W}_{\mathbb{D}}$ of $\mathbb{D}$-equivalences is a basic localizer.*

Proof. We have already checked all axioms: axiom [**BL1**] follows from Proposition 5.1.2, axiom [**BL2**] from Proposition 5.1.4 and axiom [**BL3**] from Proposition 5.1.7. □

Finally, we compute the class $\mathcal{W}_{\mathbb{D}_\mathcal{C}}$ for all represented derivators $\mathbb{D}_\mathcal{C}$, where $\mathcal{C}$ is a cocomplete and complete category.

Proposition 5.1.9. *Let $\mathcal{C}$ be a cocomplete and complete category and $\mathbb{D}_\mathcal{C}$ be the represented derivator of $\mathcal{C}$. Then the class of $\mathbb{D}_\mathcal{C}$-equivalences is*

$$\mathcal{W}_{\mathbb{D}_\mathcal{C}} \quad = \quad \begin{cases} \mathcal{W}_0 & \text{if } \mathcal{C} \text{ is not equivalent to a preorder category;} \\ \mathcal{W}_{\mathrm{ind}} & \text{if } \mathcal{C} \text{ is equivalent to a non-empty preorder category but } \mathcal{C} \not\simeq \mathbb{1}; \\ \mathcal{W}_{\mathrm{tr}} & \text{if } \mathcal{C} \simeq \mathbb{1} \text{ or } \mathcal{C} \text{ is empty.} \end{cases}$$

Proof. Let A be a small category. Then $\Delta_A\colon \mathcal{C} \to \mathcal{C}^A$ is the usual diagonal functor. Let $c, d \in \mathcal{C}$. We have that

$$\mathcal{C}^A(\Delta_A c, \Delta_A d) \cong \mathcal{C}(c,d)^{\pi_0(A)}.$$

Hence a functor $u\colon A \to B$ in Cat is a $\mathcal{W}_{\mathbb{D}_\mathcal{C}}$-equivalence if and only if the map

$$\mathcal{C}(c,d)^{\pi_0(B)} \longrightarrow \mathcal{C}(c,d)^{\pi_0(A)}, \quad f \mapsto f \circ \pi_0(u) \tag{2}$$

is a bijection for every $c, d \in \mathcal{C}$.

If $\mathcal{C}$ is not equivalent to a preordered category, there exists two objects $c, d \in \mathcal{C}$ such that $\mathcal{C}(c,d)$ has at least two elements. Then the map (2) is a bijection for this particular pair of objects if and only if $\pi_0(u)$ is a bijection. Therefore $\mathcal{W}_{\mathbb{D}_\mathcal{C}} = \mathcal{W}_0$.

If $\mathcal{C}$ is equivalent to a non-empty preordered category but $\mathcal{C} \not\simeq \mathbb{1}$, there exists two objects $c, d \in \mathcal{C}$ such that $\mathcal{C}(c,d)$ is empty. Then the map (2) is a bijection for this particular pair of objects if and only $\pi_0(A)$ and $\pi_0(B)$ are both empty or both non-empty. This is equivalent to saying that the categories A and B are both empty or both non empty. Hence $\mathcal{W}_{\mathbb{D}_\mathcal{C}} = \mathcal{W}_{\mathrm{ind}}$.

The cases $\mathcal{C} = \emptyset$ and $\mathcal{C} \simeq \mathbb{1}$ are trivial. □

5.2 $\mathbb{D}$-Beck-Chevalley Squares

Let $\mathbb{D}$ be a derivator. We define a class of squares in Cat, called the $\mathbb{D}$-Beck-Chevalley squares, which are the squares in Cat such that their image under $\mathbb{D}$ is Beck-Chevalley. We prove that the $\mathbb{D}$-Beck-Chevalley squares are exactly the weak $\mathbb{D}$-exact squares. Since we have seen that the notion of weak $\mathbb{D}$-exact squares and $\mathbb{D}$-exact squares are most of the time the same, except when $\mathcal{W}_{\mathbb{D}} = \mathcal{W}_{\text{ind}}$, this implies that in most cases the $\mathbb{D}$-Beck-Chevalley squares are the $\mathbb{D}$-exact squares. But, this is always true that the class of $\mathbb{D}$-Beck-Chevalley squares contains the $\mathbb{D}$-exact squares. We first show this result by using Theorem 4.2.10.

Definition 5.2.1. A square $\mathcal{D}$ in Cat

$$\mathcal{D} = \begin{tikzcd} C \arrow[r, "p"] \arrow[d, "v"'] \arrow[dr, phantom, "\alpha \swArrow"] & A \arrow[d, "u"] \\ D \arrow[r, "q"'] & B \end{tikzcd}$$

is **$\mathbb{D}$-Beck-Chevalley** if its image under $\mathbb{D}$ is a Beck-Chevalley square. In other words, the square $\mathcal{D}$ is $\mathbb{D}$-Beck-Chevalley if the mates

$$\begin{tikzcd} \mathbb{D}(C) \arrow[d, "v_!"'] \arrow[dr, phantom, "\seArrow \alpha_!"] & \mathbb{D}(A) \arrow[l, "p^*"'] \arrow[d, "u_!"] \\ \mathbb{D}(D) & \mathbb{D}(B) \arrow[l, "q^*"] \end{tikzcd} \qquad \begin{tikzcd} \mathbb{D}(C) \arrow[r, "p_*"] \arrow[dr, phantom, "\nwArrow \alpha_*"] & \mathbb{D}(A) \\ \mathbb{D}(D) \arrow[u, "v^*"] \arrow[r, "q_*"'] & \mathbb{D}(B) \arrow[u, "u^*"'] \end{tikzcd}$$

associated to the natural transformation $\alpha^*\colon p^*u^* \Rightarrow v^*q^*$ are natural isomorphisms.

Remark 5.2.2. The $\mathbb{D}$-Beck-Chevalley squares are well-defined since, if $\mathbb{D}$ is a derivator, the functors u^*, v^*, p^* and q^* admit left and right adjoints by axiom **[D3]**.

The following results follow immediately from the definition.

Remark 5.2.3. If a functor $u\colon A \to B$ in Cat admits a right adjoint $v\colon B \to A$, the square

$$\mathcal{D} = \begin{tikzcd} A \arrow[r] \arrow[d, "u"'] \arrow[dr, phantom, "= \swArrow"] & \mathbb{1} \arrow[d, equal] \\ B \arrow[r] & \mathbb{1} \end{tikzcd}$$

is $\mathbb{D}$-Beck-Chevalley since the left mate of its image under $\mathbb{D}$ is $u_!\Delta_A \cong v^*\Delta_A \Rightarrow \Delta_B$, which is the identity. Dually, if a functor $u\colon A \to B$ admits a left adjoint $v\colon B \to A$, the square

$$\mathcal{D} = \begin{tikzcd} A \arrow[r, "u"] \arrow[d] \arrow[dr, phantom, "=\swarrow"] & B \arrow[d] \\ \mathbb{1} \arrow[r, equal] & \mathbb{1} \end{tikzcd}$$

is $\mathbb{D}$-Beck-Chevalley since the right mate of its image under $\mathbb{D}$ is $\Delta_B \Rightarrow u_*\Delta_A \cong v^*\Delta_A$, which is the identity.

In order to use Theorem 4.2.10, we need to show that the class of $\mathbb{D}$-Beck-Chevalley squares is stable under horizontal pasting and horizontal descent and that it contains the comma squares and the squares of the form

$$\begin{tikzcd} A \arrow[r, "u"] \arrow[d] \arrow[dr, phantom, "=\swarrow"] & B \arrow[d] \\ \mathbb{1} \arrow[r, equal] & \mathbb{1} \end{tikzcd}$$

where the functor u is $\mathbb{D}$-aspherical. The next proposition gives us the first three conditions. But, we first need a lemma.

Lemma 5.2.4. *Let J be a set and $\{q_j : D_j \to D\}_{j \in J}$ be a family of functors in* Cat *such that* $\mathrm{Ob}(D) = \bigcup_{j\in J} q_j(\mathrm{Ob}(D_j))$. *Then the family $\{q_j^* : \mathbb{D}(D) \to \mathbb{D}(D_j)\}_{j\in J}$ is conservative.*

Proof. Consider a morphism ϕ in $\mathbb{D}(D)$ such that $q_j^*(\phi)$ is an isomorphism in $\mathbb{D}(D_j)$ for every $j \in J$. Let $d \in D$. By hypothesis, there exists some $j \in J$ and $d_j \in D_j$ such that $d = q_j(d_j)$. Then $d^*(\phi) = d_j^* q_j^*(\phi)$ is an isomorphism since $q_j^*(\phi)$ is an isomorphism. Since this holds for every $d \in D$, it follows from axiom **[D2]** that ϕ is an isomorphism. □

Proposition 5.2.5. *The class of $\mathbb{D}$-Beck-Chevalley squares in* Cat

(i) is stable under horizontal and vertical pasting;

(ii) is stable under horizontal and vertical descent;

(iii) contains all comma squares of the form

$$\begin{tikzcd} u\downarrow b \arrow[r, "\pi^b"] \arrow[d] \arrow[dr, phantom, "\nu\swarrow"] & A \arrow[d, "u"] \\ \mathbb{1} \arrow[r, "b"'] & B \end{tikzcd} \qquad \begin{tikzcd} b\downarrow u \arrow[r] \arrow[d, "\pi_b"'] \arrow[dr, phantom, "\mu\swarrow"] & \mathbb{1} \arrow[d, "b"] \\ A \arrow[r, "u"'] & B \end{tikzcd}$$

(iv) contains all comma squares.

Proof. (i) Immediate from the fact that the class of Beck-Chevalley squares is stable under horizontal and vertical pasting (see Remark 3.1.5).
(ii) Let J be a set and

$$\mathcal{D} = \begin{tikzcd} C \arrow[r, "p"] \arrow[d, "v"'] \arrow[dr, phantom, "\overset{\alpha}{\swarrow}"] & A \arrow[d, "u"] \\ D \arrow[r, "q"'] & B \end{tikzcd} \qquad \mathcal{D}_j = \begin{tikzcd} C_j \arrow[r, "p_j"] \arrow[d, "v_j"'] \arrow[dr, phantom, "\overset{\alpha_j}{\swarrow}"] & C \arrow[d, "v"] \\ D_j \arrow[r, "q_j"'] & D \end{tikzcd}$$

be squares in Cat such that $\mathrm{Ob}(D) = \bigcup_{j\in J} q_j(\mathrm{Ob}(D_j))$ and the square $\mathcal{D}_j$ and the horizontal pasting $\mathcal{D}_j \circ \mathcal{D}$ are $\mathbb{D}$-Beck-Chevalley for every $j \in J$. Denote by γ_j the natural transformation of the square $\mathcal{D}_j \circ \mathcal{D}$ for every $j \in J$. We have the following diagram

$$\begin{tikzcd} \mathbb{D}(C_j) \arrow[d, "(v_j)_!"'] & \mathbb{D}(C) \arrow[l, "p_j^*"'] \arrow[d, "v_!"] \arrow[dl, phantom, "\searrow(\alpha_j)_!"] & \mathbb{D}(A) \arrow[l, "p^*"'] \arrow[d, "u_!"] \arrow[dl, phantom, "\searrow\alpha_!"] \\ \mathbb{D}(D_j) & \mathbb{D}(D) \arrow[l, "q_j^*"] & \mathbb{D}(B) \arrow[l, "q^*"] \end{tikzcd} = \begin{tikzcd} \mathbb{D}(C_j) \arrow[d, "(v_j)_!"'] & \mathbb{D}(A) \arrow[l, "(pp_j)^*"'] \arrow[d, "u_!"] \arrow[dl, phantom, "\searrow(\gamma_j)_!"] \\ \mathbb{D}(D_j) & \mathbb{D}(B) \arrow[l, "(qq_j)^*"] \end{tikzcd}$$

i.e. $(\gamma_j)_! = q_j^*\alpha_! \circ (\alpha_j)_! p^*$, for every $j \in J$. Since $(\gamma_j)_!$ and $(\alpha_j)_!$ are natural isomorphisms, this implies that $q_j^*\alpha_!$ is a natural isomorphism for every $j \in J$. By Lemma 5.2.4, the family of functors $\{q_j^*\}_{j\in J}$ is conservative. This implies that the mate $\alpha_!$ is a natural isomorphism and hence that the square $\mathcal{D}$ is $\mathbb{D}$-Beck-Chevalley. This shows the stability under horizontal descent.

The proof of stability under vertical descent is dual.
(iii) This is axiom **[D4]**.
(iv) According to (i), (ii) and (iii) and Remark 5.2.3, the proof of (iv) is essentially the same as the proof of Proposition 4.2.8 (iv). □

The next three lemmas give examples of squares that are $\mathbb{D}$-exact if and only if they are $\mathbb{D}$-Beck-Chevalley. In particular, the third lemma shows that the last condition of Theorem 4.2.10 is satisfied by the class of $\mathbb{D}$-Beck-Chevalley squares.

Lemma 5.2.6. *Let A be a small category and $\mathcal{D}$ be the following square.*

$$\mathcal{D} = \begin{tikzcd} A \arrow[r] \arrow[d] \arrow[dr, phantom, "\overset{=}{\swarrow}"] & \mathbb{1} \arrow[d, equal] \\ \mathbb{1} \arrow[r, equal] & \mathbb{1} \end{tikzcd}$$

The following are equivalent:

(i) The category A is $\mathbb{D}$-aspherical.

(ii) The functor $\Delta_A \colon \mathbb{D}(\mathbb{1}) \to \mathbb{D}(A)$ is fully faithful.

(iii) The square $\mathcal{D}$ is $\mathbb{D}$-exact.

(iv) The square $\mathcal{D}$ is $\mathbb{D}$-Beck-Chevalley.

(v) The counit $\epsilon\colon \operatorname{colim}_A \Delta_A \Rightarrow 1_{\mathbb{D}(\mathbb{1})}$ is a natural isomorphism.

(vi) The unit $\eta\colon 1_{\mathbb{D}(\mathbb{1})} \Rightarrow \lim_A \Delta_A$ is a natural isomorphism.

Proof. The equivalence of (i) and (ii) follows immediately from the definition and the equivalence of (i) and (iii) was proved in Corollary 4.2.7. Moreover, the natural transformations in (v) and (vi) are the mates of the image under $\mathbb{D}$ of the square $\mathcal{D}$ which proves the equivalence of (iv) with (v) and (vi). Finally, Lemma 3.2.9 implies that (ii) is equivalent to (v) and (vi). $\square$

Lemma 5.2.7. *Let $u\colon A \to B$ be a functor in* Cat *and $\mathcal{D}$ be the following square.*

$$\mathcal{D} = \begin{tikzcd} A \arrow[r] \arrow[d, "u"'] \arrow[dr, phantom, "\overset{=}{\swarrow}"] & \mathbb{1} \arrow[d, equal] \\ B \arrow[r] & \mathbb{1} \end{tikzcd}$$

The following are equivalent:

(i) The functor u is $\mathbb{D}$-aspherical.

(ii) The square $\mathcal{D}$ is $\mathbb{D}$-exact.

(iii) The square $\mathcal{D}$ is $\mathbb{D}$-Beck-Chevalley.

(iv) The natural transformation $u_!\Delta_A \Rightarrow \Delta_B$ induced by the counit $\epsilon\colon u_!u^* \Rightarrow 1_{\mathbb{D}(B)}$ is a natural isomorphism.

(v) The natural transformation $\lim_B \Rightarrow \lim_B u_*u^* \cong \lim_A u^*$ induced by the unit $\eta\colon 1_{\mathbb{D}(B)} \Rightarrow u_*u^*$ is a natural isomorphism.

Proof. The equivalence of (i) and (ii) was proved in Proposition 4.2.5. Moreover, the natural transformations in (iv) and (v) are the mates of the image under $\mathbb{D}$ of the square $\mathcal{D}$ which proves the equivalence of (iii) with (iv) and (v). The equivalence of (i) with (iv) and (v) comes from Proposition 5.1.6, since a $\mathbb{D}$-aspherical functor is in particular $\mathbb{D}$-local over B. $\square$

Lemma 5.2.8. *Let $u\colon A \to B$ be a functor in* Cat *and $\mathcal{D}$ be the following square.*

$$\mathcal{D} = \begin{tikzcd} A \arrow[r, "u"] \arrow[d] \arrow[dr, phantom, "\overset{=}{\swarrow}"] & B \arrow[d] \\ \mathbb{1} \arrow[r, equal] & \mathbb{1} \end{tikzcd}$$

The following are equivalent:

(i) *The functor u is $\mathbb{D}$-coaspherical.*

(ii) *The square $\mathcal{D}$ is $\mathbb{D}$-exact.*

(iii) *The square $\mathcal{D}$ is $\mathbb{D}$-Beck-Chevalley.*

(iv) *The natural transformation $\Delta_B \Rightarrow u_*\Delta_A$ induced by the unit $\eta\colon 1_{\mathbb{D}(B)} \Rightarrow u_*u^*$ is a natural isomorphism.*

(v) *The natural transformation $\operatorname{colim}_A u^* \cong \operatorname{colim}_B u_!u^* \Rightarrow \operatorname{colim}_B$ induced by the counit $\epsilon\colon u_!u^* \Rightarrow 1_{\mathbb{D}(B)}$ is a natural isomorphism.*

Proof. The proof is dual to the proof of Lemma 5.2.7. □

It follows from this result that the class of $\mathbb{D}$-Beck-Chevalley squares contains all $\mathbb{D}$-exact squares.

Theorem 5.2.9. *Every $\mathbb{D}$-exact square in* Cat *is a $\mathbb{D}$-Beck-Chevalley square.*

Proof. By Proposition 5.2.5 and Lemma 5.2.8, the class of $\mathbb{D}$-Beck-Chevalley squares in Cat satisfies all conditions of Theorem 4.2.10 with respect to the basic localizer $\mathcal{W}_{\mathbb{D}}$ and hence contains all $\mathbb{D}$-exact squares. □

Finally, we shows that the notion of $\mathbb{D}$-Beck-Chevalley squares and the notion of weak $\mathbb{D}$-exact squares are equivalent.

Theorem 5.2.10. *A square in* Cat *is $\mathbb{D}$-Beck-Chevalley if and only if it is weak $\mathbb{D}$-exact.*

Proof. Consider a square $\mathcal{D}$ in Cat

$$\mathcal{D} = \begin{tikzcd} C \arrow[r, "p"] \arrow[d, "v"'] \arrow[dr, phantom, "\overset{\alpha}{\swarrow}"] & A \arrow[d, "u"] \\ D \arrow[r, "q"'] & B \end{tikzcd}$$

and the left mate $\alpha_!\colon v_!p^* \Rightarrow q^*u_!$ of its image under $\mathbb{D}$. By axiom **[D2]**, this mate is a natural isomorphism if and only if $d^*\alpha_!\colon d^*v_!p^* \Rightarrow d^*q^*u_!$ is a natural isomorphism for every $d \in D$. Let $d \in D$. We have the following diagrams.

$$\begin{tikzcd}
v\downarrow d \arrow[r, "\pi^d"] \arrow[d] \arrow[dr, phantom, "\nu \swArrow"] & C \arrow[r, "p"] \arrow[d, "v"] \arrow[dr, phantom, "\alpha \swArrow"] & A \arrow[d, "u"] \\
\mathbb{1} \arrow[r, "d"'] & D \arrow[r, "q"'] & B
\end{tikzcd}
=
\begin{tikzcd}
v\downarrow d \arrow[dr, "p^d"] \arrow[drr, bend left, "p\circ\pi^d"] \arrow[ddr, bend right] & & \\
 & u\downarrow q(d) \arrow[r, "\pi^{q(d)}"] \arrow[d] \arrow[dr, phantom, "\mu \swArrow"] & A \arrow[d, "u"] \\
 & \mathbb{1} \arrow[r, "q(d)"'] & B
\end{tikzcd}$$

By axiom **[D4]**, the image under $\mathbb{D}$ of the two comma squares are Beck-Chevalley, i.e. we have two natural isomorphisms

$$\nu_!\colon \operatorname{colim}_{v\downarrow d}(\pi^d)^* \Rightarrow d^*v_! \quad \text{and} \quad \mu_!\colon \operatorname{colim}_{u\downarrow q(d)}(\pi^{q(d)})^* \Rightarrow q(d)^*u_!.$$

Considering the left mates of the image under $\mathbb{D}$ of the diagrams above, we have the following commutative diagram of natural transformations.

$$\begin{tikzcd}
\operatorname{colim}_{v\downarrow d}(\pi^d)^*p^* \cong \operatorname{colim}_{u\downarrow q(d)}(p^d)_!(p^d)^*(\pi^{q(d)})^* \arrow[r, "\operatorname{colim}_{u\downarrow q(d)}\epsilon(\pi^{q(d)})^*"] \arrow[d, "\nu_!p^*"', "\cong"] & \operatorname{colim}_{u\downarrow q(d)}(\pi^{q(d)})^* \arrow[d, "\cong"', "\mu_!"] \\
d^*v_!p^* \arrow[r, "d^*\alpha_!"'] & d^*q^*u_!
\end{tikzcd}$$

where ϵ denotes the counit $\epsilon\colon (p^d)_!(p^d)^* \Rightarrow 1_{\mathbb{D}(u\downarrow q(d))}$. Hence $d^*\alpha_!$ is a natural isomorphism if and only if the natural transformation

$$\operatorname{colim}_{v\downarrow d}(p\pi^d)^* \Rightarrow \operatorname{colim}_{u\downarrow q(d)}(\pi^{q(d)})^*$$

induced by ϵ is a natural isomorphism. By Proposition 5.1.6, it is equivalent to saying that the functor $p^d\colon v\downarrow d \Rightarrow u\downarrow q(d)$ is $\mathbb{D}$-colocal over A or, in other words, to saying that the square $\mathcal{D}$ is weak $\mathbb{D}$-exact since this holds for every $d \in D$. This shows that a square is $\mathbb{D}$-Beck-Chevalley if and only if it is weak $\mathbb{D}$-exact. $\square$

6 Homotopy Derivators and the Basic Localiser $\mathcal{W}_\infty$

For every model category $\mathcal{M}$, there is a homotopy derivator sending a small category A to the homotopy category of $\mathcal{M}^A$, where the weak equivalences are defined levelwise. In Section 6.1, we check that this actually defines a derivator for model categories that are combinatorial. In Section 6.2, we show that, if there is a Quillen equivalence between two combinatorial model categories, their derivators are equivalent and they have the same basic localizer. We use this result, in Section 6.4, to compute the basic localizer of the homotopy derivator of the category of simplicial sets equipped with the Quillen model structure. If the category of small categories is equipped with the Thomason model structure where the weak equivalences are defined as the functors whose nerve is a weak homotopy equivalence of simplicial sets, there exists a Quillen equivalence between these two model categories. This shows that the basic localizer associated to the homotopy derivator of the category of simplicial sets consists of all functors whose nerve is a weak homotopy equivalence, in other words it is the basic localizer $\mathcal{W}_\infty$, which we defined more explicitly in Section 6.3.

6.1 Homotopy Derivator of a Combinatorial Model Category

In this section, we consider combinatorial model categories. These kind of model categories are useful since, if $\mathcal{M}$ is a combinatorial model category, we can equip the category $\mathcal{M}^A$ with projective and injective model structures, for every small category A. In this case, the homotopy Kan extensions of every functor in Cat can be constructed explicitly. In fact, for every functor $u\colon A \to B$, the adjunctions

$$\begin{tikzcd} \mathcal{M}^A \arrow[r, bend left, "u_!"] \arrow[r, phantom, "\perp"] & \mathcal{M}^B \arrow[l, bend left, "u^*"] \end{tikzcd} \qquad \begin{tikzcd} \mathcal{M}^B \arrow[r, bend left, "u^*"] \arrow[r, phantom, "\perp"] & \mathcal{M}^A \arrow[l, bend left, "u_*"] \end{tikzcd}$$

are Quillen pairs when considering the projective and injective model structures respectively and hence their total derived functors induce adjunctions between the homotopy categories. We first define what it means for a model category to be combinatorial.

Definition 6.1.1. A model category $\mathcal{M}$ is **cofibrantly generated** if there exists two sets of morphisms I and J in $\mathcal{M}$ such that

- the set I permits the small object argument and a morphism in $\mathcal{M}$ has the right lifting property with respect to I if and only if it is a trivial fibration, and
- the set J permits the small object argument and a morphism in $\mathcal{M}$ has the right lifting property with respect to J if and only if it is a fibration.

Definition 6.1.2. A model category is **combinatorial** if it is cofibrantly generated and locally presentable.

Let $\mathcal{M}$ be a combinatorial model category. Fibrations and weak equivalences are defined levelwise in the projective model structure on $\mathcal{M}^A$, while cofibrations and weak equivalences are defined levelwise in the injective one, for A a small category. Since the model category $\mathcal{M}$ is combinatorial, this actually defines model structures on $\mathcal{M}^A$.

Definition 6.1.3. Let A be a small category.

(i) A morphism in $\mathcal{M}^A$ is a **weak equivalence** if it is a levelwise weak equivalence.

(ii) A morphism in $\mathcal{M}^A$ is a **projective fibration** if it is a levelwise fibration.

(iii) A morphism in $\mathcal{M}^A$ is a **projective cofibration** if it has the left lifting property with respect to all projective trivial fibrations.

(iv) A morphism in $\mathcal{M}^A$ is an **injective cofibration** if it is a levelwise cofibration.

(v) A morphism in $\mathcal{M}^A$ is an **injective fibration** if it has the right lifting property with respect to all injective trivial cofibrations.

Theorem 6.1.4. *Let A be a small category. There exist two model structures on $\mathcal{M}^A$.*

*(i) the **projective model structure** $\mathcal{M}^A_{\mathrm{proj}}$ determined by the levelwise weak equivalences, the projective fibrations and the projective cofibrations, and*

*(ii) the **injective model structure** $\mathcal{M}^A_{\mathrm{inj}}$ determined by the levelwise weak equivalences, the injective cofibrations and the injective fibrations.*

Proof. See Proposition A.2.8.2 in [Lur09]. □

Remark 6.1.5. If A is a small category, the projective model structure and the injective model structure induce the same homotopy category $\mathrm{Ho}(\mathcal{M}^A)$ of $\mathcal{M}^A$ since they have the same weak equivalences.

The following result shows that, for every functor $u\colon A \to B$ in Cat, the adjunctions of $u^*\colon \mathcal{M}^B \to \mathcal{M}^A$ with its left and right Kan extensions are Quillen pairs when considering the projective and injective model structures respectively. By Theorem 2.5.6, this gives adjunctions between the homotopy categories, which define the left and right homotopy Kan extensions.

Proposition 6.1.6 (Axiom **[D3]**). *Let $u\colon A \to B$ be a functor in* Cat*. The following adjunctions hold.*

$$\begin{tikzcd}[column sep=huge] \mathrm{Ho}(\mathcal{M}^A) \arrow[r, bend left=50, "\mathbb{L}u_!"] \arrow[r, bend right=50, "\mathbb{R}u_*"'] & \mathrm{Ho}(\mathcal{M}^B) \arrow[l, "\mathbb{R}u^* = \mathbb{L}u^*" description] \end{tikzcd}$$

Proof. The adjunctions

$$\begin{tikzcd} \mathcal{M}^A_{\mathrm{proj}} \arrow[r, bend left, "u_!"] \arrow[r, phantom, "\perp"] & \mathcal{M}^B_{\mathrm{proj}} \arrow[l, bend left, "u^*"] \end{tikzcd} \qquad \begin{tikzcd} \mathcal{M}^B_{\mathrm{inj}} \arrow[r, bend left, "u^*"] \arrow[r, phantom, "\perp"] & \mathcal{M}^A_{\mathrm{inj}} \arrow[l, bend left, "u_*"] \end{tikzcd}$$

are Quillen pairs since the functor u^* preserves projective (trivial) fibrations and injective (trivial) cofibrations. By Theorem 2.5.6, this implies that we have two adjunctions

$$\begin{tikzcd} \mathrm{Ho}(\mathcal{M}^A) \arrow[r, bend left, "\mathbb{L}u_!"] \arrow[r, phantom, "\perp"] & \mathrm{Ho}(\mathcal{M}^B) \arrow[l, bend left, "\mathbb{R}u^*"] \end{tikzcd} \quad \begin{tikzcd} \mathrm{Ho}(\mathcal{M}^B) \arrow[r, bend left, "\mathbb{L}u^*"] \arrow[r, phantom, "\perp"] & \mathrm{Ho}(\mathcal{M}^A) \arrow[l, bend left, "\mathbb{R}u_*"] \end{tikzcd}$$

between the homotopy categories of $\mathcal{M}^A$ and $\mathcal{M}^B$, one using the total right derived functor of u^* and the other its total left derived functor. Moreover, since the functor u^* preserves weak equivalences, this implies that its total left and right derived functors are equal and such that the following diagram commutes.

$$\begin{tikzcd} \mathcal{M}^B \arrow[r, "u^*"] \arrow[d] & \mathcal{M}^A \arrow[d] \\ \mathrm{Ho}(\mathcal{M}^B) \arrow[r, "\mathbb{R}u^* = \mathbb{L}u^*"'] & \mathrm{Ho}(\mathcal{M}^A) \end{tikzcd}$$

This shows the result. □

Finally, we check the remaining axioms for the homotopy derivator of a combinatorial model category.

Theorem 6.1.7. *Let $\mathcal{M}$ be a combinatorial model category. Then the prederivator*

$$\mathbb{D}_{\mathcal{M}} \colon \mathrm{Cat}^{\mathrm{op}} \to \mathrm{CAT}, \quad A \mapsto \mathrm{Ho}(\mathcal{M}^A)$$

is a derivator, called the ***homotopy derivator*** *of $\mathcal{M}$.*

Proof. Axiom [**D1**] follows from the fact that

$$\mathrm{Ho}(\mathcal{M}^{\coprod_{i\in I} A_i}) \cong \mathrm{Ho}(\prod_{i\in I} \mathcal{M}^{A_i}) \cong \prod_{i\in I} \mathrm{Ho}(\mathcal{M}^{A_i})$$

for every family $\{A_i\}_{i\in I}$ of small categories. Axiom [**D2**] is immediate since the weak equivalences are defined levelwise. Axiom [**D3**] is proven as Proposition 6.1.6.

Finally, we prove axiom [**D4**]. Since the prederivator $\mathcal{M}^{(-)} \colon \mathrm{Cat}^{\mathrm{op}} \to \mathrm{CAT}$ is a derivator, for every functor $u \colon A \to B$, we have a natural isomorphism

$$\begin{tikzcd}
\mathcal{M}_{\mathrm{proj}}^{u\downarrow b} \arrow[d, "\mathrm{colim}_{u\downarrow b}"'] \arrow[dr, phantom, "\searrow\cong"] & \mathcal{M}_{\mathrm{proj}}^{A} \arrow[l, "(\pi^b)^*"'] \arrow[d, "u_*"] \\
\mathcal{M}_{\mathrm{proj}} & \mathcal{M}_{\mathrm{proj}}^{B} \arrow[l, "b^*"]
\end{tikzcd}$$

We want to replace each functor in the square by its total left derived functor, in oder to obtain the required natural isomorphism. We know that the total left derived functors of u_* and $\mathrm{colim}_{u\downarrow b}$ exist and are absolute by Proposition 6.1.6. Moreover, since the functors b^* and $(\pi^b)^*$ preserve weak equivalences, their total left derived functors also exist, but we need to show that they are absolute in order to have a natural isomorphism $\mathbb{L}\,\mathrm{colim}_{u\downarrow b}\,\mathbb{L}(\pi^b)^* \Rightarrow \mathbb{L}b^*\mathbb{L}u_*$. To see this, it suffices to show that the adjunctions

$$\begin{tikzcd}
\mathcal{M}_{\mathrm{proj}}^{B} \arrow[r, bend left, "b^*"] \arrow[r, phantom, "\perp"] & \mathcal{M}_{\mathrm{proj}} \arrow[l, bend left, "b_*"]
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{M}_{\mathrm{proj}}^{A} \arrow[r, bend left, "(\pi^b)^*"] \arrow[r, phantom, "\perp"] & \mathcal{M}_{\mathrm{proj}}^{u\downarrow b} \arrow[l, bend left, "(\pi^b)_*"]
\end{tikzcd}$$

are Quillen pairs, by Theorem 2.5.6. Since the functors b^* and $(\pi^b)^*$ preserve weak equivalences, it remains to show that they preserve projective cofibrations, or equivalently that the functors b_* and $(\pi^b)_*$ preserve projective trivial fibrations.

We first show that b_* preserves projective trivial fibrations. By axiom **[D4]** applied to the represented derivator of $\mathcal{M}$, we have that

$$(b')^*b_*(\phi) \cong \lim_{b'\downarrow b} \Delta_{b'\downarrow b}(\phi) \cong \prod_{B(b',b)} \phi,$$

for every morphism ϕ in $\mathcal{M}_{\mathrm{proj}}$ and every $b' \in B$. If a morphism ϕ in $\mathcal{M}_{\mathrm{proj}}$ is a trivial fibration, then this product morphism is also a trivial fibration. Hence, since the evaluation functors create projective trivial fibrations, this implies that the functor b_* preserves them.

We now show that the functor $(\pi^b)_*$ preserves projective trivial fibrations. First note that, if $a \in A$, the functor

$$l\colon \pi^b_a \to a \downarrow \pi^b, \quad (a, u(a) \xrightarrow{f} b) \mapsto (a, u(a) \xrightarrow{f} b, a \xrightarrow{1_A} a),$$

where π^b_a denotes the discrete fiber of π^b over a, admits a right adjoint

$$a \downarrow \pi^b \to \pi^b_a, \quad (a', u(a') \xrightarrow{g} b, a \xrightarrow{h} a') \mapsto (a, u(a) \xrightarrow{u(h)} u(a') \xrightarrow{g} b).$$

In particular, we have a natural isomorphism $\lim_{a\downarrow\pi^b} \Rightarrow \lim_{\pi^b_a} l^*$, since this natural transformation is conjugate to the natural isomorphism in Remark 5.2.3. As before, it follows that

$$a^*(\pi^b)_*(\beta) \cong \lim_{a\downarrow\pi^b} \beta\pi^a \cong \lim_{\pi^b_a} \beta(\pi^a \circ l) \cong \prod_{f\in B(u(a),b)} \beta_{(a,f)},$$

for every natural transformation β in $\mathcal{M}^{u\downarrow b}_{\text{proj}}$ and every $a \in A$. If a natural transformation β in $\mathcal{M}^{u\downarrow b}_{\text{proj}}$ is a projective trivial fibration, then each of its component $\beta_{(a,f)}$ is a trivial fibration for every $(a, f) \in u \downarrow b$ and hence this product map is also a trivial fibration. Since the evaluation functors create projective trivial fibrations, this implies that $(\pi^b)_*$ preserves them.

Finally, we obtain the required natural isomorphism

$$\begin{tikzcd} \mathrm{Ho}(\mathcal{M}^{u\downarrow b}) \arrow[d, "\mathbb{L}\operatorname{colim}_{u\downarrow b}"'] \arrow[dr, phantom, "\Swarrow\cong"] & \mathrm{Ho}(\mathcal{M}^A) \arrow[l, "\mathbb{L}(\pi^b)^*"'] \arrow[d, "\mathbb{L}u_*"] \\ \mathrm{Ho}(\mathcal{M}) & \mathrm{Ho}(\mathcal{M}^B) \arrow[l, "\mathbb{L}b^*"] \end{tikzcd}$$

The second part of axiom **[D4]** is dual. □

In particular, for every small category A, this implies that the left and right adjoints of the diagonal functor $\Delta_A\colon \mathrm{Ho}(\mathcal{M}) \to \mathrm{Ho}(\mathcal{M}^A)$ exist. They are called the homotopy colimit and limit functors of diagrams of shape A.

Definition 6.1.8. Let A be a small category.

(i) The **homotopy colimit functor** of diagrams of shape A is the total left derived functor

$$\operatorname{hocolim}_A = \mathbb{L}\operatorname{colim}_A\colon \mathrm{Ho}(\mathcal{M}^A) \to \mathrm{Ho}(\mathcal{M})$$

of the colimit functor of diagrams of shape A.

(ii) The **homotopy limit functor** of diagrams of shape A is the total right derived functor

$$\operatorname{holim}_A = \mathbb{R}\lim_A\colon \mathrm{Ho}(\mathcal{M}^A) \to \mathrm{Ho}(\mathcal{M})$$

of the limit functor of diagrams of shape A.

Finally, we present the example of the homotopy derivator of the category of simplicial sets equipped with the Quillen model structure.

Example 6.1.9. Consider the category sSet of simplicial sets and define the following classes of maps.

- The weak equivalences are the maps of simplicial sets such that their geometric realization in the category of topological spaces is a weak homotopy equivalence, called **weak homotopy equivalences** of simplicial sets.
- The cofibrations are the monomorphisms.
- The fibrations are the maps of simplicial sets that have the right lifting property with respect to the inclusions of k-horns $\Lambda^n_k \hookrightarrow \Delta^n$ for every $0 \leq k \leq n$ and $n \in \mathbb{N}$, called **Kan fibrations**.

This defines a model structure on sSet, called the Quillen model structure (see [JT99]). Since the category sSet is locally presentable, it is combinatorial with

$$I = \{\delta\Delta^n \hookrightarrow \Delta^n \mid n \in \mathbb{N}\} \quad \text{and} \quad J = \{\Lambda^n_k \hookrightarrow \Delta^n \mid 0 \le k \le n, n \in \mathbb{N}\}.$$

By Theorem 6.1.7, this implies that $\mathbb{D}_{\mathrm{sSet}} \colon \mathrm{Cat}^{\mathrm{op}} \to \mathrm{CAT}, A \mapsto \mathrm{Ho}(\mathrm{sSet}^A)$ is a derivator.

6.2 Quillen Equivalences and Derivators

The aim here is to show that, if we have a Quillen equivalence between two combinatorial model categories, their derivators are equivalent and thus give rise to the same basic localizer. The definition of an equivalence between derivators that we give here comes from [Gro13], Proposition 2.9.

Definition 6.2.1. Two derivators $\mathbb{D}$ and $\mathbb{E}$ are **equivalent** if there exists a pseudo-natural transformation $\tau \colon \mathbb{D} \to \mathbb{E}$ such that $\tau_A \colon \mathbb{D}(A) \to \mathbb{E}(A)$ is an equivalence of categories for every small category A.

The next result shows that, if two derivators are equivalent, they have the same basic localizer.

Theorem 6.2.2. *Let $\mathbb{D}$ and $\mathbb{E}$ be two equivalent derivators. Then $\mathcal{W}_{\mathbb{D}} = \mathcal{W}_{\mathbb{E}}$.*

Proof. Let $u \colon A \to B$ be a $\mathbb{D}$-equivalence in Cat. By Proposition 5.1.5, this implies that the natural transformation $\lim^{\mathbb{D}}_B \Delta^{\mathbb{D}}_B \Rightarrow \lim^{\mathbb{D}}_A \Delta^{\mathbb{D}}_A$ induced by the unit $\eta \colon 1_{\mathbb{D}(B)} \Rightarrow u^{\mathbb{D}}_* u^*_{\mathbb{D}}$ is a natural isomorphism. By equivalence of $\mathbb{D}$ and $\mathbb{E}$, we have the following diagram.

$$\begin{tikzcd}
\mathbb{E}(\mathbb{1}) \arrow[r, "\Delta^{\mathbb{E}}_B"] \arrow[d, "\simeq"'] \arrow[dd, bend right=60, phantom, "\cong"] \arrow[ddd, bend right=80, "\Delta^{\mathbb{E}}_A"'] \arrow[dr, phantom, "\cong"] & \mathbb{E}(B) \arrow[d, "\simeq"] \arrow[ddd, bend left=80, "\lim^{\mathbb{E}}_B"] \\
\mathbb{D}(\mathbb{1}) \arrow[r, "\Delta^{\mathbb{D}}_B"] \arrow[d, "\Delta^{\mathbb{D}}_A"'] \arrow[dr, phantom, "\overset{\cong}{\swarrow}"] & \mathbb{D}(B) \arrow[d, "\lim^{\mathbb{D}}_B \cong"] \\
\mathbb{D}(A) \arrow[r, "\lim^{\mathbb{D}}_A"'] \arrow[d, "\simeq"'] \arrow[dr, phantom, "\cong"] & \mathbb{D}(\mathbb{1}) \arrow[d, "\simeq"] \\
\mathbb{E}(A) \arrow[r, "\lim^{\mathbb{E}}_A"'] & \mathbb{E}(\mathbb{1})
\end{tikzcd}$$

It follows from this diagram that the natural transformation $\lim^{\mathbb{E}}_B \Delta^{\mathbb{E}}_B \Rightarrow \lim^{\mathbb{E}}_A \Delta^{\mathbb{E}}_A$ induced by the unit $\eta \colon 1_{\mathbb{E}(B)} \Rightarrow u^{\mathbb{E}}_* u^*_{\mathbb{E}}$ is a natural isomorphism and hence the functor u is an $\mathbb{E}$-equivalence, by Proposition 5.1.5. Similarly, we can show that if a functor in Cat is an $\mathbb{E}$-equivalence, then it is a $\mathbb{D}$-equivalence. This shows that $\mathcal{W}_{\mathbb{D}} = \mathcal{W}_{\mathbb{E}}$. $\square$

To prove that a Quillen equivalence between two combinatorial model categories $\mathcal{M}$ and $\mathcal{N}$ induces an equivalence between their homotopy derivators, we first check that it induces Quillen equivalences between the categories $\mathcal{M}^A$ and $\mathcal{N}^A$ equipped with the projective and injective model structures, for every small category A. Theorem 2.5.8 then gives us an equivalence between the homotopy categories $\mathrm{Ho}(\mathcal{M}^A)$ and $\mathrm{Ho}(\mathcal{N}^A)$, for every small category A.

Lemma 6.2.3. *Let $F\colon \mathcal{M} \to \mathcal{N}$ and $G\colon \mathcal{N} \to \mathcal{M}$ be functors between combinatorial model categories such that $F \dashv G$ is a Quillen equivalence. For every small category A, the adjunctions*

$$\begin{tikzcd} \mathcal{M}^A_{\mathrm{proj}} \arrow[r, bend left, "F_*"] \arrow[r, phantom, "\perp"] & \mathcal{N}^A_{\mathrm{proj}} \arrow[l, bend left, "G_*"] \end{tikzcd} \qquad \begin{tikzcd} \mathcal{M}^A_{\mathrm{inj}} \arrow[r, bend left, "F_*"] \arrow[r, phantom, "\perp"] & \mathcal{N}^A_{\mathrm{inj}} \arrow[l, bend left, "G_*"] \end{tikzcd}$$

are Quillen equivalences.

Proof. We show that the first adjunction is a Quillen equivalence and then the second follows by duality. Since the weak equivalences and the projective fibrations are defined levelwise in the projective model structure on $\mathcal{M}^A$, the functor G_* preserves fibrations and trivial fibrations and the adjunction $F_* \dashv G_*$ is a Quillen pair. Now consider a cofibrant object $X \in \mathcal{M}^A$ and a fibrant object $Y \in \mathcal{N}^A$. Since projective fibrations are defined levelwise and projective cofibrations are in particular levelwise cofibrations, the object $X(a) \in \mathcal{M}$ is cofibrant and the object $Y(a) \in \mathcal{N}$ is fibrant, for every $a \in A$. It follows from this that $f\colon X \Rightarrow G_*Y$ is a weak equivalence if and only if $f_a\colon X(a) \to G(Y(a))$ is a weak equivalence for every $a \in A$ if and only if $f^\#\colon F(X(a)) \to Y(a)$ is a weak equivalence for every $a \in A$ if and only if $f^\#\colon F_*X \Rightarrow Y$ is a weak equivalence, since $F \dashv G$ is a Quillen equivalence. Hence the adjunction $F_* \dashv G_*$ is a Quillen equivalence between $\mathcal{M}^A_{\mathrm{proj}}$ and $\mathcal{N}^A_{\mathrm{proj}}$. ☐

Corollary 6.2.4. *Let $F\colon \mathcal{M} \to \mathcal{N}$ and $G\colon \mathcal{N} \to \mathcal{M}$ be functors between combinatorial model categories such that $F \dashv G$ is a Quillen equivalence. For every small category A, we have an equivalence of categories*

$$\begin{tikzcd} \mathrm{Ho}(\mathcal{M}^A) \arrow[r, bend left, "\mathbb{L}F_*"] \arrow[r, phantom, "\simeq"] & \mathrm{Ho}(\mathcal{N}^A). \arrow[l, bend left, "\mathbb{R}G_*"] \end{tikzcd}$$

Proof. This follows immediately from Lemma 6.2.3 and Theorem 2.5.8. ☐

Finally, we show that a Quillen equivalence gives rise to an equivalence between homotopy derivators.

Theorem 6.2.5. *Let $F\colon \mathcal{M} \to \mathcal{N}$ and $G\colon \mathcal{N} \to \mathcal{M}$ be functors between combinatorial model categories such that $F \dashv G$ is a Quillen equivalence. Then the homotopy derivators*

$$\mathbb{D}_{\mathcal{M}}\colon \mathrm{Cat}^{\mathrm{op}} \to \mathrm{CAT}, \quad A \mapsto \mathrm{Ho}(\mathcal{M}^A) \quad \textit{and} \quad \mathbb{D}_{\mathcal{N}}\colon \mathrm{Cat}^{\mathrm{op}} \to \mathrm{CAT}, \quad A \mapsto \mathrm{Ho}(\mathcal{N}^A)$$

are equivalent. Moreover, the equality $\mathcal{W}_{\mathbb{D}_{\mathcal{M}}} = \mathcal{W}_{\mathbb{D}_{\mathcal{N}}}$ holds.

Proof. Since $\mathbb{L}F_*\colon \mathrm{Ho}(\mathcal{M}^A) \simeq \mathrm{Ho}(\mathcal{N}^A)$ is an equivalence of categories for every $A \in \mathrm{Cat}$ by Corollary 6.2.4 and all functors $\mathbb{L}F_*$ are absolute Kan extensions by Proposition 2.4.7, it follows that the pseudo-natural transformation given by $\mathbb{L}F_*\colon \mathrm{Ho}(\mathcal{M}^A) \to \mathrm{Ho}(\mathcal{N}^A)$, for every $A \in \mathrm{Cat}$, defines an equivalence between the derivators $\mathbb{D}_{\mathcal{M}}$ and $\mathbb{D}_{\mathcal{N}}$. The equality $\mathcal{W}_{\mathbb{D}_{\mathcal{M}}} = \mathcal{W}_{\mathbb{D}_{\mathcal{N}}}$ follows from Theorem 6.2.2. $\square$

6.3 Definition of $\mathcal{W}_\infty$ and Minimality

We introduce here the basic localizer $\mathcal{W}_\infty$ which consists of all functors in Cat whose nerve is a weak homotopy equivalence of simplicial sets. Cisinski has proved that the class $\mathcal{W}_\infty$ is actually a basic localizer and that it is the minimal one, i.e. every basic localizer contains $\mathcal{W}_\infty$. In this section, we give a sketch of Cisinski's proof and more details can be found in [Cis04]. We first give the definition of the nerve functor.

Definition 6.3.1. The **nerve functor** $N\colon \mathrm{Cat} \to \mathrm{sSet}$ carries

- a small category A to its nerve $NA \in \mathrm{sSet}$ defined by

$$(NA)_n = \mathrm{Cat}(\Delta^n, A),$$

  for every $n \in \mathbb{N}$, where $\Delta^n = 0 \to 1 \to \ldots \to n$, and

- a functor $u\colon A \to B$ in Cat to the map of simplicial sets $Nu\colon NA \to NB$ such that

$$(Nu)_n\colon \mathrm{Cat}(\Delta^n, A) \to \mathrm{Cat}(\Delta^n, B),$$

  is the post-composition by u, for every $n \in \mathbb{N}$.

We are now ready to define the basic localizer $\mathcal{W}_\infty$.

Definition 6.3.2. We define $\mathcal{W}_\infty$ to be the class of functors in Cat whose nerve is a weak homotopy equivalence of simplicial sets. In other words, if $\mathcal{W}_Q$ denotes the class of weak homotopy equivalences of sSet, we set $\mathcal{W}_\infty = N^{-1}\mathcal{W}_Q$.

Remark 6.3.3. A small category A is $\mathcal{W}_\infty$-aspherical if its nerve is homotopy equivalent to a point. To see this, a small category A is $\mathcal{W}_\infty$-aspherical if its nerve is weak homotopy equivalent to a point by definition and, since the geometric realization of a simplicial set is a CW-complex, this is equivalent to say that it is homotopy equivalent to a point, by Whitehead's Theorem. Then a functor $u\colon A \to B$ in Cat is $\mathcal{W}_\infty$-aspherical if the nerve of the comma category $u \downarrow b$ is homotopy equivalent to a point for every $b \in B$, i.e. if the functor u is homotopy initial. Dually, the $\mathcal{W}_\infty$-coaspherical functors correspond to the homotopy final functors.

The two next results say that the class $\mathcal{W}_\infty$ is the minimal basic localizer of Cat.

Theorem 6.3.4. *The class $\mathcal{W}_\infty$ of functors in* Cat *is a basic localizer.*

Proof. See Theorem 2.1.13 in [Cis04]. $\square$

Theorem 6.3.5 (Cisinski)**.** *The basic localizer $\mathcal{W}_\infty$ is the minimal one.*

Sketch of proof. Let $i_\Delta \colon \mathrm{sSet} \to \mathrm{Cat}$ be the functor carrying

- a simplicial set X to the category $\Delta \downarrow X$ of simplices of X whose
  - objects are the pairs $(\Delta^n, \Delta^n \to X)$ for $n \in \mathbb{N}$, and
  - morphisms $(\Delta^n, \Delta^n \to X) \to (\Delta^m, \Delta^m \to X)$ are morphisms $\Delta^n \to \Delta^m$ such that the following diagram commutes.

$$\begin{tikzcd} \Delta^n \arrow[rr] \arrow[dr] & & \Delta^m \arrow[dl] \\ & X & \end{tikzcd}$$

- a map $f \colon X \to Y$ in sSet to the induced functor $\Delta \downarrow f \colon \Delta \downarrow X \to \Delta \downarrow Y$.

Cisinski proves that $\mathcal{W}_Q = i_\Delta^{-1}\mathcal{W}_\infty$ (Theorem 2.1.16 in [Cis04]), where $\mathcal{W}_Q$ is the class of weak homotopy equivalences in sSet. To show that the basic localizer $\mathcal{W}_\infty$ is the minimal one, he shows that, for every basic localizer $\mathcal{W}$, we have $\mathcal{W} = N^{-1}i_\Delta^{-1}\mathcal{W}$ and $\mathcal{W}_Q \subseteq i_\Delta^{-1}\mathcal{W}$, which implies that

$$\mathcal{W}_\infty = N^{-1}i_\Delta^{-1}\mathcal{W}_\infty = N^{-1}\mathcal{W}_Q \subseteq N^{-1}i_\Delta^{-1}\mathcal{W} = \mathcal{W}.$$

For more details, see [Cis04], Section 2.2. $\square$

6.4 Quillen Equivalence between Cat and sSet

The nerve functor $N \colon \mathrm{Cat} \to \mathrm{sSet}$ admits a left adjoint and this adjunction induces an equivalence between the homotopy categories of Cat and sSet. But, this adjunction is not a Quillen equivalence and hence it does not induce an equivalence between the homotopy derivators of Cat and sSet. To define the Quillen equivalence between Cat and sSet, we need to postcompose the nerve functor twice with the right adjoint of the subdivision endofunctor of sSet, called the extension endofunctor. In this section, we present the results of R. W. Thomason (see [Tho80]) and R. Fritsch and D. M. Latch (see [FL79]), which we use to show that the basic localizer of sSet is $\mathcal{W}_\infty$. We first define the left adjoint to the nerve functor.

Definition 6.4.1. We define the functor $c \colon \mathrm{sSet} \to \mathrm{Cat}$ to be the functor sending

- a simplicial set X to the category $c(X)$ defined by:
  - the objects of $c(X)$ are the 0-simplicies of X;

  - the morphisms of $c(X)$ are freely generated by the 1-simplicies of X;
  - imposing the relation $d_1 x = d_0 x \circ d_2 x$ for every 2-simplex x.

- a map $f\colon X \to Y$ of simplicial sets to the functor $c(f)\colon c(X) \to c(Y)$ defined as
  - $f_0\colon X_0 \to Y_0$ on the objects;
  - $f_1\colon X_1 \to Y_1$ on the morphisms.

This induces an adjunction

$$\begin{tikzcd} \mathrm{sSet} \arrow[r, bend left, "c"] \arrow[r, phantom, "\perp"] & \mathrm{Cat} \arrow[l, bend left, "N"] \end{tikzcd}$$

which gives rise to an equivalence between the homotopy categories of sSet and Cat. Moreover, the subdivision endofunctor $\mathrm{Sd}\colon \mathrm{sSet} \to \mathrm{sSet}$ admits as right adjoint the extensions endofunctor $\mathrm{Ex}\colon \mathrm{sSet} \to \mathrm{sSet}$. Hence we also have an adjunction

$$\begin{tikzcd} \mathrm{sSet} \arrow[r, bend left, "\mathrm{Sd}"] \arrow[r, phantom, "\perp"] & \mathrm{sSet}. \arrow[l, bend left, "\mathrm{Ex}"] \end{tikzcd}$$

By composing the first adjunction twice with this one, we finally obtain an adjunction

$$\begin{tikzcd} \mathrm{sSet} \arrow[r, bend left, "c\mathrm{Sd}^2"] \arrow[r, phantom, "\perp"] & \mathrm{Cat}. \arrow[l, bend left, "\mathrm{Ex}^2 N"] \end{tikzcd}$$

which is shown to be a Quillen equivalence in [FL79], once we have equipped the category Cat with the Thomason model structure. We first define this model structure.

Theorem 6.4.2 (Thomason model structure on Cat)**.** *Consider the following class of functors in* Cat*.*

- *The weak equivalences are the functors u in* Cat *such that $Ex^2 N(u)$ is a weak homotopy equivalence in* sSet*.*
- *The fibrations are the functors u in* Cat *such that $Ex^2 N(u)$ is a Kan fibration in* sSet*.*
- *The cofibrations are the functors in* Cat *that have the left lifting property with respect to all trivial fibrations.*

This defines a model structure on Cat*.*

Proof. See [Tho80], Sections 3 and 4. □

It follows directly from the definition of the Thomason model structure that the adjunction

$$\begin{tikzcd} \mathrm{sSet} \arrow[r, bend left, "c\mathrm{Sd}^2"] \arrow[r, phantom, "\perp"] & \mathrm{Cat}. \arrow[l, bend left, "\mathrm{Ex}^2 N"] \end{tikzcd}$$

is a Quillen pair. At first glance, bearing in mind the Thomason model structure on Cat, one might expect that the class of weak equivalences is not $\mathcal{W}_\infty$. However, as was shown by Thomason, these classes coincide.

Proposition 6.4.3. *A functor in* Cat *is a weak equivalence for the Thomason model structure if and only if its nerve is a weak homotopy equivalence of simplicial sets. In other words, the class of weak equivalences of the Thomason model structure on* Cat *is* $\mathcal{W}_\infty$.

Proof. See Proposition 2.4 in [Tho80]. □

Moreover, it follows from the definition that the fibrations in the Thomason model structure on Cat are exactly the functors which have the right lifting property with respect to the set of functors $\{c\mathrm{Sd}^2\Lambda^n_k \to c\mathrm{Sd}^2\Delta^n \mid 0 \le k \le n, n \in \mathbb{N}\}$ and that the trivial fibrations are exactly the ones which have the right lifting property with the set of functors $\{c\mathrm{Sd}^2\delta\Delta^n \to c\mathrm{Sd}^2\Delta^n \mid n \in \mathbb{N}\}$. This implies that the Thomason model structure is cofibrantly generated. Therefore, the category Cat equipped with the Thomason model structure is a combinatorial model category and we obtain a homotopy derivator

$$\mathbb{D}_{\mathrm{Cat}} \colon \mathrm{Cat}^{\mathrm{op}} \to \mathrm{CAT} \quad A \mapsto \mathrm{Ho}(\mathrm{Cat}^A).$$

Theorem 6.4.4. *The prederivator*

$$\mathbb{D}_{\mathrm{Cat}} \colon \mathrm{Cat}^{\mathrm{op}} \to \mathrm{CAT} \quad A \mapsto \mathrm{Ho}(\mathrm{Cat}^A).$$

is a derivator, when the category Cat *is equipped with the Thomason model structure.*

Now we want to compute the basic localizer of this derivator and, as expected, Maltsiniotis shows that it is $\mathcal{W}_\infty$.

Proposition 6.4.5. *The basic localizer of the homotopy derivator* $\mathbb{D}_{\mathrm{Cat}}$ *is* $\mathcal{W}_\infty$*, when the category* Cat *is equipped with the Thomason model structure.*

Proof. See Proposition 3.1.10 in [Mal05]. □

Finally, Fritsch and Latch show that the adjunction

$$\begin{tikzcd} \mathrm{sSet} \arrow[r, bend left, "c\mathrm{Sd}^2"] \arrow[r, phantom, "\perp"] & \mathrm{Cat}. \arrow[l, bend left, "\mathrm{Ex}^2 N"] \end{tikzcd}$$

is a Quillen equivalence and it follows from the results of Section 6.2 that the basic localizer associated to the homotopy derivator of the category sSet is $\mathcal{W}_\infty$.

Theorem 6.4.6. *Suppose the category* sSet *is equipped with the Quillen model structure and the category* Cat *is equipped with the Thomason model structure. Then the Quillen pair*

$$\begin{tikzcd} \mathrm{sSet} \arrow[r, bend left, "cSd^2"] \arrow[r, phantom, "\perp"] & \mathrm{Cat}. \arrow[l, bend left, "Ex^2 N"] \end{tikzcd}$$

is a Quillen equivalence and the equalities $\mathcal{W}_{\mathbb{D}_{\mathrm{sSet}}} = \mathcal{W}_{\mathbb{D}_{\mathrm{Cat}}} = \mathcal{W}_\infty$ *hold.*

Proof. For the first part of the theorem, see [FL79]. The second part follows from Proposition 6.4.5 and Theorem 6.2.5. □

Part III
Calculus of Limits and Colimits

7 Cartesian Squares

We introduce here general pullback squares, called cartesian squares, with respect to every derivator and solve three problems about them. In Section 7.1, we first give a characterization of cartesian squares in terms of right $\mathbb{D}$-Beck-Chevalley squares at a diagram. Sections 7.2, 7.3 and 7.4 give the resolutions of the three problems, which we called respectively "Pullback composition and cancellation", "Pullback iteration" and "Pullback cube". For example,the first problem consists of showing that, if a diagram of the form

$$\begin{tikzcd}
X_{00} \arrow[r] \arrow[d] & X_{10} \arrow[r] \arrow[d] & X_{20} \arrow[d] \\
X_{01} \arrow[r] & X_{11} \arrow[r] & X_{21}
\end{tikzcd}$$

is such that the right square is a cartesian square, then the left square is cartesian if and only if the exterior square is cartesian, with respect to every derivator. This problem is well-known with respect to the represented derivator, where the cartesian squares correspond to the usual pullback squares. Of course, one could dualize every result in this section and obtain similar problems for generalized pushout squares, called cocartesian squares.

7.1 Characterization of Cartesian Squares

We define in this section pullback squares with repect to every derivator. As expected, pullback squares are diagrams of shape the square category $\square$ such that the object in the top left corner is the limit of the full subdiagram in which this latter was removed.

Definition 7.1.1. We define the category $\square$ to be the small category

$$\begin{tikzcd}
00 \arrow[r] \arrow[d] & 10 \arrow[d] \\
01 \arrow[r] & 11
\end{tikzcd}$$

and the category $\lrcorner$ to be the full subcategory of $\square$ without the object 00.

$$\begin{tikzcd}
 & 10 \arrow[d] \\
01 \arrow[r] & 11
\end{tikzcd}$$

Definition 7.1.2. Let $\mathbb{D}$ be a derivator. A diagram $X \in \mathbb{D}(\square)$ is called a **square**. We can see this object as a diagram in $\mathbb{D}(\mathbb{1})$ of the form

$$\begin{tikzcd} X_{00} \arrow[r] \arrow[d] & X_{10} \arrow[d] \\ X_{01} \arrow[r] & X_{11} \end{tikzcd}$$

where X_{00}, X_{10}, X_{01} and X_{11} denote the images of X under the appropriate evaluation functors.

Definition 7.1.3. Let $\mathbb{D}$ be a derivator and $i_\lrcorner \colon \lrcorner \to \square$ denote the inclusion functor. A square $X \in \mathbb{D}(\square)$ is **cartesian** (or a **pullback**) if the square X lies in the essential image of $(i_\lrcorner)_* \colon \mathbb{D}(\lrcorner) \to \mathbb{D}(\square)$.

We want to find criteria to check that a diagram lies in the essential image of the right Kan extension of a functor in Cat. In particular, we are interested in fully faithful functors in Cat, which give rise to fully faithful left and right Kan extensions.

Proposition 7.1.4. *Let $\mathcal{W}$ be a basic localizer. If $u \colon A \to B$ is a fully faithful functor in* Cat*, the square*

$$\mathcal{D} = \begin{tikzcd} A \arrow[r, equal] \arrow[d, equal] \arrow[dr, phantom, "\overset{=}{\swarrow}"] & A \arrow[d, "u"] \\ A \arrow[r, "u"'] & B \end{tikzcd}$$

is weak $\mathcal{W}$-exact.

Proof. The induced functor of Definition 4.2.1 (i) is the functor

$$A \downarrow a \longrightarrow u \downarrow u(a), \quad (a', a' \xrightarrow{f} a) \mapsto (a', u(a') \xrightarrow{u(f)} u(a))$$

for every $a \in A$. Since the functor u is fully faithful, it is an isomorphism and, in particular, a $\mathcal{W}$-colocal functor over A. This shows that the square $\mathcal{D}$ is weak $\mathcal{W}$-exact. $\square$

Let us fix a derivator $\mathbb{D}$.

Remark 7.1.5. By Theorem 5.2.10, Proposition 7.1.4 implies that, if $u \colon A \to B$ is a fully faithful functor in Cat, the square

$$\mathcal{D} = \begin{tikzcd} A \arrow[r, equal] \arrow[d, equal] \arrow[dr, phantom, "\overset{=}{\swarrow}"] & A \arrow[d, "u"] \\ A \arrow[r, "u"'] & B \end{tikzcd}$$

is $\mathbb{D}$-Beck-Chevalley.

Corollary 7.1.6. *If $u\colon A \to B$ is a fully faithful functor in* Cat*, the functors*

$$u_*, u_! \colon \mathbb{D}(A) \to \mathbb{D}(B)$$

are fully faithful.

Proof. By Remark 7.1.5, since the functor u is fully faithful, the square

$$\mathcal{D} = \begin{tikzcd} A \arrow[r, equal] \arrow[d, equal] \arrow[dr, phantom, "\overset{=}{\swarrow}"] & A \arrow[d, "u"] \\ A \arrow[r, "u"'] & B \end{tikzcd}$$

is $\mathbb{D}$-Beck-Chevalley. This means that the square

$$\mathbb{D}(\mathcal{D}) = \begin{tikzcd} \mathbb{D}(A) \arrow[r, equal] \arrow[d, equal] \arrow[dr, phantom, "\overset{=}{\swarrow}"] & \mathbb{D}(A) \\ \mathbb{D}(A) & \mathbb{D}(B) \arrow[l, "u^*"] \arrow[u, "u^*"'] \end{tikzcd}$$

is Beck-Chevalley, i.e. the mates $\eta\colon 1_{\mathbb{D}(A)} \Rightarrow u^*u_!$ and $\epsilon\colon u^*u_* \Rightarrow 1_{\mathbb{D}(A)}$ are natural isomorphisms. By Lemma 3.2.9, this implies that the functors $u_*, u_! \colon \mathbb{D}(A) \to \mathbb{D}(B)$ are fully faithful. $\square$

We want to compute the essential image of the right Kan extension $u_* \colon \mathbb{D}(A) \to \mathbb{D}(B)$ of a fully faithful functor $u\colon A \to B$ in Cat. This essential image consists of all diagrams such that the component of the unit of the adjunction $u^* \dashv u_*$ at these diagrams is an isomorphism. We give this result in terms of mates and Beck-Chevalley squares. In order to do this, we introduce the notion of right $\mathbb{D}$-Beck-Chevalley squares at a diagram. One could also define the notion of left $\mathbb{D}$-Beck-Chevalley squares at a diagram and dualize the results to left Kan extensions.

Definition 7.1.7. Consider a square $\mathcal{D}$ in Cat

$$\mathcal{D} = \begin{tikzcd} C \arrow[r, "p"] \arrow[d, "v"'] \arrow[dr, phantom, "\overset{\alpha}{\swarrow}"] & A \arrow[d, "u"] \\ D \arrow[r, "q"'] & B \end{tikzcd}$$

and a diagram $X \in \mathbb{D}(D)$. We say that the square $\mathcal{D}$ is **right $\mathbb{D}$-Beck-Chevalley** at X if the component $(\alpha_*)_X \colon u^*q_*(X) \to p_*v^*(X)$ of the right mate is an isomorphism in $\mathbb{D}(A)$.

Remark 7.1.8. In particular, a $\mathbb{D}$-Beck-Chevalley square in Cat is right $\mathbb{D}$-Beck-Chevalley at every diagram.

The following result shows that a diagram lies in the essential image of the right Kan extension of a fully faithful functor if and only if the component of the unit at this diagram is an isomorphism.

Proposition 7.1.9. *Let $u\colon A \to B$ be a fully faithful functor in* Cat*. A diagram $X \in \mathbb{D}(B)$ lies in the essential image of $u_*\colon \mathbb{D}(A) \to \mathbb{D}(B)$ if and only if the square*

$$\mathcal{D} = \begin{tikzcd} A \arrow[r, "u"] \arrow[d, "u"'] \arrow[dr, phantom, "\overset{=}{\swArrow}"] & B \arrow[d, equal] \\ B \arrow[r, equal] & B \end{tikzcd}$$

is right $\mathbb{D}$-Beck-Chevalley at X.

Proof. Recall that, since the functor u is fully faithful, the functor u_* is also fully faithful (Corollary 7.1.6) and hence the counit $\epsilon\colon u^*u_* \Rightarrow 1_{\mathbb{D}(A)}$ is a natural isomorphism (Lemma 3.2.9). By definition, the square $\mathcal{D}$ is right $\mathbb{D}$-Beck-Chevalley at $X \in \mathbb{D}(D)$ if and only if the component $\eta_X\colon X \to u_*u^*(X)$ of the unit is an isomorphism in $\mathbb{D}(B)$. We show that this condition is equivalent to the fact that the diagram X lies in the essential image of u_*. If η_X is an isomorphism, then $X \cong u_*u^*(X)$ lies in the essential image of u_*.

Conversely, suppose that $X \cong u_*(Y)$ for some $Y \in \mathbb{D}(A)$. Then the diagram

$$\begin{tikzcd} u_*(Y) \arrow[r, "\eta_{u_*(Y)}"] \arrow[dr, equal] & u_*u^*u_*(Y) \arrow[d, "u_*(\epsilon_Y)", "\cong"'] \\ & u_*(Y) \end{tikzcd}$$

commutes by the triangle identity and hence $\eta_{u_*(Y)}$ is an isomorphism, since ϵ_Y is one. By naturality of η, the diagram

$$\begin{tikzcd} X \arrow[r, "\eta_X"] \arrow[d, "\cong"'] & u_*u^*(X) \arrow[d, "\cong"] \\ u_*(Y) \arrow[r, "\cong", "\eta_{u_*(Y)}"'] & u_*u^*u_*(Y) \end{tikzcd}$$

commutes and this shows that η_X is an isomorphism. $\square$

When the functor $u_*\colon \mathbb{D}(A) \to \mathbb{D}(B)$ is fully faithful, the counit of the adjunction $u^* \dashv u_*$ is a natural isomorphism. Hence it follows from the triangle identities that, if we want to show that a diagram $X \in \mathbb{D}(B)$ lies in the essential image of u_*, we only need to check that some components of the unit $\eta_X\colon X \Rightarrow u_*u^*(X)$ are isomorphisms.

Proposition 7.1.10. *Let $u\colon A \to B$ be a fully faithful functor in* Cat. *A diagram $X \in \mathbb{D}(B)$ lies in the essential image of $u_*\colon \mathbb{D}(A) \to \mathbb{D}(B)$ if and only if the pasting*

$$\begin{tikzcd}
b \downarrow u \arrow[r] \arrow[d, "\pi_b"'] \arrow[dr, phantom, "\swArrow"] & \mathbb{1} \arrow[d, "b"] \\
A \arrow[r, "u"] \arrow[d, "u"'] \arrow[dr, phantom, "=\swArrow"] & B \arrow[d, equal] \\
B \arrow[r, equal] & B
\end{tikzcd}$$

is right $\mathbb{D}$-Beck-Chevalley at X for every $b \in B \setminus u(A)$.

Proof. By Proposition 7.1.9, a diagram $X \in \mathbb{D}(B)$ lies in the essential image of u_* if and only if the square

$$\mathcal{D} = \begin{tikzcd}
A \arrow[r, "u"] \arrow[d, "u"'] \arrow[dr, phantom, "=\swArrow"] & B \arrow[d, equal] \\
B \arrow[r, equal] & B
\end{tikzcd}$$

is right $\mathbb{D}$-Beck-Chevalley at X or equivalently if $\eta_X\colon X \to u_*u^*(X)$ is an isomorphism. Hence, since comma squares are $\mathbb{D}$-Beck-Chevalley, the first implication is clear.

Suppose now that the pasting is right $\mathbb{D}$-Beck-Chevalley at X for every $b \in B \setminus u(A)$, i.e. the morphism

$$b^*(X) \xrightarrow{b^*(\eta_X)} b^*u_*u^*(X) \xrightarrow{\cong} \lim_{b\downarrow u} \pi_b^* u^*(X)$$

is an isomorphism for every $b \in B \setminus u(A)$ or equivalently $b^*(\eta_X)$ is an isomorphism for every $b \in B \setminus u(A)$. By axiom **[D2]**, it suffices to prove that $u(a)^*(\eta_X) = a^*u^*(\eta_X)$ is an isomorphism for every $a \in A$ in order to show that η_X is an isomorphism. By the triangle identity, the diagram

$$\begin{tikzcd}
a^*u^*(X) \arrow[r, "a^*u^*(\eta_X)"] \arrow[dr, equal] & a^*u^*u_*u^*(X) \arrow[d, "a^*(\epsilon_{u^*(X)})", "\cong"'] \\
& a^*u^*(X)
\end{tikzcd}$$

commutes and hence $u(a)^*(\eta_X)$ is an isomorphism since $\epsilon\colon u^*u_* \Rightarrow 1_{\mathbb{D}(A)}$ is a natural isomorphism. This shows that η_X is an isomorphism and that the diagram X lies in the essential image of u_*. $\square$

Applying these results to the functor $i_{\lrcorner}\colon \lrcorner \to \square$, we obtain the following outcome, which implies that the object in the top left corner of a pullback square is the limit of the full subdiagram in which this latter was removed.

Corollary 7.1.11. *A square $X \in \mathbb{D}(\square)$ is cartesian if and only if the pasting*

$$\begin{tikzcd}
\lrcorner \arrow[r] \arrow[d, equal] \arrow[dr, phantom, "\swArrow"] & \mathbb{1} \arrow[d, "00"] \\
\lrcorner \arrow[r, "i_\lrcorner"] \arrow[d, "i_\lrcorner"'] \arrow[dr, phantom, "=\swArrow"] & \square \arrow[d, equal] \\
\square \arrow[r, equal] & \square
\end{tikzcd}$$

is right $\mathbb{D}$-Beck-Chevalley at X.

Proof. Since the functor $i_\lrcorner \colon \lrcorner \to \square$ is fully faithfull, $\square \setminus i_\lrcorner(\lrcorner) = \{00\}$ and the comma category $00 \downarrow i_\lrcorner$ is $\lrcorner$, the result follows directly from Proposition 7.1.10. $\square$

Remark 7.1.12. Corollary 7.1.11 implies that a square $X \in \mathbb{D}(\square)$ is cartesian if and only if $X_{00} \cong \lim i_\lrcorner^*(X)$, where

$$i_\lrcorner^*(X) = \begin{tikzcd}
 & X_{10} \arrow[d] \\
X_{01} \arrow[r] & X_{11}
\end{tikzcd}$$

This can be seen by computing the right mate of the pasting in the corollary.

In particular, if we consider the represented derivator of a cocomplete and complete category $\mathcal{C}$, the condition $X_{00} \cong \lim i_\lrcorner^*(X)$ corresponds to the expected condition for a pullback square in $\mathcal{C}$.

Similarly, for a combinatorial model category $\mathcal{M}$, a cartesian square with respect to the homotopy derivator of $\mathcal{M}$ corresponds to a homotopy pullback square in $\mathcal{M}$.

7.2 Pullback Composition and Cancellation

Let $\mathbb{D}$ be a derivator. For the first problem, we consider diagrams of shape the small category $\square\square$ which is defined by

$$\begin{tikzcd}
00 \arrow[r] \arrow[d] & 10 \arrow[r] \arrow[d] & 20 \arrow[d] \\
01 \arrow[r] & 11 \arrow[r] & 21
\end{tikzcd}$$

Let $X \in \mathbb{D}(\square\square)$, i.e. X is a diagram of the form

$$\begin{tikzcd}
X_{00} \arrow[r] \arrow[d] & X_{10} \arrow[r] \arrow[d] & X_{20} \arrow[d] \\
X_{01} \arrow[r] & X_{11} \arrow[r] & X_{21}
\end{tikzcd}$$

We want to show that if the diagram $X \in \mathbb{D}(\square\square)$ is such that its right square is cartesian, then its left square is cartesian if and only if its exterior square is cartesian.

For this problem, we also consider the category $\_\square$ defined as the full subcategory of $\square\square$ without the object 00

$$\begin{tikzcd}
 & 10 \arrow[r] \arrow[d] & 20 \arrow[d] \\
01 \arrow[r] & 11 \arrow[r] & 21
\end{tikzcd}$$

and the category $\_\lrcorner$ defined as the full subcategory of $\_\square$ without the object 10.

$$\begin{tikzcd}
 & & 20 \arrow[d] \\
01 \arrow[r] & 11 \arrow[r] & 21
\end{tikzcd}$$

We adopt the following notations:

- $i_{01} \colon \square \to \square\square$ for the inclusion of the square in the left square, and similarly $i_{01} \colon \lrcorner \to \_\square$;
- $i_{12} \colon \square \to \square\square$ for the inclusion of the square in the right square, and similarly $i_{12} \colon \square \to \_\square$ and $i_{12} \colon \lrcorner \to \_\lrcorner$;
- $i_{02} \colon \square \to \square\square$ for the inclusion of the square in the exterior square, and similarly $i_{02} \colon \lrcorner \to \_\lrcorner$.

It follows from this that, if $X \in \mathbb{D}(\square\square)$, the diagram given by the left square of the diagram X is the square $i_{01}^*(X) \in \mathbb{D}(\square)$ and similarly for the other squares. We also consider the inclusions

$$i \colon \_\lrcorner \longrightarrow \_\square \quad \text{and} \quad j \colon \_\square \longrightarrow \square\square$$

and the composite of these two inclusions

$$k \colon \_\lrcorner \longrightarrow \square\square.$$

To solve our problem, we first prove that a diagram lies in the essential image of the right Kan extension of one of these three inclusions if and only if some of its squares are cartesian.

Lemma 7.2.1. *A diagram $X \in \mathbb{D}(\_\square)$ lies in the essential image of*

$$i_* \colon \mathbb{D}(\_\lrcorner) \to \mathbb{D}(\_\square)$$

if and only if its right square $i_{12}^*(X)$ is cartesian.

Proof. Since the functor $i \colon \_\lrcorner \to \_\square$ is fully faithful, we apply Proposition 7.1.10. We have that $\_\square \setminus i(\_\lrcorner) = \{10\}$ and that the comma category $10 \downarrow i = \lrcorner$, where $\lrcorner$ denotes the left part of $\_\lrcorner$. Hence a diagram $X \in \mathbb{D}(\_\square)$ lies in the essential image of $i_* \colon \mathbb{D}(\_\lrcorner) \to \mathbb{D}(\_\square)$ if and only if the following left-hand diagram is right $\mathbb{D}$-Beck-Chevalley at X.

$$\begin{tikzcd}
\lrcorner \arrow[r] \arrow[d, "i_{12}"'] \arrow[dr, phantom, "\swarrow"] & \mathbb{1} \arrow[d, "10"] \\
_\lrcorner \arrow[r, "i"] \arrow[d, "i"'] \arrow[dr, phantom, "\overset{=}{\swarrow}"] & _\square \arrow[d, equal] \\
_\square \arrow[r, equal] & _\square
\end{tikzcd}
=
\begin{tikzcd}
\lrcorner \arrow[r] \arrow[d, equal] \arrow[dr, phantom, "\swarrow"] & \mathbb{1} \arrow[d, "00"] \\
\lrcorner \arrow[r, "i_\lrcorner"] \arrow[d, "i_\lrcorner"'] \arrow[dr, phantom, "\overset{=}{\swarrow}"] & \square \arrow[d, equal] \\
\square \arrow[r, equal] \arrow[d, "i_{12}"'] \arrow[dr, phantom, "\overset{=}{\swarrow}"] & \square \arrow[d, "i_{12}"] \\
_\square \arrow[r, equal] & _\square
\end{tikzcd}$$

Since the right-hand diagram is equal to the left-hand diagram, this is equivalent to saying that the former is right $\mathbb{D}$-Beck-Chevalley at X. But this means that the pasting of the two upper squares is right $\mathbb{D}$-Beck-Chevalley at $i_{12}^*(X)$, which is equivalent to the fact that the square $i_{12}^*(X)$ is cartesian by Corollary 7.1.11. $\square$

Lemma 7.2.2. *A diagram $X \in \mathbb{D}(\square\square)$ lies in the essential image of*

$$j_* \colon \mathbb{D}(\_\square) \to \mathbb{D}(\square\square)$$

if and only if its left square $i_{01}^*(X)$ is cartesian.

Proof. Since the functor $j \colon \_\square \to \square\square$ is fully faithful, we apply Proposition 7.1.10. We have that $\square\square \setminus j(\_\square) = \{00\}$ and that the comma category $00 \downarrow j = \_\square$. Hence a diagram $X \in \mathbb{D}(\square\square)$ lies in the essential image of $j_* \colon \mathbb{D}(\_\square) \to \mathbb{D}(\square\square)$ if and only if the following diagram is right $\mathbb{D}$-Beck-Chevalley at X.

$$
\begin{tikzcd}
\_\square \arrow[r] \arrow[d, equal] \arrow[dr, phantom, "\swarrow"] & \mathbb{1} \arrow[d, "00"] \\
\_\square \arrow[r, "j"] \arrow[d, "j"'] \arrow[dr, phantom, "\overset{=}{\swarrow}"] & \square\square \arrow[d, equal] \\
\square\square \arrow[r, equal] & \square\square
\end{tikzcd}
\tag{1}
$$

Now consider the functor $i_{01} \colon \lrcorner \to \_\square$ sending $10 \mapsto 10$, $11 \mapsto 11$ and $01 \mapsto 01$. This functor admits a right adjoint $\_\square \to \lrcorner$ defined by $10, 20 \mapsto 10$, $11, 21 \mapsto 11$ and $01 \mapsto 01$. The details are left to the reader. It follows from this that the square

$$
\begin{tikzcd}
\lrcorner \arrow[r] \arrow[d, "i_{01}"'] \arrow[dr, phantom, "\overset{=}{\swarrow}"] & \mathbb{1} \arrow[d, equal] \\
\_\square \arrow[r] & \mathbb{1}
\end{tikzcd}
$$

is $\mathbb{D}$-Beck-Chevalley (see Remark 5.2.3) and that (1) is right $\mathbb{D}$-Beck-Chevalley at X if and only if the following left-hand diagram is right $\mathbb{D}$-Beck-Chevalley at X.

$$
\begin{tikzcd}
\lrcorner \arrow[r] \arrow[d, "i_{01}"'] \arrow[dr, phantom, "\overset{=}{\swarrow}"] & \mathbb{1} \arrow[d, equal] \\
\_\square \arrow[r] \arrow[d, equal] \arrow[dr, phantom, "\swarrow"] & \mathbb{1} \arrow[d, "00"] \\
\_\square \arrow[r, "j"] \arrow[d, "j"'] \arrow[dr, phantom, "\overset{=}{\swarrow}"] & \square\square \arrow[d, equal] \\
\square\square \arrow[r, equal] & \square\square
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
\lrcorner \arrow[r] \arrow[d, equal] \arrow[dr, phantom, "\swarrow"] & \mathbb{1} \arrow[d, "00"] \\
\lrcorner \arrow[r, "i_{\lrcorner}"] \arrow[d, "i_{\lrcorner}"'] \arrow[dr, phantom, "\overset{=}{\swarrow}"] & \square \arrow[d, equal] \\
\square \arrow[r, equal] \arrow[d, "i_{01}"'] \arrow[dr, phantom, "\overset{=}{\swarrow}"] & \square \arrow[d, "i_{01}"] \\
\square\square \arrow[r, equal] & \square\square
\end{tikzcd}
$$

Since the right-hand diagram is equal to the left-hand diagram, this is equivalent to saying that the former is right $\mathbb{D}$-Beck-Chevalley at X. But this means that the pasting of the two upper squares is right $\mathbb{D}$-Beck-Chevalley at $i_{01}^*(X)$, which is equivalent to the fact that the square $i_{01}^*(X)$ is cartesian by Corollary 7.1.11. $\square$

Corollary 7.2.3. *A diagram $X \in \mathbb{D}(\square\square)$ lies in the essential image of*

$$k_* \colon \mathbb{D}(\_\_\lrcorner) \to \mathbb{D}(\square\square)$$

if and only if both squares $i_{01}^*(X)$ and $i_{12}^*(X)$ are cartesian.

Proof. Suppose first that a diagram $X \in \mathbb{D}(\square\square)$ is such that the squares $i_{01}^*(X)$ and $i_{12}^*(X)$ are cartesian. By Lemma 7.2.2, since the square $i_{01}^*(X)$ is cartesian, the diagram X lies in the essential image of $j_* \colon \mathbb{D}(\_\square) \to \mathbb{D}(\square\square)$, i.e. there exists a diagram $Y \in \mathbb{D}(\_\square)$ such that $X \cong j_*(Y)$. Then the square $i_{12}^*(Y) \cong i_{12}^*(X)$ is cartesian and, by Lemma 7.2.1, we have that the diagram Y lies in the essential image of $i_* \colon \mathbb{D}(\_\_\lrcorner) \to \mathbb{D}(\_\square)$, i.e. there exists a diagram $Z \in \mathbb{D}(\_\_\lrcorner)$ such that $Y \cong i_*(Z)$. Finally, this implies that the diagram

$$X \cong j_*(Y) \cong j_* i_*(Z) \cong k_*(Z)$$

lies in the essential image of $k_* \colon \mathbb{D}(\_\_\lrcorner) \to \mathbb{D}(\square\square)$ since $k = j \circ i$.

Suppose now that a diagram $X \in \mathbb{D}(\square\square)$ lies in the essential image of k_*. Since $k_* \cong j_* i_*$, the diagram X lies in particular in the essential image of j_*. By Lemma 7.2.2, this means that the square $i_{01}^*(X)$ is cartesian. Moreover, since the diagram X lies in the essential image of k_*, the following left-hand diagram is right $\mathbb{D}$-Beck-Chevalley at X (see Proposition 7.1.9).

$$\begin{tikzcd}
__\lrcorner \arrow[r, "k"] \arrow[d, "k"'] \arrow[dr, phantom, "=\!\swarrow"] & \square\square \arrow[d, equal] \\
\square\square \arrow[r, equal] & \square\square
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
__\lrcorner \arrow[r, "i"] \arrow[d, "i"'] \arrow[dr, phantom, "=\!\swarrow"] & _\square \arrow[d, equal] \\
_\square \arrow[r, equal] \arrow[d, equal] \arrow[dr, phantom, "=\!\swarrow"] & _\square \arrow[d, "j"] \\
_\square \arrow[r, "j"] \arrow[d, "j"'] \arrow[dr, phantom, "=\!\swarrow"] & \square\square \arrow[d, equal] \\
\square\square \arrow[r, equal] & \square\square
\end{tikzcd}$$

Hence the right-hand diagram is also right $\mathbb{D}$-Beck-Chevalley at X. By Remark 7.1.5, the middle square is $\mathbb{D}$-Beck-Chevalley since j is fully faithful and, by Lemma 7.1.9, the lower square is right $\mathbb{D}$-Beck-Chevalley at X since the diagram X lies in the essential image of j_*. It follows from this that the upper square is right $\mathbb{D}$-Beck-Chevalley at $j^*(X)$. By Proposition 7.1.9, this means that the diagram $j^*(X)$ lies in the essential image of i_* and hence that the square $i_{12}^* j^*(X) = i_{12}^*(X)$ is cartesian by Lemma 7.2.1. $\square$

We now have a criterion to check that the left and right squares of a diagram in $\mathbb{D}(\square\square)$ are cartesian. It remains to show that, if the right square is cartesian, this criterion coincide with the fact that the exterior square is cartesian.

Theorem 7.2.4. *Let $X \in \mathbb{D}(\square\square)$ be such that its right square $i_{12}^*(X)$ is cartesian. Then its left square $i_{01}^*(X)$ is cartesian if and only if its exterior square $i_{02}^*(X)$ is cartesian.*

Proof. If the square $i_{12}^*(X)$ is cartesian, then the square $i_{01}^*(X)$ is cartesian if and only if the diagram X lies in the essential image of $k_* \colon \mathbb{D}(\_\_\lrcorner) \to \mathbb{D}(\square\square)$ by Corollary 7.2.3.

Since the functor k is fully faithful and $\square\!\square \backslash k(\_\!\_\lrcorner) = \{00, 10\}$, this is equivalent to saying that the following diagrams are right $\mathbb{D}$-Beck-Chevalley at X by Proposition 7.1.10.

$$
\begin{tikzcd}
\_\!\_\lrcorner \arrow[r] \arrow[d, equal] \arrow[dr, phantom, "\swarrow"] & \mathbb{1} \arrow[d, "00"] \\
\_\!\_\lrcorner \arrow[r, "k"] \arrow[d, "k"'] \arrow[dr, phantom, "\overset{=}{\swarrow}"] & \square\!\square \arrow[d, equal] \\
\square\!\square \arrow[r, equal] & \square\!\square
\end{tikzcd}
\quad (2) \qquad
\begin{tikzcd}
\lrcorner \arrow[r] \arrow[d, "i_{12}"'] \arrow[dr, phantom, "\swarrow"] & \mathbb{1} \arrow[d, "10"] \\
\_\!\_\lrcorner \arrow[r, "k"] \arrow[d, "k"'] \arrow[dr, phantom, "\overset{=}{\swarrow}"] & \square\!\square \arrow[d, equal] \\
\square\!\square \arrow[r, equal] & \square\!\square
\end{tikzcd}
=
\begin{tikzcd}
\lrcorner \arrow[r] \arrow[d, equal] \arrow[dr, phantom, "\swarrow"] & \mathbb{1} \arrow[d, "00"] \\
\lrcorner \arrow[r, "i_{\lrcorner}"] \arrow[d, "i_{\lrcorner}"'] \arrow[dr, phantom, "\overset{=}{\swarrow}"] & \square \arrow[d, equal] \\
\square \arrow[r, equal] \arrow[d, "i_{12}"'] \arrow[dr, phantom, "\overset{=}{\swarrow}"] & \square \arrow[d, "i_{12}"] \\
\square\!\square \arrow[r, equal] & \square\!\square
\end{tikzcd}
\quad (3)
$$

Diagram (3) is right $\mathbb{D}$-Beck-Chevalley at X by hypothesis, since the square $i_{12}^*(X)$ is cartesian, which means that the right-hand diagram is right $\mathbb{D}$-Beck-Chevalley at X. Hence the diagram X lies in the essential image of k_* if and only if (2) is right $\mathbb{D}$-Beck-Chevalley at X. Now consider the inclusion $i_{02} \colon \lrcorner \to \_\!\_\lrcorner$ sending $10 \mapsto 20$, $11 \mapsto 21$ and $01 \mapsto 01$. This functor admits a right adjoint $\_\!\_\lrcorner \to \lrcorner$ given by $20 \mapsto 10$, $21 \mapsto 11$ and $01, 11 \mapsto 01$. The details are left to the reader. Then diagram (2) is right $\mathbb{D}$-Beck-Chevalley at X if and only if the following left-hand diagram is right $\mathbb{D}$-Beck-Chevalley at X.

$$
\begin{tikzcd}
\lrcorner \arrow[r] \arrow[d, "i_{02}"'] \arrow[dr, phantom, "\overset{=}{\swarrow}"] & \mathbb{1} \arrow[d, equal] \\
\_\!\_\lrcorner \arrow[r] \arrow[d, equal] \arrow[dr, phantom, "\swarrow"] & \mathbb{1} \arrow[d, "00"] \\
\_\!\_\lrcorner \arrow[r, "k"] \arrow[d, "k"'] \arrow[dr, phantom, "\overset{=}{\swarrow}"] & \square\!\square \arrow[d, equal] \\
\square\!\square \arrow[r, equal] & \square\!\square
\end{tikzcd}
=
\begin{tikzcd}
\lrcorner \arrow[r] \arrow[d, equal] \arrow[dr, phantom, "\swarrow"] & \mathbb{1} \arrow[d, "00"] \\
\lrcorner \arrow[r, "i_{\lrcorner}"] \arrow[d, "i_{\lrcorner}"'] \arrow[dr, phantom, "\overset{=}{\swarrow}"] & \square \arrow[d, equal] \\
\square \arrow[r, equal] \arrow[d, "i_{02}"'] \arrow[dr, phantom, "\overset{=}{\swarrow}"] & \square \arrow[d, "i_{02}"] \\
\square\!\square \arrow[r, equal] & \square\!\square
\end{tikzcd}
$$

Since the right-hand diagram is equal to the left-hand diagram, this is equivalent to saying that the former is right $\mathbb{D}$-Beck-Chevalley at X. But this means that the pasting of the two upper squares is right $\mathbb{D}$-Beck-Chevalley at $i_{02}^*(X)$, which is equivalent to the fact that the square $i_{02}^*(X)$ is cartesian by Corollary 7.1.11. This proves the statement. $\square$

This theorem allows us to compute limits of diagrams in $\mathbb{D}(\_\!\_\lrcorner)$.

Corollary 7.2.5. *Let $X \in \mathbb{D}(\lrcorner)$. If we construct*

$$\begin{tikzcd}
X_{00} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X_{10} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X_{20} \arrow[d] \\
X_{01} \arrow[r] & X_{11} \arrow[r] & X_{21}
\end{tikzcd} \tag{4}$$

such that X_{10} is first constructed to be the pullback of $X_{11} \to X_{21} \leftarrow X_{20}$ and then X_{00} is constructed to be the pullback of $X_{01} \to X_{11} \leftarrow X_{10}$, we have that

$$\lim X \cong X_{00}.$$

Proof. By Corollary 7.2.3, diagram (4) corresponds up to isomorphism to the diagram $k_*(X) \in \mathbb{D}(\square\square)$ which obviouly lies in the essential image of $k_* \colon \mathbb{D}(\lrcorner) \to \mathbb{D}(\square\square)$. In particular, Proposition 7.1.10 implies that the following diagram is right $\mathbb{D}$-Beck-Chevalley at $k_*(X)$.

$$\begin{tikzcd}
\lrcorner \arrow[r] \arrow[d, equal] \arrow[dr, phantom, "\swarrow"] & \mathbb{1} \arrow[d, "00"] \\
\lrcorner \arrow[r, "k"] \arrow[d, "k"'] \arrow[dr, phantom, "="] \arrow[dr, phantom, "\swarrow"] & \square\square \arrow[d, equal] \\
\square\square \arrow[r, equal] & \square\square
\end{tikzcd}$$

By computing the right mate, we obtain $X_{00} \cong \lim k^*k_*(X)$. Moreover, since the functor k is fully faithful, the functor k_* is fully faithful by Corollary 7.1.6 and $k^*k_*(X) \cong X$ by Lemma 3.2.9. Finally, we obtain $X_{00} \cong \lim X$. $\square$

7.3 Strongly Cartesian Cube

Let $\mathbb{D}$ be a derivator. For the second problem, we consider diagrams of shape the small category $\lrcorner$ which is defined by

$$\begin{tikzcd}
 & & 110 \arrow[dd] \\
 & 101 \arrow[dr] & \\
011 \arrow[rr] & & 111
\end{tikzcd}$$

We want to compute the limits of such diagrams.

Let $X \in \mathbb{D}(\lrcorner)$, i.e. X is a diagram of the form

$$\begin{tikzcd}
& & X_{110} \arrow[dd] \\
& X_{101} \arrow[dr] & \\
X_{011} \arrow[rr] & & X_{111}
\end{tikzcd}$$

We first construct the pullbacks of $X_{011} \to X_{111} \leftarrow X_{101}$ and $X_{101} \to X_{111} \leftarrow X_{110}$. If $\lrcorner\!\!\lrcorner$ denotes the small category

$$\begin{tikzcd}
& & 100 \arrow[dr] \arrow[dd] & \\
& & & 110 \arrow[dd] \\
001 \arrow[rr] \arrow[dr] & & 101 \arrow[dr] & \\
& 011 \arrow[rr] & & 111
\end{tikzcd}$$

we now have a diagram

$$\begin{tikzcd}
& & X_{100} \arrow[dr] \arrow[dd] & \\
& & & X_{110} \arrow[dd] \\
X_{001} \arrow[rr] \arrow[dr] & & X_{101} \arrow[dr] & \\
& X_{011} \arrow[rr] & & X_{111}
\end{tikzcd}$$

of shape $\lrcorner\!\!\lrcorner$ such that both squares are cartesian. Again, we can construct the pullback of $X_{001} \to X_{101} \leftarrow X_{100}$. If $\square\!\!\lrcorner$ is the small category

$$\begin{tikzcd}
000 \arrow[rr] \arrow[dd] & & 100 \arrow[dr] \arrow[dd] & \\
& & & 110 \arrow[dd] \\
001 \arrow[rr] \arrow[dr] & & 101 \arrow[dr] & \\
& 011 \arrow[rr] & & 111
\end{tikzcd}$$

we obtain this time a diagram

$$\begin{tikzcd}
X_{000} \arrow[rr] \arrow[dd] & & X_{100} \arrow[dr] \arrow[dd] & \\
& & & X_{110} \arrow[dd] \\
X_{001} \arrow[rr] \arrow[dr] & & X_{101} \arrow[dr] & \\
& X_{011} \arrow[rr] & & X_{111}
\end{tikzcd}$$

of shape $\square_3$ such that the three squares are cartesian. We want to show that

$$\lim X \cong X_{000}.$$

We call the squares in the category $\square_3$ the right square, the bottom square and the back square respectively (look at it as a part of a cube). We adopt the following notations:

- $i_r\colon \square \to \square_3$ for the inclusion of the square in the right square, and similarly $i_r\colon \square \to \square_2$ and $i_r\colon \lrcorner \to \lrcorner_3$;
- $i_{bo}\colon \square \to \square_3$ for the inclusion of the square in the bottom square, and similarly $i_{bo}\colon \square \to \square_2$ and $i_{bo}\colon \lrcorner \to \lrcorner_3$;
- $i_{bk}\colon \square \to \square_3$ for the inclusion of the square in the back square, and similarly $i_{bk}\colon \lrcorner \to \square_2$.

We also consider the inclusions

$$i\colon \lrcorner_3 \longrightarrow \square_2 \quad \text{and} \quad j\colon \square_2 \longrightarrow \square_3$$

and the composite of these two inclusions

$$k\colon \lrcorner_3 \longrightarrow \square_3.$$

As in the first problem, we first find criteria for these three squares to be cartesian in terms of essential images of the right Kan extensions of these inclusion functors.

Lemma 7.3.1. *A diagram $X \in \mathbb{D}(\square_2)$ lies in the essential image of*

$$i_*\colon \mathbb{D}(\lrcorner_3) \to \mathbb{D}(\square_2)$$

if and only if the right square $i_r^*(X)$ and the bottom square $i_{bo}^*(X)$ are cartesian.

Proof. By Proposition 7.1.10, since the functor i is fully faithful and $\square_2 \backslash i(\lrcorner_3) = \{100, 001\}$, a diagram $X \in \mathbb{D}(\square_2)$ lies in the essential image of i_* if and only if the two following left-hand diagrams are right $\mathbb{D}$-Beck-Chevalley at X.

$$\begin{tikzcd} \lrcorner \arrow[r] \arrow[d, "i_r"'] \arrow[dr, phantom, "\swArrow"] & \mathbb{1} \arrow[d, "100"] \\ \lrcorner_3 \arrow[r, "i"] \arrow[d, "i"'] \arrow[dr, phantom, "=\swArrow"] & \square_2 \arrow[d, equal] \\ \square_2 \arrow[r, equal] & \square_2 \end{tikzcd} = \begin{tikzcd} \lrcorner \arrow[r] \arrow[d, equal] \arrow[dr, phantom, "\swArrow"] & \mathbb{1} \arrow[d, "00"] \\ \lrcorner \arrow[r, "i_{\lrcorner}"] \arrow[d, "i_{\lrcorner}"'] \arrow[dr, phantom, "=\swArrow"] & \square \arrow[d, equal] \\ \square \arrow[r, equal] \arrow[d, "i_r"'] \arrow[dr, phantom, "=\swArrow"] & \square \arrow[d, "i_r"] \\ \square_2 \arrow[r, equal] & \square_2 \end{tikzcd} \qquad \begin{tikzcd} \lrcorner \arrow[r] \arrow[d, "i_{bo}"'] \arrow[dr, phantom, "\swArrow"] & \mathbb{1} \arrow[d, "100"] \\ \lrcorner_3 \arrow[r, "i"] \arrow[d, "i"'] \arrow[dr, phantom, "=\swArrow"] & \square_2 \arrow[d, equal] \\ \square_2 \arrow[r, equal] & \square_2 \end{tikzcd} = \begin{tikzcd} \lrcorner \arrow[r] \arrow[d, equal] \arrow[dr, phantom, "\swArrow"] & \mathbb{1} \arrow[d, "00"] \\ \lrcorner \arrow[r, "i_{\lrcorner}"] \arrow[d, "i_{\lrcorner}"'] \arrow[dr, phantom, "=\swArrow"] & \square \arrow[d, equal] \\ \square \arrow[r, equal] \arrow[d, "i_{bo}"'] \arrow[dr, phantom, "=\swArrow"] & \square \arrow[d, "i_{bo}"] \\ \square_2 \arrow[r, equal] & \square_2 \end{tikzcd}$$

Hence this is equivalent to saying that the two right-hand diagrams are right $\mathbb{D}$-Beck-Chevalley at X, i.e. the squares $i_r^*(X)$ and $i_{bo}^*(X)$ are cartesian (Corollary 7.1.11). $\square$

Lemma 7.3.2. *A diagram $X \in \mathbb{D}(\text{⧉})$ lies in the essential image of*

$$j_* \colon \mathbb{D}(\text{⟓}) \to \mathbb{D}(\text{⧉})$$

if and only if the back square $i_{bk}^*(X)$ is cartesian.

Proof. By Proposition 7.1.10, since the functor j is fully faithful and $\text{⧉} \setminus j(\text{⟓}) = \{000\}$, a diagram $X \in \mathbb{D}(\text{⧉})$ lies in the essential image of j_* if and only if the following diagram is right $\mathbb{D}$-Beck-Chevalley at X.

$$\begin{tikzcd}
\text{⟓} \arrow[r] \arrow[d, equal] \arrow[dr, phantom, "\swarrow"] & \mathbb{1} \arrow[d, "000"] \\
\text{⟓} \arrow[r, "j"] \arrow[d, "j"'] \arrow[dr, phantom, "\overset{=}{\swarrow}"] & \text{⧉} \arrow[d, equal] \\
\text{⧉} \arrow[r, equal] & \text{⧉}
\end{tikzcd} \tag{5}$$

Now consider the functor $i_{bk} \colon \lrcorner \to \text{⟓}$ sending $10 \mapsto 100$, $11 \mapsto 101$ and $01 \mapsto 001$. This functor admits a right adjoint $\text{⟓} \to \lrcorner$ defined by $100, 110 \mapsto 10$, $101, 111 \mapsto 11$ and $001, 011 \mapsto 01$. The details are left to the reader. It follows from this that (5) is right $\mathbb{D}$-Beck-Chevalley at X if and only if the following left-hand diagram is right $\mathbb{D}$-Beck-Chevalley at X.

$$\begin{tikzcd}
\lrcorner \arrow[r] \arrow[d, "i_{bk}"'] \arrow[dr, phantom, "\overset{=}{\swarrow}"] & \mathbb{1} \arrow[d, equal] \\
\text{⟓} \arrow[r] \arrow[d, equal] \arrow[dr, phantom, "\swarrow"] & \mathbb{1} \arrow[d, "000"] \\
\text{⟓} \arrow[r, "j"] \arrow[d, "j"'] \arrow[dr, phantom, "\overset{=}{\swarrow}"] & \text{⧉} \arrow[d, equal] \\
\text{⧉} \arrow[r, equal] & \text{⧉}
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
\lrcorner \arrow[r] \arrow[d, equal] \arrow[dr, phantom, "\swarrow"] & \mathbb{1} \arrow[d, "00"] \\
\lrcorner \arrow[r, "i_{\lrcorner}"] \arrow[d, "i_{\lrcorner}"'] \arrow[dr, phantom, "\overset{=}{\swarrow}"] & \square \arrow[d, equal] \\
\square \arrow[r, equal] \arrow[d, "i_{bk}"'] \arrow[dr, phantom, "\overset{=}{\swarrow}"] & \square \arrow[d, "i_{bk}"] \\
\text{⧉} \arrow[r, equal] & \text{⧉}
\end{tikzcd}$$

Hence this is equivalent to saying that the right-hand diagram is right $\mathbb{D}$-Beck-Chevalley at X, i.e. the square $i_{bk}^*(X)$ is cartesian (Corollary 7.1.11). $\square$

Corollary 7.3.3. *A diagram $X \in \mathbb{D}(\boxcube)$ lies in the essential image of*

$$k_* \colon \mathbb{D}(\lrcorner) \to \mathbb{D}(\boxcube)$$

if and only if the three squares $i_r^*(X)$, $i_{bo}^*(X)$ and $i_{bk}^*(X)$ are cartesian.

Proof. Since Lemmas 7.3.1 and 7.3.2 hold and $k = j \circ i$, the proof is essentially the same as the proof of Corollary 7.2.3. $\square$

Finally, we check that if a diagram $X \in \mathbb{D}(\boxcube)$ contains only cartesian squares, the object X_{000} is the limit of the subdiagram $k^*(X) \in \mathbb{D}(\lrcorner)$.

Theorem 7.3.4. *Let $X \in \mathbb{D}(\lrcorner)$. If we construct*

$$\begin{tikzcd}
X_{000} \arrow[r] \arrow[dd] & X_{100} \arrow[dd] \arrow[dr] & \\
& & X_{110} \arrow[dd] \\
X_{001} \arrow[r] \arrow[dr] & X_{101} \arrow[dr] & \\
& X_{011} \arrow[r] & X_{111}
\end{tikzcd} \tag{6}$$

such that X_{001} and X_{100} are first constructed to be the pullbacks of $X_{011} \to X_{111} \leftarrow X_{101}$ and $X_{101} \to X_{111} \leftarrow X_{110}$ respectively and then X_{000} is constructed to be the pullback of $X_{001} \to X_{101} \leftarrow X_{100}$, we have that

$$\lim X \cong X_{000}.$$

Proof. By Corollary 7.3.3, diagram (6) corresponds up to isomorphism to the object $k_*(X) \in \mathbb{D}(\boxcube)$ which obviously lies in the essential image of $k_* \colon \mathbb{D}(\lrcorner) \to \mathbb{D}(\boxcube)$. In particular, Proposition 7.1.10 implies that the following diagram is right $\mathbb{D}$-Beck-Chevalley at $k_*(X)$.

$$\begin{tikzcd}
\lrcorner \arrow[r] \arrow[d, equal] \arrow[dr, phantom, "\swarrow"] & \mathbb{1} \arrow[d, "000"] \\
\lrcorner \arrow[r, "k"] \arrow[d, "k"'] \arrow[dr, phantom, "\overset{=}{\swarrow}"] & \boxcube \arrow[d, equal] \\
\boxcube \arrow[r, equal] & \boxcube
\end{tikzcd}$$

By computing the right mate, we obtain $X_{000} \cong \lim k^* k_*(X)$. Moreover, since the functor k is fully faithful, the functor k_* is fully faithful by Corollary 7.1.6 and $k^* k_*(X) \cong X$ by Lemma 3.2.9. Finally, we obtain $X_{000} \cong \lim X$. $\square$

7.4 Pullback Cube

Let $\mathbb{D}$ be a derivator. For the last problem, we consider diagrams of shape the small category $⧉$

$$\begin{tikzcd}
000 \arrow[rr] \arrow[dr] \arrow[dd] & & 100 \arrow[dr] \arrow[dd] & \\
& 010 \arrow[rr] \arrow[dd] & & 110 \arrow[dd] \\
001 \arrow[rr] \arrow[dr] & & 101 \arrow[dr] & \\
& 011 \arrow[rr] & & 111
\end{tikzcd}$$

which we call **cubes**. We actually consider diagrams in $\mathbb{D}(⧉)$ such that the front and back faces are cartesian squares. Let $X \in \mathbb{D}(⧉)$ be a cube

$$\begin{tikzcd}
X_{000} \arrow[rr] \arrow[dr] \arrow[dd] & & X_{100} \arrow[dr] \arrow[dd] & \\
& X_{010} \arrow[rr] \arrow[dd] & & X_{110} \arrow[dd] \\
X_{001} \arrow[rr] \arrow[dr] & & X_{101} \arrow[dr] & \\
& X_{011} \arrow[rr] & & X_{111}
\end{tikzcd}$$

such that the front and back faces are cartesian. We first construct the pullbacks of the left and right face of the cube X, in other words the pullbacks of $X_{001} \to X_{011} \leftarrow X_{010}$ and $X_{101} \to X_{111} \leftarrow X_{110}$. If we denote by K the small category

$$\begin{tikzcd}
000 \arrow[rrr] \arrow[drr] \arrow[ddr] \arrow[ddd] & & & 100 \arrow[drr] \arrow[ddr] \arrow[ddd] & & \\
& & 010 \arrow[rrr] \arrow[ddd] & & & 110 \arrow[ddd] \\
& B \arrow[ur] \arrow[rrr] \arrow[dl] & & & A \arrow[ur] \arrow[dl] & \\
001 \arrow[rrr] \arrow[drr] & & & 101 \arrow[drr] & & \\
& & 011 \arrow[rrr] & & & 111
\end{tikzcd}$$

we obtain a diagram

$$\begin{tikzcd}
X_{000} \arrow[rrr] \arrow[drr] \arrow[ddr] \arrow[ddd] & & & X_{100} \arrow[drr] \arrow[ddr] \arrow[ddd] & & \\
& & X_{010} \arrow[rrr] \arrow[ddd] & & & X_{110} \arrow[ddd] \\
& X_B \arrow[ur] \arrow[rrr] \arrow[dl] & & & X_A \arrow[ur] \arrow[dl] & \\
X_{001} \arrow[rrr] \arrow[drr] & & & X_{101} \arrow[drr] & & \\
& & X_{011} \arrow[rrr] & & & X_{111}
\end{tikzcd}$$

such that the squares

$$
\begin{tikzcd}
X_A \arrow[r] \arrow[d] & X_{110} \arrow[d] \\
X_{101} \arrow[r] & X_{111}
\end{tikzcd}
\qquad (7) \qquad\qquad
\begin{tikzcd}
X_B \arrow[r] \arrow[d] & X_{010} \arrow[d] \\
X_{001} \arrow[r] & X_{011}
\end{tikzcd}
\qquad (8)
$$

are cartesian. The morphisms $X_{000} \to X_B$, $X_{100} \to X_A$ and $X_B \to X_A$ are given here by the universal property of pullbacks. We want to show that the square

$$
\begin{tikzcd}
X_{000} \arrow[r] \arrow[d] & X_{100} \arrow[d] \\
X_B \arrow[r] & X_A
\end{tikzcd}
\qquad (9)
$$

obtained by this construction is also cartesian. We adopt the following notations:

- $i \colon \square \to K$ for the inclusion of the square in the square (9);
- $i_f \colon \square \to \text{⧉}$ for the inclusion of the square in the front face, and similarly $i_f \colon \square \to K$;
- $i_b \colon \square \to \text{⧉}$ for the inclusion of the square in the back face, and similarly $i_b \colon \lrcorner \to K$;
- $i_A \colon \square \to K$ for the inclusion of the square in the square (7);
- $i_B \colon \square \to K$ for the inclusion of the square in the square (8).

We also consider the inclusions

$$j \colon \lrcorner\!\lrcorner \longrightarrow \text{⧉} \quad \text{and} \quad k \colon \text{⧉} \longrightarrow K,$$

where $\lrcorner\!\lrcorner$ is the category defined in the pullback iteration problem, and the composite of these two inclusions

$$f \colon \lrcorner\!\lrcorner \longrightarrow K.$$

To solve this problem, we will use the following criterion.

Proposition 7.4.1. *Let K be a small category that contains a square and J be a subcategory of K. Let $i \colon \square \to K$ be the inclusion of the square in K and $f \colon J \to K$ be the inclusion of J in K. Suppose that the object $i(00)$ does not lie in the image of $f \colon J \to K$ and that the functor $\bar{i} \colon \lrcorner \to i(00) \downarrow (K \setminus \{i(00)\})$ induced by i admits a right adjoint. Then, if a diagram $X \in \mathbb{D}(K)$ lies in the essential image of $f_* \colon \mathbb{D}(J) \to \mathbb{D}(K)$, the square $i^*(X)$ is cartesian.*

Proof. Let $X \in \mathbb{D}(K)$ be a diagram that lies in the essential image of f_*, i.e. there exists a diagram $Y \in \mathbb{D}(J)$ such that $X \cong f_*(Y)$. Since $i(00)$ does not belong to the image of f, the functor f factors through

$$f \colon J \xrightarrow{\bar{f}} K \setminus \{i(00)\} \xrightarrow{l} K$$

and $X \cong f_*(Y) \cong l_* \overline{f}_*(Y)$ lies in the essential image of $l_* \colon \mathbb{D}(K \setminus \{i(00)\}) \to \mathbb{D}(K)$. Then the following left-hand diagram is right $\mathbb{D}$-Beck-Chevalley at X

$$
\begin{tikzcd}
\lrcorner \arrow[r] \arrow[d, "\bar{i}"'] \arrow[dr, phantom, "\overset{=}{\swarrow}"] & \mathbb{1} \arrow[d, equal] \\
i(00) \downarrow (K \setminus \{i(00)\}) \arrow[r] \arrow[d, equal] \arrow[dr, phantom, "\swarrow"] & \mathbb{1} \arrow[d, "i(00)"] \\
K \setminus i(00) \arrow[r, "l"] \arrow[d, "l"'] \arrow[dr, phantom, "\overset{=}{\swarrow}"] & K \arrow[d, equal] \\
K \arrow[r, equal] & K
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
\lrcorner \arrow[r] \arrow[d, equal] \arrow[dr, phantom, "\swarrow"] & \mathbb{1} \arrow[d, "00"] \\
\lrcorner \arrow[r, "i_\lrcorner"] \arrow[d, "i_\lrcorner"'] \arrow[dr, phantom, "\overset{=}{\swarrow}"] & \square \arrow[d, equal] \\
\square \arrow[r, equal] \arrow[d, "i"'] \arrow[dr, phantom, "\overset{=}{\swarrow}"] & \square \arrow[d, "i"] \\
K \arrow[r, equal] & K
\end{tikzcd}
$$

since the functor $\bar{i} \colon \lrcorner \to i(00) \downarrow (K \setminus \{i(00)\})$ admits a right adjoint by hypothesis. Hence the right-hand diagram is also $\mathbb{D}$-Beck-Chevalley at X, which means that the square $i^*(X)$ is cartesian. $\square$

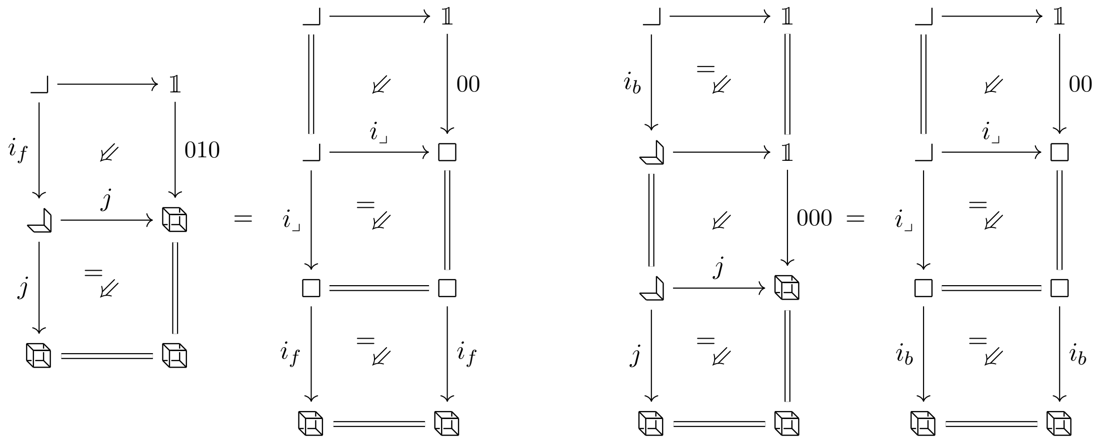
We can apply this criterion to $i \colon \square \to K$ and $f \colon \lrcorner\!\!\square \to K$ defined above since the object $i(00) = 000$ does not lies in the image of f and the functor $\lrcorner \to 000 \downarrow K \setminus \{000\}$ admits a right adjoint, as shown later. Hence it suffices to show that a diagram $X \in \mathbb{D}(K)$ lies in the essential image of $f_* \colon \mathbb{D}(\lrcorner\!\!\square) \to \mathbb{D}(K)$ when its squares $i_f^*(X)$, $i_b^*(X)$, $i_A^*(X)$ and $i_B^*(X)$ are cartesian.

Lemma 7.4.2. *A diagram $X \in \mathbb{D}(\boxplus)$ lies in the essential image of*

$$j_* \colon \mathbb{D}(\lrcorner\!\!\square) \to \mathbb{D}(\boxplus)$$

if and only if its front square $i_f^*(X)$ and its back square $i_b^*(X)$ are cartesian.

Proof. By Proposition 7.1.10, since the functor j is fully faithful and since we have $\boxplus \setminus j(\lrcorner\!\!\square) = \{000, 010\}$, a diagram $X \in \mathbb{D}(\boxplus)$ lies in the essential image of j_* if and only if the two following left-hand diagrams are right $\mathbb{D}$-Beck-Chevalley at X,



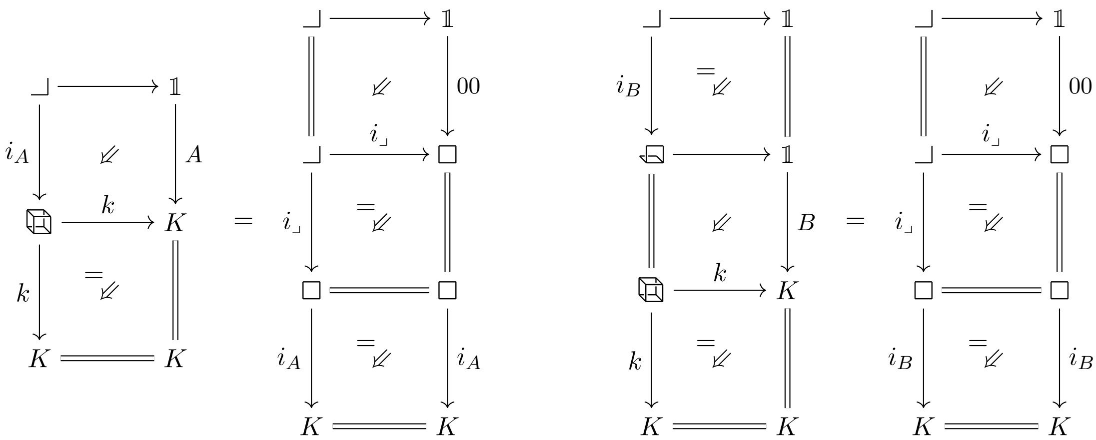
since the functor $i_f\colon \lrcorner \to \lrcorner\!\!\lrcorner$ sending $01 \mapsto 001$, $11 \mapsto 111$ and $10 \mapsto 100$ admits a right adjoint $\lrcorner\!\!\lrcorner \to \lrcorner$ given by $001, 011 \mapsto 01, 101, 111 \mapsto 11$ and $100, 110 \mapsto 10$. Hence the two right-hand diagrams are also right $\mathbb{D}$-Beck-Chevalley at X, which means that the squares $i_f^*(X)$ and $i_b^*(X)$ are cartesian. $\square$

Lemma 7.4.3. *A diagram $X \in \mathbb{D}(K)$ lies in the essential image of*

$$k_*\colon \mathbb{D}(⧉) \to \mathbb{D}(K)$$

if and only if its squares $i_A^*(X)$ and $i_B^*(X)$ are cartesian.

Proof. By Proposition 7.1.10, since the functor k is fully faithful and $K \setminus k(⧉) = \{A, B\}$, a diagram $X \in \mathbb{D}(K)$ lies in the essential image of k_* if and only if the two following left-hand diagrams are right $\mathbb{D}$-Beck-Chevalley at X,



since the functor $i_B\colon \lrcorner \to \boxminus$ sending $01 \mapsto 001$, $11 \mapsto 011$ and $10 \mapsto 010$ admits a right adjoint $\boxminus \to \lrcorner$ given by $001, 101 \mapsto 01, 011, 111 \mapsto 11$ and $010, 110 \mapsto 10$. Hence the two right-hand diagrams are also right $\mathbb{D}$-Beck-Chevalley at X, which means that the squares $i_A^*(X)$ and $i_B^*(X)$ are cartesian. $\square$

Corollary 7.4.4. *A diagram $X \in \mathbb{D}(K)$ lies in the essential image of*

$$f_* \colon \mathbb{D}(\lrcorner\!\square) \to \mathbb{D}(K)$$

if and only if its squares $i_f^*(X)$, $i_b^*(X)$, $i_A^*(X)$ and $i_B^*(X)$ are cartesian.

Proof. Since Lemmas 7.4.2 and 7.4.3 hold and $f = k \circ j$, the proof is essentially the same as the proof of Corollary 7.2.3. $\square$

Back to our problem, the cube $X \in \mathbb{D}(\boxtimes)$ that we consider gives rise to an object in $\mathbb{D}(K)$ that lies in the essential image of $f_* \colon \mathbb{D}(\lrcorner\!\square) \to \mathbb{D}(K)$, by hypothesis on the front and back faces and by construction of the pullbacks X_A and X_B. Hence the following result shows that, under this construction, the square

$$\begin{tikzcd}
X_{000} \arrow[r] \arrow[d] & X_{100} \arrow[d] \\
X_B \arrow[r] & X_A
\end{tikzcd}$$

is a cartesian square.

Theorem 7.4.5. *Let $X \in \mathbb{D}(K)$ be a cube such that the four squares $i_f^*(X)$, $i_b^*(X)$, $i_A^*(X)$ and $i_B^*(X)$ are cartesian. Then the square $i^*(X)$ is also cartesian.*

Proof. By Corollary 7.4.4, the diagram X lies in the essential image of $f_* \colon \lrcorner\!\square \to K$. Since $i(00) = 000$ does not lie in the image of f, by Proposition 7.4.1, it suffices to show that the functor $\bar{i} \colon \lrcorner \to (000 \downarrow K \setminus \{000\})$ admits a right adjoint to prove that the square $i^*(X)$ is cartesian. Note first that the category $000 \downarrow K \setminus \{000\}$ is the full subcategory $K \setminus \{000\}$ of K, i.e. the small category

$$\begin{tikzcd}
& & & 100 \arrow[drr] \arrow[ddr] \arrow[ddd] & & \\
& & 010 \arrow[rrr] \arrow[ddd] & & & 110 \arrow[ddd] \\
& B \arrow[ur] \arrow[rrr] \arrow[dl] & & & A \arrow[ur] \arrow[dl] & \\
001 \arrow[rrr] \arrow[drr] & & & 101 \arrow[drr] & & \\
& & 011 \arrow[rrr] & & & 111
\end{tikzcd}$$

and the functor $\bar{i} \colon \lrcorner \to K \setminus \{000\}$ is the functor sending $01 \mapsto B$, $00 \mapsto A$ and $10 \mapsto 100$. This functor admits a right adjoint $K \setminus \{000\} \to \lrcorner$ given by $B, 010, 001, 011 \mapsto 01$, $A, 110101, 111 \mapsto 00$ and $100 \mapsto 10$. This shows the result. $\square$

8 Limits and Colimits

As another application, we characterize the initial and final functors and the homotopy initial and homotopy final functors in terms of limits and colimits of derivators. In Section 8.1, the initial functors are defined as the functors $u\colon A \to B$ in Cat such that the comma category $u \downarrow b$ is non-empty and connected for every $b \in B$, but they are also often defined as the functors $u\colon A \to B$ in Cat which preserve limits under precomposition with respect to every complete category $\mathcal{C}$, i.e. there is an isomorphism

$$\lim_B F \cong \lim_A F \circ u$$

for every diagram $F\colon B \to \mathcal{C}$. More generally, we show in this section that the initial functors are the ones which preserve the limits under precomposition with respect to every derivator whose basic localizer contains the fundamental one.

Similarly, the homotopy initial functors are defined in Section 8.2 as the functors $u\colon A \to B$ in Cat such that the nerve of the comma category $u \downarrow b$ is homotopy equivalent to a point for every $b \in B$, but they are also often defined as the functors $u\colon A \to B$ in Cat which preserve homotopy limits under precomposition with respect to every model category $\mathcal{M}$, i.e. there is an isomorphism

$$\operatorname{holim}_B F \cong \operatorname{holim}_A F \circ u$$

for every diagram $F \in \operatorname{Ho}(\mathcal{M}^B)$. We show that they are actually the functors which preserve limits under precomposition with respect to every derivator.

Finally, in Section 8.3, we prove general results about limits and colimits in derivators, such as Fubini's theorem for limits and colimits and how to compute limits and colimits of shape the coproduct of two small categories with respect to every derivator.

8.1 Initial and Final Functors

The aim here is to show that initial (resp. final) functors in Cat are exactly the functors $u\colon A \to B$ in Cat that preserve limits (resp. colimits) under precomposition with respect to every derivator whose basic localizer contains the fundamental one. We first recall the definitions of initial and final functors.

Definition 8.1.1. Let $u\colon A \to B$ be a functor in Cat.

(i) The functor u is **initial** if for every $b \in B$ the comma category $u \downarrow b$ is non-empty and connected.

(ii) The functor u is **final** if for every $b \in B$ the comma category $b \downarrow u$ is non-empty and connected.

Remark 8.1.2. In other words, a functor in Cat is initial if it is $\mathcal{W}_0$-aspherical and it is final if it is $\mathcal{W}_0$-coaspherical.

The fact that the initial and final functors are exactly the ones which preserve limits or colimits under precomposition with respect to every derivator $\mathbb{D}$ such that $\mathcal{W}_0 \subseteq \mathcal{W}_{\mathbb{D}}$ follows from the fact that there exist derivators whose basic localizer is $\mathcal{W}_0$, for example the represented derivator of the categories Cat and Set.

Theorem 8.1.3. *Let $u\colon A \to B$ be a functor in* Cat*.*

(i) The functor u is initial if and only if, for every derivator $\mathbb{D}$ such that $\mathcal{W}_0 \subseteq \mathcal{W}_{\mathbb{D}}$, we have a natural isomorphism

$$\lim_B \cong \lim_A u^*.$$

(ii) The functor u is final if and only if, for every derivator $\mathbb{D}$ such that $\mathcal{W}_0 \subseteq \mathcal{W}_{\mathbb{D}}$, we have a natural isomorphism

$$\operatorname{colim}_A u^* \cong \operatorname{colim}_B .$$

Proof. (i) Let $u\colon A \to B$ be an initial functor in Cat, i.e. a $\mathcal{W}_0$-aspherical functor, and let $\mathbb{D}$ be a derivator such that $\mathcal{W}_0 \subseteq \mathcal{W}_{\mathbb{D}}$. Then the functor u is in particular $\mathbb{D}$-aspherical and Lemma 5.2.7 implies that we have a natural isomorphism $\lim_B \cong \lim_A u^*$.

Conversely, if a functor $u\colon A \to B$ in Cat is such that $\lim_B \cong \lim_A u^*$ for every derivator $\mathbb{D}$ such that $\mathcal{W}_0 \subseteq \mathcal{W}_{\mathbb{D}}$, it is in particular true for the represented derivator associated to a cocomplete and complete category that is not equivalent to a preorder category, e.g. the categories Set and Cat. Since the basic localizer associated to this derivator is $\mathcal{W}_0$ (see Proposition 5.1.9), it follows from Lemma 5.2.7 that the functor u is $\mathcal{W}_0$-aspherical, i.e. initial.
(ii) The proof is dual to (i). □

As a corollary of this theorem, we have the usual characterization of initial (resp. final) functors, i.e. for every cocomplete and complete category $\mathcal{C}$ and every diagram $F\colon B \to \mathcal{C}$, there is an isomorphism

$$\lim_B F \cong \lim_A F \circ u \quad (\text{resp. } \operatorname{colim}_A F \circ u \cong \operatorname{colim}_B F).$$

where $u\colon A \to B$ is an initial (resp. final) functor in Cat.

Corollary 8.1.4. *Let $u\colon A \to B$ be a functor in* Cat*.*

(i) The functor u is initial if and only if, for every cocomplete and complete category $\mathcal{C}$ and every diagram $F\colon B \to \mathcal{C}$, we have an isomorphism

$$\lim_B F \cong \lim_A F \circ u.$$

(ii) The functor u is final if and only if, for every cocomplete and complete category $\mathcal{C}$ and every diagram $F\colon B \to \mathcal{C}$, we have an isomorphism

$$\operatorname{colim}_A F \circ u \cong \operatorname{colim}_B F.$$

Proof. Since the basic localizer associated to the represented derivator of a cocomplete and complete category contains the fundamental basic localizer $\mathcal{W}_0$, by Theorem 4.1.23 and Proposition 5.1.9, and since there exists a represented derivator whose basic localizer is $\mathcal{W}_0$, the result follows immediately from Theorem 8.1.3. □

Remark 8.1.5. It is actually sufficient to suppose that the category $\mathcal{C}$ is just complete in (i) and that $\mathcal{C}$ is just cocomplete in (ii).

As an application of this theorem, we can compute the limit of diagrams of shape Δ with respect to every derivator whose basic localizer contains the fundamental one, where Δ denotes the category of non-empty ordinal numbers and order-preserving maps, i.e. the small category

$$\Delta = \quad \begin{tikzcd} {[0]} \arrow[r, shift left=2] \arrow[r, shift right=2] & {[1]} \arrow[l] \arrow[r, shift left=4] \arrow[r] \arrow[r, shift right=4] & {[2]} \arrow[l, shift left=2] \arrow[l, shift right=2] & \cdots \end{tikzcd}$$

Corollary 8.1.6. *Let $\mathbb{D}$ be a derivator such that $\mathcal{W}_0 \subseteq \mathcal{W}_{\mathbb{D}}$ and let Δ denote the category of non-empty ordinal numbers. For every diagram $F \in \mathbb{D}(\Delta)$,*

$$\lim_{\Delta} F = \mathrm{eq}(F_0 \overset{\delta_0}{\underset{\delta_1}{\rightrightarrows}} F_1).$$

Proof. Consider the category

$$I = \quad \begin{tikzcd} {[0]} \arrow[r, shift left, "0"] \arrow[r, shift right, "1"'] & {[1]} \end{tikzcd}$$

and the inclusion functor $i\colon I \to \Delta$. We show that the functor i is initial. Let $n \in \mathbb{N}$. We have to prove that the category $i \downarrow [n]$ is non-empty and connected. If $n = 0$, the category $i \downarrow [0]$ has two objects, namely $([0], [0] \to [0])$ and $([1], [1] \to [0])$, and these are clearly connected. If $n \geq 1$, the category $i \downarrow [n]$ has objects

$$([0], [0] \xrightarrow{k} [n]) \quad \text{and} \quad ([1], [1] \xrightarrow{(l,m)} [n])$$

where $0 \leq k \leq n$ and $0 \leq l \leq m \leq n$. Let us prove, for example, that the objects $([0], [0] \xrightarrow{k} [n])$ and $([0], [0] \xrightarrow{l} [n])$ are connected, where $0 \leq k \leq l \leq n$. The following commutative diagram actually gives a zig-zag of morphisms between these objects.

$$\begin{tikzcd} {[0]} \arrow[d, "0"'] \arrow[dr, "k"] & \\ {[1]} \arrow[r, "{(k,l)}"] & {[n]} \\ {[0]} \arrow[u, "1"] \arrow[ur, "l"'] & \end{tikzcd}$$

The fact that the other objects of $i \downarrow [n]$ are connected follows from this in a similar way. Hence $i\colon I \to \Delta$ is initial and, by Theorem 8.1.3, since $\mathcal{W}_0 \subseteq \mathcal{W}_{\mathbb{D}}$, we have that $\lim_{\Delta} \cong \lim_I i^*$. In other words, this means that $\lim_{\Delta} F = \mathrm{eq}(F_0 \rightrightarrows F_1)$, for every diagram $F \in \mathbb{D}(\Delta)$. $\square$

8.2 Homotopy Initial and Homotopy Final Functors

Similarly, we characterize the homotopy initial (resp. final) functors as the ones which preserve limits (resp. colimits) under precomposition with respect to every derivator. Here is a reminder of the definitions of homotopy initial and homotopy final functors.

Definition 8.2.1. Let $u\colon A \to B$ be a functor in Cat.

(i) The functor u is **homotopy initial** if for every $b \in B$ the nerve of the comma category $u \downarrow b$ is homotopy equivalent to a point.

(ii) The functor u is **homotopy final** if for every $b \in B$ the nerve of the comma category $b \downarrow u$ is homotopy equivalent to a point.

Remark 8.2.2. In other words, a functor in Cat is homotopy initial if it is $\mathcal{W}_\infty$-aspherical and it is homotopy final if it is $\mathcal{W}_\infty$-coaspherical.

As for initial and final functors, the fact that the homotopy initial and homotopy final functors are exactly the ones which preserve limits or colimits under precomposition with respect to every derivator follows from the fact that there exist derivators whose basic localizer is $\mathcal{W}_\infty$, for example the homotopy derivator of sSet.

Theorem 8.2.3. *Let $u\colon A \to B$ be a functor in* Cat*.*

(i) The functor u is homotopy initial if and only if, for every derivator $\mathbb{D}$, we have a natural isomorphism

$$\lim_B \cong \lim_A u^*.$$

(ii) The functor u is homotopy final if and only if, for every derivator $\mathbb{D}$, we have a natural isomorphism

$$\operatorname{colim}_A u^* \cong \operatorname{colim}_B .$$

Proof. (i) Let $u\colon A \to B$ be a homotopy initial functor in Cat, i.e. a $\mathcal{W}_\infty$-aspherical functor, and let $\mathbb{D}$ be a derivator. By Theorem 6.3.5, we have that $\mathcal{W}_\infty \subseteq \mathcal{W}_{\mathbb{D}}$ and hence the functor u is in particular $\mathbb{D}$-aspherical. It follows from Lemma 5.2.7 that we have a natural isomorphism $\lim_B \cong \lim_A u^*$.

Conversely, if a functor $u\colon A \to B$ in Cat is such that $\lim_B \cong \lim_A u^*$ for every derivator $\mathbb{D}$, it is in particular true for the homotopy derivator associated to the model category sSet equipped with the Quillen model structure. Since its basic localizer is $\mathcal{W}_\infty$ (see Theorem 6.4.6), it follows from Lemma 5.2.7 that the functor u is $\mathcal{W}_\infty$-aspherical, i.e. homotopy initial.
(ii) The proof is dual to (i). □

Remark 8.2.4. If a functor in Cat admits a right adjoint, it is homotopy initial, since it is $\mathcal{W}_\infty$-aspherical by Proposition 4.1.14. By Theorem 8.2.3, this shows that left adjoint functors preserve limits under precomposition with respect to every derivator. Dually, if a functor in Cat admits a left adjoint, it is homotopy final, since it is $\mathcal{W}_\infty$-coaspherical by Proposition 4.1.14. By Theorem 8.2.3, this shows that right adjoint functors preserve colimits under precomposition with respect to every derivator.

As a corollary of Theorem 8.2.3, we obtain a characterization of the homotopy initial (resp. final) functors in terms of homotopy limits (resp. colimits).

Corollary 8.2.5. *Let $u\colon A \to B$ be a functor in* Cat.

(i) The functor u is homotopy initial if and only if, for every combinatorial model category $\mathcal{M}$ and every $F \in \mathrm{Ho}(\mathcal{M}^B)$, we have an isomorphism

$$\operatorname{holim}_B F \cong \operatorname{holim}_A F \circ u.$$

(ii) The functor u is homotopy final if and only if, for every combinatorial model category $\mathcal{M}$ and every $F \in \mathrm{Ho}(\mathcal{M}^B)$, we have an isomorphism

$$\operatorname{hocolim}_A F \circ u \cong \operatorname{hocolim}_B F.$$

Proof. Since the basic localizer $\mathcal{W}_\infty$ is the minimal one, by Theorem 6.4.6, and since there exists a homotopy derivator whose basic localizer is $\mathcal{W}_\infty$, the result follows immediately from Theorem 8.2.3. □

8.3 General Results about Limits and Colimits

Finally, we show three general results about limits and colimits. The first one is Fubini's theorem for limits and colimits.

Theorem 8.3.1. *Let $\mathbb{D}$ be a derivator. For all small categories A and B, we have*

(i) $\lim_{A\times B} \cong \lim_A \lim_B \cong \lim_B \lim_A$*, and*

(ii) $\operatorname{colim}_{A\times B} \cong \operatorname{colim}_A \operatorname{colim}_B \cong \operatorname{colim}_B \operatorname{colim}_A$.

Proof. (i) Let $p\colon A \to \mathbb{1}$ and $q\colon B \to \mathbb{1}$. The following diagram commutes.

$$\begin{tikzcd} A \times B \arrow[r, "1\times q"] \arrow[d, "p\times 1"'] \arrow[dr] & A \arrow[d, "p"] \\ B \arrow[r, "q"'] & \mathbb{1} \end{tikzcd}$$

Since the derivator $\mathbb{D}$ is a 2-functor, the following diagram also commutes.

$$\begin{tikzcd} \mathbb{D}(A \times B) & \mathbb{D}(A) \arrow[l, "(1\times q)^*"'] \\ \mathbb{D}(B) \arrow[u, "(p\times 1)^*"] & \mathbb{D}(\mathbb{1}) \arrow[l, "\Delta_B"] \arrow[u, "\Delta_A"'] \arrow[ul, "\Delta_{A\times B}"'] \end{tikzcd}$$

In particular, by unicity of right adjoints,

$$\begin{aligned} \lim_{A\times B} &\cong \lim_A (1\times q)_* = \lim_A \lim_B \\ &\cong \lim_B (p\times 1)_* = \lim_B \lim_A . \end{aligned}$$

(ii) The proof is dual to (i). $\square$

Now we want to compute limits and colimits of shape the coproduct of two small categories. The solution is that the limit of shape the coproduct of two small categories is the product of the two limits of shape these categories and the colimit of shape the coproduct of two small categories is the coproduct of the two colimits of shape these categories. To show this, we first need a lemma.

Lemma 8.3.2. *Let $\mathbb{D}$ be a derivator and let $u\colon A\to B$ and $v\colon C\to D$ be two functors in* Cat*. The following diagram commutes*

$$\begin{tikzcd} \mathbb{D}(B \amalg D) \arrow[r, "(u\amalg v)^*"] \arrow[d, "\simeq"'] & \mathbb{D}(A\amalg C) \arrow[d, "\simeq"] \\ \mathbb{D}(B)\times\mathbb{D}(D) \arrow[r, "(u^*, v^*)"'] & \mathbb{D}(A)\times\mathbb{D}(C) \end{tikzcd}$$

where the vertical arrows are the canonical equivalences given by axiom **[D1]**.

Proof. Let $i_A\colon A\to A\amalg C$, $i_C\colon C\to A\amalg C$, $i_B\colon B\to B\amalg D$ and $i_D\colon D\to B\amalg D$ denote the inclusions. The canonical equivalences of axiom **[D1]** are given explicitly by $(i_A^*, i_C^*)\colon \mathbb{D}(A\amalg C)\to\mathbb{D}(A)\times\mathbb{D}(C)$ and $(i_B^*, i_D^*)\colon \mathbb{D}(B\amalg D)\to\mathbb{D}(B)\times\mathbb{D}(D)$. By definition of the coproduct of two functors, the following diagrams commute.

$$\begin{tikzcd} A \arrow[r, "u"] \arrow[d, "i_A"'] & B \arrow[d, "i_B"] \\ A\amalg C \arrow[r, "u\amalg v"'] & B\amalg D \end{tikzcd} \qquad \begin{tikzcd} C \arrow[r, "v"] \arrow[d, "i_C"'] & D \arrow[d, "i_D"] \\ A\amalg C \arrow[r, "u\amalg v"'] & B\amalg D \end{tikzcd}$$

This gives us the following equality of commutative diagrams.

$$\begin{tikzcd} & \mathbb{D}(B\amalg D) \arrow[d, "(u\amalg v)^*"] & \\ & \mathbb{D}(A\amalg C) \arrow[dl, "i_A^*"'] \arrow[d, "(i_A^*, i_C^*)"] \arrow[dr, "i_C^*"] & \\ \mathbb{D}(A) & \mathbb{D}(A)\times\mathbb{D}(C) \arrow[l] \arrow[r] & \mathbb{D}(C) \end{tikzcd} = \begin{tikzcd} & \mathbb{D}(B\amalg D) \arrow[dl, "i_B^*"'] \arrow[d, "(i_B^*, i_D^*)"] \arrow[dr, "i_D^*"] & \\ \mathbb{D}(B) \arrow[d, "u^*"'] & \mathbb{D}(B)\times\mathbb{D}(D) \arrow[l] \arrow[r] \arrow[d, "(u^*, v^*)"] & \mathbb{D}(D) \arrow[d, "v^*"] \\ \mathbb{D}(A) & \mathbb{D}(A)\times\mathbb{D}(C) \arrow[l] \arrow[r] & \mathbb{D}(C) \end{tikzcd}$$

By the universal property of products, this implies that the diagram

$$\begin{tikzcd}
\mathbb{D}(B \amalg D) \arrow[r, "(u \amalg v)^*"] \arrow[d, "(i_B^*, i_D^*)"'] & \mathbb{D}(A \amalg C) \arrow[d, "(i_A^*, i_C^*)"] \\
\mathbb{D}(B) \times \mathbb{D}(D) \arrow[r, "(u^*, v^*)"'] & \mathbb{D}(A) \times \mathbb{D}(C)
\end{tikzcd}$$

commutes. □

Here are the definitions of the product of two limit functors and the coproduct of two colimit functors.

Definition 8.3.3. Let $\mathbb{D}$ be a derivator. If A and B are small categories, we can define

(i) the functor $\lim_A \times \lim_B : \mathbb{D}(A \amalg B) \to \mathbb{D}(\mathbb{1})$ as the composite

$$\mathbb{D}(A \amalg B) \xrightarrow{\simeq} \mathbb{D}(A) \times \mathbb{D}(B) \xrightarrow{(\lim_A, \lim_B)} \mathbb{D}(\mathbb{1}) \times \mathbb{D}(\mathbb{1}) \xrightarrow{\simeq} \mathbb{D}(\mathbb{1} \amalg \mathbb{1}) \xrightarrow{\times} \mathbb{D}(\mathbb{1}),$$

where the equivalences are given by axiom **[D1]** and $\times = \lim_{\mathbb{1} \amalg \mathbb{1}}$, and

(ii) the functor $\operatorname{colim}_A \amalg \operatorname{colim}_B : \mathbb{D}(A \amalg B) \to \mathbb{D}(\mathbb{1})$ as the composite

$$\mathbb{D}(A \amalg B) \xrightarrow{\simeq} \mathbb{D}(A) \times \mathbb{D}(B) \xrightarrow{(\operatorname{colim}_A, \operatorname{colim}_B)} \mathbb{D}(\mathbb{1}) \times \mathbb{D}(\mathbb{1}) \xrightarrow{\simeq} \mathbb{D}(\mathbb{1} \amalg \mathbb{1}) \xrightarrow{\amalg} \mathbb{D}(\mathbb{1}),$$

where the equivalences are given by axiom **[D1]** and $\amalg = \operatorname{colim}_{\mathbb{1} \amalg \mathbb{1}}$.

Finally, we show that these functors are naturally isomorphic to the limit and colimit functors of shape the coproduct of the two small categories respectively.

Theorem 8.3.4. *Let $\mathbb{D}$ be a derivator. For all small categories A and B, we have*

(i) $\lim_{A \amalg B} \cong \lim_A \times \lim_B$, *and*

(ii) $\operatorname{colim}_{A \amalg B} \cong \operatorname{colim}_A \amalg \operatorname{colim}_B$.

Proof. (i) Let $p : A \to \mathbb{1}$ and $q : B \to \mathbb{1}$. Then, by Lemma 8.3.2, the following diagram commutes.

$$\begin{tikzcd}
\mathbb{D}(\mathbb{1} \amalg \mathbb{1}) \arrow[r, "(p \amalg q)^*"] \arrow[d, "\simeq"'] & \mathbb{D}(A \amalg B) \arrow[d, "\simeq"] \\
\mathbb{D}(\mathbb{1}) \times \mathbb{D}(\mathbb{1}) \arrow[r, "(\Delta_A, \Delta_B) = (p^*, q^*)"'] & \mathbb{D}(A) \times \mathbb{D}(B)
\end{tikzcd}$$

By taking the right adjoint of three of the functors in the diagram, we obtain a diagram

$$
\begin{tikzcd}
\mathbb{D}(\mathbb{1} \amalg \mathbb{1}) \arrow[dr, phantom, "\cong"] & \mathbb{D}(A \amalg B) \arrow[l, "(p \amalg q)_*"'] \arrow[d, "\simeq"] \\
\mathbb{D}(\mathbb{1}) \times \mathbb{D}(\mathbb{1}) \arrow[u, "\simeq"] & \mathbb{D}(A) \times \mathbb{D}(B) \arrow[l, "(\lim_A, \lim_B)"]
\end{tikzcd}
$$

that commutes up to a natural isomorphism. Moreover, since the following left-hand diagram commutes,

$$
\begin{tikzcd}
A \amalg B \arrow[rr, "p \amalg q"] \arrow[dr] & & \mathbb{1} \amalg \mathbb{1} \arrow[dl] \\
& \mathbb{1} &
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{D}(A \amalg B) & & \mathbb{D}(\mathbb{1} \amalg \mathbb{1}) \arrow[ll, "(p \amalg q)^*"'] \\
& \mathbb{D}(\mathbb{1}) \arrow[ul] \arrow[ur] &
\end{tikzcd}
$$

the right-hand diagram also commutes, and this gives us a natural isomorphism

$$
\begin{tikzcd}
& \mathbb{D}(\mathbb{1}) \arrow[d, phantom, "\cong"] & \\
\mathbb{D}(\mathbb{1} \amalg \mathbb{1}) \arrow[ur, "\times"] \arrow[drr, phantom, "\cong"] & & \mathbb{D}(A \amalg B) \arrow[ll, "(p \amalg q)_*"'] \arrow[ul, "\lim_{A \amalg B}"'] \arrow[d, "\simeq"] \\
\mathbb{D}(\mathbb{1}) \times \mathbb{D}(\mathbb{1}) \arrow[u, "\simeq"] & & \mathbb{D}(A) \times \mathbb{D}(B) \arrow[ll, "(\lim_A, \lim_B)"]
\end{tikzcd}
$$

By definition of $\lim_A \times \lim_B$, this means that $\lim_{A \amalg B} \cong \lim_A \times \lim_B$.
(ii) The proof is dual to (i). $\square$

Let A be a small category and let $[1]$ denote the small category $0 \to 1$. Consider the two functors $i_0 \colon A \to A \times [1], a \mapsto (a, 0)$ and $i_1 \colon A \to A \times [1], a \mapsto (a, 1)$. Then there exists a **cone category** $A^{\lhd}$ of A and a **cocone category** $A^{\rhd}$ of A that are constructed as the pushouts of the following squares.

$$
\begin{tikzcd}
A \arrow[r] \arrow[d, "i_0"'] & \mathbb{1} \arrow[d, "\varnothing"] \\
A \times [1] \arrow[r] & A^{\lhd}
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[r] \arrow[d, "i_1"'] & \mathbb{1} \arrow[d, "*"] \\
A \times [1] \arrow[r] & A^{\rhd}
\end{tikzcd}
$$

The cone category $A^{\lhd}$ has the same objects and morphisms as A plus an initial object $\varnothing$, and the cocone category $A^{\rhd}$ has the same objects and morphisms as A plus a terminal object $*$. The last result says that the right Kan extension of the inclusion functor $i \colon A \to A^{\lhd}$ sends a diagram of shape A to its limit cone, and dually for colimits.

Theorem 8.3.5. *Let $\mathbb{D}$ be a derivator and let A be a small category.*

(i) If $i\colon A \to A^{\triangleleft}$ is the inclusion functor, for every diagram $X \in \mathbb{D}(A)$, we have

$$\lim_A X \cong i_*(X)_{\varnothing},$$

where $\varnothing$ denotes the initial object of $A^{\triangleleft}$.

(ii) If $j\colon A \to A^{\triangleright}$ is the inclusion functor, for every diagram $X \in \mathbb{D}(A)$, we have

$$\operatorname{colim}_A X \cong j_!(X)_*,$$

where $*$ denotes the terminal object of $A^{\triangleright}$.

Proof. (i) Note first that the square

$$\begin{tikzcd}
A \arrow[r] \arrow[d, equal] \arrow[dr, phantom, "\swarrow"] & \mathbb{1} \arrow[d, "\varnothing"] \\
A \arrow[r, "i"'] & A^{\triangleleft}
\end{tikzcd}$$

is a comma square. Let $X \in \mathbb{D}(A)$ be a diagram. Since the functor i is fully faithful and $A^{\triangleleft} \setminus i(A) = \{\varnothing\}$, by Proposition 7.1.10, the following diagram is right $\mathbb{D}$-Beck-Chevalley at $i_*(X)$.

$$\begin{tikzcd}
A \arrow[r] \arrow[d, equal] \arrow[dr, phantom, "\swarrow"] & \mathbb{1} \arrow[d, "\varnothing"] \\
A \arrow[r, "i"] \arrow[d, "i"'] \arrow[dr, phantom, "\overset{=}{\swarrow}"] & A^{\triangleleft} \arrow[d, equal] \\
A^{\triangleleft} \arrow[r, equal] & A^{\triangleleft}
\end{tikzcd}$$

By computing the right mate, we obtain $i_*(X)_{\varnothing} \cong \lim i^* i_*(X)$. Moreover, since the functor i is fully faithful, the functor i_* is fully faithful by Corollary 7.1.6 and $i^* i_*(X) \cong X$ by Lemma 3.2.9. Finally, we obtain $i_*(X)_{\varnothing} \cong \lim_A X$.
(ii) The proof is dual to (i). $\square$

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