HOMOTOPICAL RELATIONS BETWEEN 2-DIMENSIONAL CATEGORIES AND THEIR ∞-ANALOGUES

LYNE MOSER

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Supervised by Jérôme Scherer and Kathryn Hess

ABSTRACT

In this thesis, we study the homotopical relations of 2-categories, double categories, and their ∞-analogues. For this, we construct homotopy theories for the objects of interest, and show that there are homotopically full embeddings of 2-categories into $(\infty, 2)$-categories, and of double categories into double $(\infty, 1)$-categories, which are compatible with the inclusions of 2-categories and $(\infty, 2)$-categories into their double categorical analogues.

In the strict setting, we first present two model structures on the category of double categories and double functors, constructed in papers by the author, Sarazola, and Verdugo. Unlike previously defined model structures for double categories, they recover Lack's model structure for 2-categories. More precisely, the horizontal embedding functor from 2-categories to double categories is homotopically well-behaved, and embeds the homotopy theory of 2-categories into that of double categories in a reflective way. While, in the first model structure, all double categories are fibrant, the fibrant objects of the second model structure are the *weakly horizontally invariant* double categories. We show that both model structures are enriched over 2-categories, and that the model structure for weakly horizontally invariant double categories is further monoidal with respect to the Gray tensor product for double categories.

Going to the ∞-world, we then consider ∞-versions of these 2-dimensional categories. Double $(\infty, 1)$-categories are defined as double Segal objects in spaces which are complete in the horizontal direction, and hence include $(\infty, 2)$-categories in the form of Barwick's 2-fold complete Segal spaces. We then construct a nerve from double categories to double $(\infty, 1)$-categories, and show that it embeds the homotopy theory for weakly horizontally invariant double categories into that of double $(\infty, 1)$-categories in a reflective way. Finally, by restricting the nerve along the horizontal embedding, we obtain a nerve from 2-categories to 2-fold complete Segal spaces, which again embeds the homotopy theory of 2-categories into that of $(\infty, 2)$-categories in a reflective way.

Keywords: 2-categories, double categories, $(\infty, 2)$-categories, double $(\infty, 1)$-categories, nerve construction, homotopy theory.

RÉSUMÉ

Dans cette thèse, on étudie les relations homotopiques des 2-catégories, doubles catégories, et leurs ∞-analogues. Pour cela, on munit les catégories d'objets d'intérêt de structures homotopiques, et on montre qu'il y a des plongements homotopiques pleins des 2-catégories dans les $(\infty, 2)$-catégories, et des doubles catégories dans les doubles $(\infty, 1)$-catégories, qui sont compatibles avec les inclusions des 2-catégories et $(\infty, 2)$-catégories dans leurs analogues doubles catégoriques.

Dans le cadre strict, on présente tout d'abord deux structures de modèles sur la catégorie des doubles catégories et doubles foncteurs, construites dans des articles par l'auteur, Sarazola, et Verdugo. Contrairement aux structures de modèles existantes sur les doubles catégories, ces nouvelles structures de modèles recouvrent la structure de modèles de Lack sur les 2-catégories. Plus précisément, le plongement horizontal des 2-catégories dans les doubles catégories a un bon comportement homotopique, et donne un plongement réflexif de la théorie d'homotopie des 2-catégories dans celle des doubles catégories. Tandis que, dans la première structure de modèles, toutes les doubles catégories sont fibrantes, les objets fibrants de la seconde structure de modèles sont les doubles catégories *faiblement et horizontalement invariantes*. On montre que les deux structures de modèles sont enrichies sur les 2-catégories et que la structure de modèles pour les doubles catégories faiblement et horizontalement invariantes est de plus monoïdale par rapport au produit tensoriel de Gray pour les doubles catégories.

En passant dans le cadre "infini", on considère ensuite des ∞-versions de ces catégories 2-dimensionnelles. Les doubles $(\infty, 1)$-catégories sont définies comme étant des doubles objets de Segal dans les espaces qui sont complets dans la direction horizontale, et contiennent ainsi les $(\infty, 2)$-catégories sous la forme d'espaces de Segal complets itérés introduits par Barwick. On construit ensuite un nerf depuis les doubles catégories vers les doubles $(\infty, 1)$-catégories et on montre qu'il donne un plongement réflexif de la théorie d'homotopie des doubles catégories faiblement et horizontalement invariantes dans celle des doubles $(\infty, 1)$-catégories. Finalement, en restreignant le nerf le long du plongement horizontal, on obtient un nerf depuis les 2-catégories vers les espaces de Segal complets itérés, qui à nouveau donne un plongement réflexif de la théorie d'homotopie des 2-catégories dans celle des $(\infty, 2)$-catégories.

Mots-clés: 2-catégories, doubles catégories, $(\infty, 2)$-catégories, doubles $(\infty, 1)$-catégories, nerf, théorie d'homotopie.

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INTRODUCTION

We give here a general introduction about the main results appearing in this thesis. We refer the reader to the beginning of each part for more specific introductions to the content of each section.

The setting

Higher category theory aims to study more structured objects than categories. While a category consists of objects and morphisms between objects which compose associatively, a higher category can be obtained by adding higher morphisms and by weakening the associativity constraint. For example, a 2-category is obtained by also adding morphisms between morphisms, called 2-*morphisms*, without changing the strictness of the associativity constraint of compositions. In particular, 2-categories are categories enriched in categories, in the sense that their morphisms and 2-morphisms form a category of morphisms between every pair of objects, rather than a set. Categories and 2-categories are often too strict to accommodate many examples that appear in nature. In the perspective of generalizing categories, an $(\infty,1)$-category is interpreted as a categorical structure that admits morphisms in all dimensions, such that all k-morphisms are invertible for $k>1$, where compositions are only defined, and associative up to higher invertible morphisms. Such a higher structure should be thought of as a homotopical version of a category. Similarly, we obtain the notion of an $(\infty,2)$-category by instead requiring that the k-morphisms are invertible for $k>2$, and this should again be interpreted as a weaker version of a 2-category. In particular, categories and 2-categories should give rise to $(\infty,1)$-categories and $(\infty,2)$-categories whose higher invertible k-morphisms are all trivial.

While the strict versions can be defined by fully describing their structure, to make sense of their ∞-analogues, we need models. The machinery used here is often that of *model categories*, which provide a good environment to do homotopy theory. Then, the $(\infty,1)$-categories and $(\infty,2)$-categories are defined as the fibrant objects of a given model structure. In particular, model categories were introduced by Quillen in [Qui67] to axiomatize the homotopy theory of spaces, which are equivalent to $(\infty,0)$-categories – also called ∞-*groupoids* – by Grothendieck's homotopy hypothesis. Hence spaces give a model of ∞-groupoids, and furthermore, the usual nerve from categories to simplicial sets restricts to a homotopically full embedding of groupoids into ∞-groupoids, in the form of Kan complexes.

Several models of $(\infty,1)$-categories have been developed: among others, there are quasi-categories, originally defined by Boardman and Vogt [BV73], and further developed by Joyal [Joy02] and Lurie [Lur09], simplicial categories, defined by Quillen [Qui67] and developed as a model for $(\infty,1)$-categories by Bergner [Ber07], and complete Segal spaces, due to Rezk [Rez01]. Furthermore, all the models of $(\infty,1)$-categories have been shown to be equivalent, and $(\infty,1)$-category theory has been developed in a model-independent way by Riehl and Verity [RV19]. Similarly to the case of ∞-groupoids, a category can be interpreted via nerve constructions as an $(\infty,1)$-category. In the model of quasi-categories, this construction is given by the usual nerve from categories to simplicial sets, while there are adapted versions for the models of simplicial categories and complete Segal spaces.

Going one dimension up, different models of $(\infty,2)$-categories generalizing the models of $(\infty,1)$-categories mentioned above have also been introduced. As a generalization of quasi-categories, we have the 2-quasi-categories, due to Ara [Ara14], while simplicial categories can be generalized to categories enriched in quasi-categories or complete Segal spaces, for example. These latter models again interpret $(\infty, 2)$-categories as categories enriched in $(\infty, 1)$-categories, which gives the ∞-analogue of the fact that a 2-category is an enriched category in categories. Finally, the model of complete Segal spaces admits two generalizations: one given by Rezk's Θ_2-spaces [Rez10], and the other by Barwick's 2-fold complete Segal spaces [Bar05]. There are also other models of $(\infty, 2)$-categories that we did not mention here, and all these models of $(\infty, 2)$-categories have also been shown to be equivalent. Hence this gives several nice contexts in which one could study the theory of $(\infty, 2)$-categories. Moreover, several nerves that fully embed the homotopy theory of 2-categories into that of $(\infty, 2)$-categories have already been constructed in different models: into saturated 2-precomplicial sets by Ozornova and Rovelli in [OR19], into 2-quasi-categories by Campbell in [Cam20], and into ∞-bicategories by Gagna, Harpaz, and Lanari in [GHL19].

Another type of 2-dimensional category, which are closely related to 2-categories but are obtained by different methods, is that of a double category. While 2-categories are extending the notion of a category by adding 2-morphisms between the morphisms and hence can be seen as a globular version of a 2-dimensional category, double categories have two directions and hence give a cubical version of a 2-dimensional category. More precisely, a double category has two kinds of morphisms between the objects – called *horizontal* and *vertical morphisms* – and its 2-morphisms are *squares*. In particular, there is a *horizontal embedding* of 2-categories into double categories, which interprets a 2-category as a horizontal double category with only trivial vertical morphisms. As in the 2-categorical case, there is an ∞-analogue of a double category modeled by Segal objects in complete Segal spaces, due to Haugseng [Hau13].

In this double categorical context, the strict and ∞-versions have not yet been compared. A natural expectation is that there is an ∞-version of the horizontal embedding, which embeds $(\infty, 2)$-categories into double $(\infty, 1)$-categories, and that 2-categories and double categories embed into their more homotopical versions, in such a way that the following diagram commutes.

$$\begin{tikzcd}
\{\text{2-categories}\} \arrow[r, hook] \arrow[d, hook] & \{(\infty,2)\text{-categories}\} \arrow[d, hook] \\
\{\text{double categories}\} \arrow[r, hook] & \{\text{double } (\infty,1)\text{-categories}\}
\end{tikzcd}$$

The aim of this thesis is to make sense of this picture by constructing homotopy theories for double categories, as well as a comparison nerve functor from double categories to double $(\infty, 1)$-categories which is homotopically well-behaved, and hence shows that the double categorical ∞-setting extends the strict one. Furthermore, since $(\infty, 2)$-categories in the form of 2-fold complete Segal spaces are in particular double $(\infty, 1)$-categories, we interpret this inclusion as the ∞-analogue of the horizontal embedding. We then show that the double categorical nerve functor restricts along the horizontal embedding of 2-categories into double categories, and provide a new nerve construction for 2-categories into the model of 2-fold complete Segal spaces.

Why double categories?

The theory of $(\infty, 1)$-categories has been the most studied until now. Since $(\infty, 1)$-categories should be thought of as a "weak" version of categories, it has been shown that all the constructions and theorems of category theory are supported in an $(\infty, 1)$-category, but in a weaker way, i.e., everything should not work strictly anymore but up to coherent homotopy. For example, the notion of limit has been constructed in each of the models of $(\infty, 1)$-categories presented above, and they correspond to each other when passing from a model to another. A well-known result in category theory characterizes a limit of a functor as a terminal object in the category of cones over this functor. Using this characterization, limits in an $(\infty, 1)$-categories have then been constructed as terminal objects in the corresponding $(\infty, 1)$-category of cones (see [Joy02, Lur09, Ras18, RV19, RV15, Rov19]).

Going one dimension up, we would also like to develop a useable theory of $(\infty, 2)$-categories, and in particular a notion of limit in an $(\infty, 2)$-category. For this, we first want to see if, in the strict setting, limits in a 2-category admit a similar characterization to the one of a limit in a (1-)category in terms of terminal objects. Since 2-categories are categories enriched in categories, a more accurate notion of limit in a 2-category is that of a 2-*limit*, first introduced by Auderset [Aud74] and Borceux-Kelly [BK75], and further developed by Street [Str76], Kelly [Kel82, Kel89] and Lack [Lac10]. A 2-limit is an enriched version of a limit, in the sense that its universal property is now expressed in terms of an isomorphism between hom-categories, rather than hom-sets. As an example, the category of algebras over a monad, also called *Eilenberg-Moore category*, can be obtained as a special kind of 2-limit (called *lax limit*). Hence having such a notion of limit in the $(\infty, 2)$-categorical context would allow us to study algebraic objects in a more homotopical way. However, as clingman and the author show in [cM20], a 2-limit cannot be characterized as a 2-terminal object in the corresponding 2-category of cones, and we might therefore not be able to adapt the construction of limits in the $(\infty, 1)$-categorical setting given above to the $(\infty, 2)$-categorical one.

This is where it becomes interesting to work with double categories. Indeed, by looking at 2-categories as horizontally embedded in double categories, results by Grandis and Paré [Gra20, GP99] show that a 2-limit of a 2-functor is equivalently a double terminal object in the double category of cones over the corresponding horizontal double functor. Hence, a passage from the 2-categorical setting to the double categorical one allows us to get the desired characterization of 2-limits in terms of some kind of terminal objects, which we might be able to import to the ∞-world.

While this issue arises in the strict context, we would like to see if it is still happening in the more homotopical setting, which is sometimes better behaved. For this, we aim to understand if the strict setting extends to the ∞-setting, by showing that 2-categories and double categories are special instances of $(\infty, 2)$-categories and double $(\infty, 1)$-categories in such a way that these inclusions are compatible with the horizontal embeddings. As mentioned above, this is the matter addressed in this thesis. Hence, the results proved here imply that one could not define a limit in an $(\infty, 2)$-category as a terminal object in the corresponding $(\infty, 2)$-category of cones, but one would rather need to pass to the double $(\infty, 1)$-categorical setting and define a limit in an $(\infty, 2)$-category as a terminal object in the corresponding double $(\infty, 1)$-category of cones. Another approach to limits in an $(\infty, 2)$-category has been taken by Gagna, Harpaz, and Lanari in [GHL20].

Homotopy theory of double categories

A first step towards showing that 2-dimensional categories are embedded in their ∞-analogues is to construct homotopy theories for 2-categories and double categories. As mentioned above, the language of model categories provides a good framework to do homotopy theory. A model structure is defined to be the data of three classes of morphisms in a category – called *weak equivalences*, *cofibrations*, *fibrations* – satisfying some axioms (see Definition 4.1.7), where the weak equivalences yield a weaker notion of invertibility between the objects of the ambient category. In particular, a fibration which is also a weak equivalence is called a *trivial fibration*, and a model structure is fully determined by the data of its classes of weak equivalences and trivial fibrations, as well as by its classes of trivial fibrations and fibrant objects, i.e., the objects such that the unique morphism to the terminal object is a fibration. We will therefore only specify such classes to give the data of the model structures considered in this introduction.

Going back to our matter, we aim to endow the categories 2Cat of 2-categories and 2-functors and DblCat of double categories and double functors with model structures. In the case of 2-categories, Lack constructs in [Lac02, Lac04] a model structure on 2Cat whose weak equivalences are the *biequivalences*, i.e., the 2-functors which admit a pseudo-inverse up to pseudo-natural equivalence. Note that all 2-categories are fibrant in this model structure. As for double categories, Fiore, Paoli, and Pronk construct several model structures on DblCat in [FP10, FPP08]. However, their model structures on DblCat do not recover the Lack's model structure on 2-categories through the horizontal embedding.

Therefore, in [MSV20a], the author, Sarazola, and Verdugo construct a new model structure on DblCat, compatible with the horizontal embedding of 2-categories into double categories. In particular, it is obtained as an induced model structure from two copies of Lack's model structure, and its weak equivalences, called suggestively *double biequivalences*, give a double categorical analogue of biequivalences for 2-categories. Under some assumption, they can indeed be characterized as the double functors which have a pseudo-inverse up to horizontal pseudo-natural equivalence. The following result summarizes the main features of this model structure and is a compilation of Theorem 7.1.3 and Propositions 7.2.5 and 7.2.10.

Theorem A. *There is a model structure on* DblCat*, in which the weak equivalences are the double biequivalences, and the trivial fibrations are the double functors which are surjective on objects, full on horizontal morphisms, surjective on vertical morphisms, and fully faithful on squares. Moreover, all double categories are fibrant.*

By construction this model structure has the correct homotopical behavior with respect to the horizontal embedding of 2-categories into double categories, which we denote by $\mathbb{H}\colon 2\mathrm{Cat} \to \mathrm{DblCat}$. Indeed, this functor is both left and right Quillen, and homotopically fully faithful from Lack's model structure to the model structure for double categories defined above. Furthermore, Lack's model structure is also both left- and right-induced along $\mathbb{H}$ from the model structure on DblCat, and this shows that the homotopy theory of 2-categories is completely determined by this homotopy theory of double categories through the horizontal embedding. This is the content of the next theorem, which appears as Theorems 7.4.1, 7.4.4, 7.4.6 and 7.4.7 in this thesis.

Theorem B. *The horizontal embedding functor* $\mathbb{H}\colon 2\mathrm{Cat} \to \mathrm{DblCat}$ *is both left and right Quillen, and homotopically fully faithful, where* 2Cat *is endowed with Lack's model structure and* DblCat *is endowed with the model structure of Theorem A. Moreover, Lack's model structure is both left- and right-induced along* $\mathbb{H}$ *from the model structure on* DblCat *of Theorem A.*

While this model structure is as compatible as possible with the horizontal embedding, it is unsurprisingly not well-behaved with respect to the vertical direction. For instance, as described in Theorem A, the trivial fibrations in this model structure are not symmetric between the horizontal and vertical directions. Indeed, while they are full on horizontal morphisms, they are only surjective on vertical morphisms. This will be an issue when trying to apply the nerve functor from double categories to double $(\infty, 1)$-categories. Indeed, it will not have the correct homotopical behavior with respect to this model structure, as for instance the model structure for double $(\infty, 1)$-category does not carry this asymmetry. Furthermore, as we will see below, the double categories whose nerve is fibrant, i.e., is a double $(\infty, 1)$-category, are precisely the *weakly horizontally invariant* ones, where a weakly horizontally invariant double category, first introduced in [MSV20b], is defined as a double category in which the vertical morphisms can be transferred along horizontal equivalences. Hence, to remedy these shortcomings, we construct in [MSV20b] another model structure on DblCat, whose trivial fibrations behave symmetrically between the horizontal and vertical directions, and whose fibrant double categories are the weakly horizontally invariant ones. This results appears as Proposition 8.1.2 and Theorem 8.1.15.

Theorem C. *There is a model structure on* DblCat*, in which the trivial fibrations are the double functors which are surjective on objects, full on horizontal and vertical morphisms, and fully faithful on squares, and the fibrant objects are the weakly horizontally invariant double categories.*

While we could not get an explicit description of the weak equivalences in general, we show in Proposition 8.1.18 that double biequivalences are in particular weak equivalences in this second model structure, and in Proposition 8.3.4 that weak equivalences between weakly horizontally invariant double categories are precisely the double biequivalences. As a consequence, the weak equivalences between fibrant objects are precisely the double functors which have a pseudo-inverse up to horizontal pseudo-natural equivalence. Furthermore, as described in the next result, the homotopy theory of weakly horizontally invariant double categories is contained in that of double categories. This appears as Theorem 8.4.1.

Theorem D. *The identity functor* $\mathrm{id}\colon \mathrm{DblCat} \to \mathrm{DblCat}$ *is right Quillen, and homotopically fully faithful, from the model structure on* DblCat *of Theorem C for weakly horizontally invariant double categories to the model structure on* DblCat *of Theorem A.*

While the horizontal embedding $\mathbb{H}\colon 2\mathrm{Cat} \to \mathrm{DblCat}$ remains left Quillen and homotopically fully faithful from Lack's model structure to this new model structure, it is not right Quillen anymore. Indeed, the horizontal double category $\mathbb{H}\mathcal{A}$ associated to a 2-category $\mathcal{A}$ is typically not weakly horizontally invariant; see Remark 8.4.5. We consider instead a more homotopical version of $\mathbb{H}$ given by the functor $\mathbb{H}^{\simeq}\colon 2\mathrm{Cat} \to \mathrm{DblCat}$, which sends a 2-category $\mathcal{A}$ to the double category $\mathbb{H}^{\simeq}\mathcal{A}$ which is defined as $\mathbb{H}\mathcal{A}$ on horizontal data and whose vertical morphisms are now given by the adjoint equivalences in $\mathcal{A}$. In particular, the inclusion $\mathbb{H}\mathcal{A} \to \mathbb{H}^{\simeq}\mathcal{A}$ is a weak equivalence, which exhibits $\mathbb{H}^{\simeq}\mathcal{A}$ as a fibrant replacement of $\mathbb{H}\mathcal{A}$ in the model structure for weakly horizontally invariant double categories. These results are summarized in the following theorem, which is a compilation of Theorems 8.4.7, 8.4.9 and 8.4.11.

Theorem E. *The homotopical horizontal embedding* $\mathbb{H}^{\simeq}\colon 2\mathrm{Cat} \to \mathrm{DblCat}$ *is right Quillen, and homotopically fully faithful, where* 2Cat *is endowed with Lack's model structure and* DblCat *is endowed with the model structure of Theorem C for weakly horizontally invariant double categories. Lack's model structure is further right-induced along* $\mathbb{H}^{\simeq}$ *from the model structure on* DblCat *of Theorem C.*

Moreover, the double category $\mathbb{H}^{\simeq}\mathcal{A}$ *provides a fibrant replacement of* $\mathbb{H}\mathcal{A}$ *in the model structure on* DblCat *of Theorem C.*

Relations between 2-dimensional categories and their ∞-analogues

As mentioned above, the models of double $(\infty,1)$-categories and $(\infty,2)$-categories we want to consider are given by Segal objects in complete Segal spaces and 2-fold complete Segal spaces. In this thesis, since we want to see $(\infty,2)$-categories as horizontally embedded in double $(\infty,1)$-categories, we actually consider a transposed notion of Haugseng's Segal objects in complete Segal spaces mentioned above, where the completeness condition is in the horizontal direction. We refer to them as *horizontally complete double* $(\infty,1)$*-category.* In particular, the horizontally complete double $(\infty,1)$-category and 2-fold complete Segal spaces are given by the fibrant objects in two different model structures on bisimplicial spaces, i.e., bisimplicial objects in the category sSet of simplicial sets and simplicial maps. Moreover, the model structure on $\mathrm{sSet}^{\Delta^{\mathrm{op}}\times\Delta^{\mathrm{op}}}$ for 2-fold complete Segal spaces can be obtained as a localization of the model structure on $\mathrm{sSet}^{\Delta^{\mathrm{op}}\times\Delta^{\mathrm{op}}}$ for horizontally complete double $(\infty,1)$-categories, and hence the identity on $\mathrm{sSet}^{\Delta^{\mathrm{op}}\times\Delta^{\mathrm{op}}}$ gives a homotopically full embedding of $(\infty,2)$-categories into double $(\infty,1)$-categories, which we interpret as the ∞-version of the horizontal embedding.

In order to compare these ∞-versions of 2-dimensional categories with their stricter versions, we construct a nerve functor $\mathbb{N}\colon \mathrm{DblCat}\to\mathrm{sSet}^{\Delta^{\mathrm{op}}\times\Delta^{\mathrm{op}}}$ from double categories into bisimplicial spaces. By studying the fibrancy of the nerves in the model structure for horizontally complete double $(\infty,1)$-categories, we can see that the nerve of a double category is fibrant if and only if the double category we started with is weakly horizontally invariant. This forces us to consider the model structure on DblCat for weakly horizontally invariant double categories for the nerve to be right Quillen. As summarized below, we show that the nerve has the right homotopical properties, embedding double categories into their ∞-analogues. This appears as Theorems 11.3.1, 11.2.7 and 11.4.8.

Theorem F. *The nerve functor* $\mathbb{N}\colon \mathrm{DblCat}\to\mathrm{DblCat}^h_\infty$ *is right Quillen, and homotopically fully faithful, where* DblCat *is endowed with the model structure of Theorem C for weakly horizontally invariant double categories, and* DblCat^h_∞ *denotes the model structure on* $\mathrm{sSet}^{\Delta^{\mathrm{op}}\times\Delta^{\mathrm{op}}}$ *for horizontally complete double* $(\infty,1)$*-categories. Moreover, the nerve of a double category* $\mathbb{A}$ *is a horizontally complete double* $(\infty,1)$*-category if and only if the double category* $\mathbb{A}$ *is weakly horizontally invariant.*

Now, recall that the horizontal double category $\mathbb{H}\mathcal{A}$ associated to a 2-category $\mathcal{A}$ is in general not weakly horizontally invariant, and hence, by the above result, the nerve $\mathbb{N}\mathbb{H}\mathcal{A}$ does not give a double $(\infty,1)$-category nor a 2-fold complete Segal space. In order to restrict the nerve $\mathbb{N}$ to a nerve from two categories, we therefore need to consider instead a horizontal homotopical embedding $\mathbb{H}^{\simeq}\colon 2\mathrm{Cat}\to\mathrm{DblCat}$. By pre-composing with this embedding, we obtain a nerve functor $\mathbb{N}\mathbb{H}^{\simeq}\colon 2\mathrm{Cat}\to\mathrm{sSet}^{\Delta^{\mathrm{op}}\times\Delta^{\mathrm{op}}}$, which embeds the homotopy theory of 2-categories into that of 2-fold complete Segal spaces. Furthermore, we construct a level-wise homotopy equivalence between the nerves $\mathbb{N}\mathbb{H}^{\simeq}\mathcal{A}$ and $\mathbb{N}\mathbb{H}\mathcal{A}$ of a 2-category $\mathcal{A}$, showing that $\mathbb{N}\mathbb{H}^{\simeq}\mathcal{A}$ exhibits a fibrant replacement of $\mathbb{N}\mathbb{H}\mathcal{A}$. The below theorem recapitulates these results, and appears as Theorems 12.1.1, 12.2.1, 12.1.3 and 12.3.5.

Theorem G. *The nerve functor* $\mathbb{N}\mathbb{H}^{\simeq}\colon 2\mathrm{Cat}\to 2\mathrm{CSS}$ *is right Quillen, and homotopically fully faithful, where* 2Cat *is endowed with Lack's model structure, and* 2CSS *denotes the model structure on* $\mathrm{sSet}^{\Delta^{\mathrm{op}}\times\Delta^{\mathrm{op}}}$ *for* 2*-fold complete Segal spaces. Lack's model structure on* 2Cat *is further right-induced from the model structure* 2CSS *for* 2*-fold complete Segal spaces.*

Moreover, the nerve $\mathbb{N}\mathbb{H}^{\simeq}\mathcal{A}$ *provides a fibrant replacement of the nerve* $\mathbb{N}\mathbb{H}\mathcal{A}$ *in* 2CSS *(and in* DblCat^h_∞*), for every* 2*-category* $\mathcal{A}$*.*

In particular, the results given in this thesis can be summarized by the following diagram of right Quillen and homotopically fully faithful functors

$$\begin{tikzcd} 2\mathrm{Cat} \arrow[r, "\mathbb{N}\mathbb{H}^{\simeq}"] \arrow[d, "\mathbb{H}"'] & 2\mathrm{CSS} \arrow[dd, "\mathrm{id}"] \\ \mathrm{DblCat} \arrow[r, phantom, "\nearrow\!\!\!\nearrow \simeq"] & \\ \mathrm{DblCat} \arrow[u, "\mathrm{id}"] \arrow[r, "\mathbb{N}"'] & \mathrm{DblCat}^h_\infty \end{tikzcd}$$

filled with a natural transformation which is level-wise a weak equivalence. This gives the expected compatibility of the different categorical objects present in this diagram.

Outline

The different parts of the thesis are organized as follows. In Part I., we first introduce 2-categories and double categories. We then present in Part II. the main aspects of the theory of model categories. The reader familiar with 2-dimensional categories and model categories may wish to skip these two first parts, except maybe Section 3.6, which presents notions of weak horizontal invertibility in a double category whose theory was developed in [MSV20a, MSV20b, Mos20]. Then, in Part III., we construct two model structures for double categories, based on results of joint work [MSV20a, MSV20b] with Maru Sarazola and Paula Verdugo. Then, in Part IV., we introduce the ∞-analogues of 2-categories and double categories and, finally, in Part V., we give the nerve construction, defined in the paper [Mos20] by the author, relating the strict notions with their ∞-versions. A more detailed outline can be found at the beginning of each part and each section.

Contents

PART I.

2-DIMENSIONAL CATEGORIES

A category consists of objects and morphisms together with an associative and unital composition on its morphisms. In particular, such a structure can be obtained by two different categorical constructions from the category Set of sets and maps. The first construction is given by "enrichment": a category is a category enriched in Set, in the sense that its morphisms between every pair of objects form a set. The second construction is given by "internalization": a category is an internal category to Set. In other words, it is a diagram in Set which consists of a set of objects and a set of morphisms, as well as maps representing source, target, identities, and compositions of morphisms. In particular, categories form themselves a category Cat with functors as morphisms.

These "enrichment" and "internalization" constructions can be iterated, and we can study the categories enriched over Cat, as well as the categories internal to Cat. While for sets, the processes of enrichment and internalization give the same notion of categories, in the case of Cat, they give rise to two different notions. A category enriched over Cat is called a 2-*category*. As a category, it has objects, but rather than a set of morphisms, it now has a category of morphisms between any pair of objects. The morphisms in these hom-categories are called 2-*morphisms*, and are morphisms between morphisms. In comparison, an internal category to Cat is called a *double category*, and it has two kinds of morphisms between its objects – called the *horizontal* and *vertical* morphisms. Its 2-morphisms sit in a square of two horizontal morphisms and two vertical morphisms, and are therefore called *squares*. In particular, a 2-category can always be seen as a horizontal double category with only trivial vertical morphisms. These 2-dimensional categories also come with a notion of morphisms preserving their structure, called 2-*functors* and *double functors*, respectively, and they form categories 2Cat and DblCat, respectively.

While 2-categories seem to be a more direct generalization of a category, many aspects of 2-category theory benefit from a passage to double categories. For example, a good notion of limit for 2-categories is that of a 2-*limit*, first introduced by Auderset [Aud74] and Borceux-Kelly [BK75], and further developed by Street [Str76], Kelly [Kel82, Kel89] and Lack [Lac10]. As clingman and the author show in [cM20], a 2-limit cannot be characterized as a 2-terminal object in the corresponding 2-category of cones, but a passage to double categories allows such a characterization by results of Grandis and Paré [Gra20, GP99]. Indeed, they show that the 2-limit of a 2-functor is double terminal in the double category of cones over the corresponding horizontal double functor.

In this first part, we introduce the theory of 2-categories and double categories and study their relations. While the theory of 2-categories is more classical, we show that it can be generalized in the context of double categories. In particular, we show that both categories 2Cat and DblCat are cartesian closed, and that they both admit a closed symmetric monoidal structure given by the *Gray tensor product*, introduced by Gray [Gra74] for 2-categories and by Böhm [Böh19] for double categories. We also show that they both have weak invertibility notions for their morphisms, by defining an *equivalence* to be a morphism which has an inverse up to 2-isomorphism. In the case of double categories, this also induces a notion of weak invertibility for its squares, which was recently introduced independently by the author, Sarazola, and Verdugo in [MSV20a], and by Grandis and Paré in [GP19]. In particular, using this terminology, we can define *weakly horizontally invariant* double categories, which correspond to the class of fibrant objects of one of the model structures on DblCat, and which were first introduced by the author, Sarazola, and Verdugo in [MSV20b].

In Section 1, we first give a brief introduction on enriched categories and locally presentable categories. We then introduce 2-categories in Section 2 and show that they have the properties mentioned above. Finally, in Section 3, we show that this theory extends to the context of double categories, and we compare the categories 2Cat and DblCat through several adjunctions.

1. Prelude on enriched categories and locally presentable categories

In this first section, we introduce some terminology about enriched categories and locally presentable categories, which will be useful in the rest of the thesis. In Section 1.1, we first recall briefly the definition of a category *enriched* over a monoidal category. We then introduce *tensored* and *cotensored* enriched categories, and show several technical results which follow from the universal properties in the definitions of tensors and cotensors. In Section 1.2, we introduce *locally presentable* categories, which gives a useful condition of "smallness" on a category preventing us running into set-theoretic issues. In particular, all categories considered in this thesis satisfy this condition and, as an example, we show that the categories Set of sets and Cat of small categories are locally presentable.

1.1. **Enriched categories.**

A category consists of a collection of objects together with a hom-set of morphisms between every pair of objects, which have an associative and unital composition. This concept can be generalized by requiring that the hom objects are objects in a monoidal category rather than sets. In this section, we introduce briefly this generalization of categories, called *enriched categories*, in order to fix some terminology for the rest of the thesis.

First, recall that a *monoidal category* $(\mathcal{T}, \otimes, I)$ is a category $\mathcal{T}$ endowed with a monoidal product $\otimes \colon \mathcal{T} \times \mathcal{T} \to \mathcal{T}$ which is associative and unital with respect to the monoidal unit $I \in \mathcal{T}$ up to canonical isomorphisms. We say that it is *symmetric monoidal* if the monoidal product is symmetric up to isomorphism. Furthermore, we say that it is closed if it admits internal homs, defined as follows.

Definition 1.1.1. A monoidal category $(\mathcal{T}, \otimes, I)$ is **closed monoidal** if, for every pair of objects $S, T \in \mathcal{T}$, there is a **internal hom object** $[S, T] \in \mathcal{T}$ and an isomorphism

$$\mathcal{T}(R \otimes S, T) \cong \mathcal{T}(R, [S, T])$$

natural in R, S, and T.

Remark 1.1.2. Using the universal property of the internal hom, one can show that it induces a functor $[-,-] \colon \mathcal{T}^{\mathrm{op}} \times \mathcal{T} \to \mathcal{T}$. In particular, given an object $S \in \mathcal{T}$, the isomorphisms in the definition above imply that the functor $[S, -] \colon \mathcal{T} \to \mathcal{T}$ is right adjoint to the functor $- \otimes S \colon \mathcal{T} \to \mathcal{T}$. As a consequence, we get that the tensor product preserves colimits in the first variable. Furthermore, if the tensor product is symmetric, it preserves colimits in both variables.

While the isomorphisms expressing the universal property of the internal hom in the definition above are isomorphisms of sets, they can be upgraded to isomorphisms in the ambient category.

Proposition 1.1.3. *Let $(\mathcal{T}, \otimes, I)$ be a closed monoidal category. For every tuple of objects R, S, and T in $\mathcal{T}$, there is an isomorphism in $\mathcal{T}$*

$$[R \otimes S, T] \cong [R, [S, T]]$$

natural in R, S, and T.

Proof. By the universal property of the internal hom $[-,-]$ and the associativity of $\otimes$, we have that

$$\begin{aligned}\mathcal{T}(Q,[R,[S,T]]) &\cong \mathcal{T}(Q\otimes R,[S,T]) \cong \mathcal{T}((Q\otimes R)\otimes S,T)\\ &\cong \mathcal{T}(Q\otimes(R\otimes S),T) \cong \mathcal{T}(Q,[R\otimes S,T]),\end{aligned}$$

for every object $Q \in \mathcal{T}$. Hence, by the Yoneda Lemma, we can conclude that there is an isomorphism $[R\otimes S,T] \cong [R,[S,T]]$ in $\mathcal{T}$, natural in R, S, and T. □

An example of monoidal structure on a category is given by the product, which exists under the assumption that the category considered is *complete*, i.e., it admits all small limits.

Remark 1.1.4. If $\mathcal{T}$ is complete, then the product $\times\colon \mathcal{T}\times\mathcal{T}\to\mathcal{T}$ defines a symmetric monoidal product on $\mathcal{T}$, with monoidal unit the terminal object $*\in\mathcal{T}$. If $\mathcal{T}$ is closed with respect to the product, we say that it is **cartesian closed**.

We now introduce the concept of categories enriched in a monoidal category.

Definition 1.1.5. Let $(\mathcal{T},\otimes,I)$ be a monoidal category. A **$\mathcal{T}$-enriched category** $\mathcal{A}$ consists of

(i) objects $A, C, E, \ldots$,
(ii) a hom object $\mathcal{A}(A,C)\in\mathcal{T}$, for every pair of objects A and C,
(iii) a composition morphism $\circ\colon \mathcal{A}(A,C)\otimes\mathcal{A}(C,E)\to\mathcal{A}(A,E)$ in $\mathcal{T}$, for every tuple of objects A, C, and E,
(iv) an identity morphism $\mathrm{id}_A\colon I\to\mathcal{A}(A,A)$, for every object A,

such that composition is associative and unital in $\mathcal{T}$. See [Rie14, Definition 3.3.1] or [Kel05, §1.2] for a complete definition.

Remark 1.1.6. In particular, a closed monoidal category $(\mathcal{T},\otimes,I)$ is enriched over itself with hom objects given by the internal homs $[-,-]$.

Given two enriched categories, we can define a notion of morphisms between those as follows.

Definition 1.1.7. Let $(\mathcal{T},\otimes,I)$ be a monoidal category, and let $\mathcal{A}$ and $\mathcal{B}$ be $\mathcal{T}$-enriched categories. A **$\mathcal{T}$-enriched functor** $F\colon\mathcal{A}\to\mathcal{B}$ consists of

(i) an assignment on objects $A\in\mathcal{A}\mapsto FA\in\mathcal{B}$,
(ii) a morphism $F_{A,C}\colon\mathcal{A}(A,C)\to\mathcal{B}(FA,FC)$ in $\mathcal{T}$, for each pair of objects $A, C\in\mathcal{A}$,

such that it is compatible with compositions and identities of $\mathcal{A}$ and $\mathcal{B}$. See [Rie14, Definition 3.5.1] or [Kel05, §1.2] for a complete definition.

Enriched categories and enriched functors assemble into a category.

Notation 1.1.8. Let $(\mathcal{T},\otimes,I)$ be a monoidal category. We write $\mathcal{T}-\mathrm{Cat}$ for the category of $\mathcal{T}$-enriched categories and $\mathcal{T}$-enriched functors.

As mentioned in the introduction, the hom objects of a category are sets, and hence they correspond to categories enriched over Set, the category of sets and maps.

Example 1.1.9. A Set-enriched category is a category and a Set-enriched functor is a functor. Hence the category Cat of categories and functors correspond to the category Set−Cat.

Every enriched category has an underlying category, which is given by applying the functor $\mathcal{T}(I,-)\colon\mathcal{T}\to\mathrm{Set}$, where I is the monoidal unit, to its hom objects.

Definition 1.1.10. Let $\mathcal{A}$ be a $\mathcal{T}$-enriched category. Its **underlying category** $\mathcal{A}_0$ is the category with the same objects as $\mathcal{A}$ and with hom-sets $\mathcal{A}_0(A,C) \coloneqq \mathcal{T}(I,\mathcal{A}(A,C))$. See [Rie14, Definition 3.4.5] or [Kel05, §1.3] for a complete definition.

When the enriching category is closed monoidal, we can further define *tensored* and *cotensored* enriched categories as follows

Definition 1.1.11. Let $\mathcal{T}$ be a closed monoidal category. A $\mathcal{T}$-enriched category $\mathcal{A}$ is said to be **tensored** over $\mathcal{T}$ if, for every object $A \in \mathcal{A}$ and every object $T \in \mathcal{T}$, there is an object $A \otimes_{\mathcal{A}} T \in \mathcal{A}$ and an isomorphism

$$\mathcal{A}(A \otimes_{\mathcal{A}} T, C) \cong [T, \mathcal{A}(A, C)],$$

for every object $C \in \mathcal{A}$, natural in A, C, and T. It is **cotensored** over $\mathcal{T}$ if, for every object $C \in \mathcal{A}$ and every object $T \in \mathcal{T}$, there is an object $C^T \in \mathcal{A}$ and an isomorphism

$$[T, \mathcal{A}(A, C)] \cong \mathcal{A}(A, C^T),$$

for every object $A \in \mathcal{A}$, natural in A, C, and T.

Remark 1.1.12. By applying the functor $\mathcal{T}(I, -) \colon \mathcal{T} \to \mathrm{Set}$ to the above isomorphisms, we obtain isomorphisms of sets

$$\mathcal{A}_0(A \otimes_{\mathcal{A}} T, C) \cong \mathcal{T}(T, \mathcal{A}(A, C)) \cong \mathcal{A}_0(A, C^T)$$

natural in A, C, and T.

Remark 1.1.13. In particular, a closed symmetric monoidal category $(\mathcal{T}, \otimes, I)$ is tensored and cotensored over itself, with tensors given by the monoidal products $\otimes$ and cotensors given by the internal homs $[-,-]$.

Remark 1.1.14. Using the universal property of the tensor and the cotensor, one can show that they induce functors $- \otimes_{\mathcal{A}} - \colon \mathcal{A} \times \mathcal{T} \to \mathcal{A}$ and $(-)^{(-)} \colon \mathcal{T}^{\mathrm{op}} \times \mathcal{A} \to \mathcal{A}$. In particular, given an object $A \in \mathcal{A}$, the functor $A \otimes_{\mathcal{A}} - \colon \mathcal{T} \to \mathcal{A}$ is left adjoint to the functor $\mathcal{A}(A, -) \colon \mathcal{A} \to \mathcal{T}$ and, for every object $S \in \mathcal{T}$, the functor $- \otimes_{\mathcal{A}} S \colon \mathcal{A} \to \mathcal{A}$ is right adjoint to the functor $(-)^S \colon \mathcal{A} \to \mathcal{A}$. As a consequence, the tensoring functor $\otimes_{\mathcal{A}}$ preserves colimits in both variables.

The following results tell us that the tensors and cotensors of an enriched category commute with the tensor of the enriching category.

Proposition 1.1.15. *Let $(\mathcal{T}, \otimes, I)$ be a closed monoidal category, and let $\mathcal{A}$ be a tensored and cotensored $\mathcal{T}$-enriched category. Then, for every object $A \in \mathcal{A}$ and every pair of objects $S, T \in \mathcal{T}$, we have isomorphisms*

$$(A \otimes_{\mathcal{A}} S) \otimes_{\mathcal{A}} T \cong A \otimes_{\mathcal{A}} (S \otimes T) \quad \textit{and} \quad (C^S)^T \cong C^{S \otimes T}$$

natural in A, S, and T.

Proof. We prove that the first isomorphism holds. By Remark 1.1.12 and since $\mathcal{T}$ is closed monoidal, we have that

$$\begin{aligned} \mathcal{A}_0((A \otimes_{\mathcal{A}} S) \otimes_{\mathcal{A}} T, C) &\cong \mathcal{A}_0(A \otimes_{\mathcal{A}} S, C^T) \cong \mathcal{T}(S, \mathcal{A}(A, C^T)) \\ &\cong \mathcal{T}(S, [T, \mathcal{A}(A, C)]) \cong \mathcal{T}(S \otimes T, \mathcal{A}(A, C)) \\ &\cong \mathcal{A}_0(A \otimes_{\mathcal{A}} (S \otimes T), C), \end{aligned}$$

for every object $C \in \mathcal{A}$. Hence, by the Yoneda lemma, we can conclude that there is an isomorphism $(A \otimes_{\mathcal{A}} S) \otimes_{\mathcal{A}} T \cong A \otimes_{\mathcal{A}} (S \otimes T)$ in $\mathcal{A}$, natural in A, S, and T. The other isomorphism can be shown to hold similarly. $\square$

Finally, we show that, if we are only given the isomorphisms of sets of Remark 1.1.12, we can upgrade them to isomorphisms in the monoidal category $\mathcal{T}$, and hence the set-level isomorphisms are enough to determine when a $\mathcal{T}$-enriched category is tensored and cotensored.

Corollary 1.1.16. *Let $(\mathcal{T}, \otimes, I)$ be a closed symmetric monoidal category, and let $\mathcal{A}$ be a $\mathcal{T}$-enriched category. If*

(i) for every pair of objects $A, C \in \mathcal{A}$ and every object $T \in \mathcal{T}$, there are objects $A \otimes_{\mathcal{A}} T$ and C^T in $\mathcal{A}$ together with isomorphisms of sets

$$\mathcal{A}_0(A \otimes_{\mathcal{A}} T, C) \cong \mathcal{T}(T, \mathcal{A}(A, C)) \cong \mathcal{A}_0(A, C^T)$$

natural in A, C, and T,

(ii) for every object $A \in \mathcal{A}$ and every pair of objects $S, T \in \mathcal{T}$, there is an isomorphism

$$(A \otimes_{\mathcal{A}} T) \otimes_{\mathcal{A}} A \cong A \otimes_{\mathcal{A}} (T \otimes S)$$

in $\mathcal{A}$ natural in A, S, and T,

then $\mathcal{A}$ is tensored and cotensored over $\mathcal{T}$.

Proof. By the above isomorphisms, and the commutativity of $\otimes$, we have that

$$\begin{aligned}\mathcal{T}(S, \mathcal{A}(A \otimes_{\mathcal{A}} T, C)) &\cong \mathcal{A}_0((A \otimes_{\mathcal{A}} T) \otimes_{\mathcal{A}} S, C) \cong \mathcal{A}_0(A \otimes_{\mathcal{A}} (T \otimes S), C) \\ &\cong \mathcal{T}(T \otimes S, \mathcal{A}(A, C)) \cong \mathcal{T}(S \otimes T, \mathcal{A}(A, C)) \\ &\cong \mathcal{T}(S, [T, \mathcal{A}(A, C)]),\end{aligned}$$

for every object $S \in \mathcal{T}$. Hence, by the Yoneda Lemma, we deduce that there is an isomorphism $\mathcal{A}(A \otimes_{\mathcal{A}} T, C) \cong [T, \mathcal{A}(A, C)]$ in $\mathcal{T}$, natural in A, C, and T. Similarly, one can show that there is an isomorphism $[T, \mathcal{A}(A, C)] \cong \mathcal{A}(A, C^T)$ in $\mathcal{T}$, natural in A, C, and T. This shows that $\mathcal{A}$ is a tensored and cotensored $\mathcal{T}$-enriched category. □

1.2. **Locally presentable categories.** In this section, we introduce conditions of "smallness" on a category, namely that of *local presentability* categories. In particular, all the categories considered in this thesis satisfy this condition. We give a characterization of these locally small categories in terms of *limit-sketches*, which will be useful to prove that the categories considered in this thesis are locally presentable. In particular, we show that the category Cat of small categories and functors is locally presentable using this characterization. The definitions and results in this section are based on [AR94, §1].

We first define what it means for an object in a category to be small.

Definition 1.2.1. Let $\mathcal{M}$ be a category and λ be a regular cardinal. An object $A \in \mathcal{M}$ is said to be **λ-small** if, for every λ-filtered colimit of a diagram $\{X_\mu\}$ in $\mathcal{M}$, there is an isomorphism $\operatorname{colim}_\mu \mathcal{M}(A, X_\mu) \cong \mathcal{M}(A, \operatorname{colim}_\mu X_\mu)$

A category is **locally presentable** when it is *cocomplete*, i.e., it admits all small colimits, and it is generated by a set of small objects, as described below.

Definition 1.2.2. A category $\mathcal{M}$ is **locally presentable** if it is cocomplete, and there is a regular cardinal λ and a set $\mathcal{G}$ of λ-small objects in $\mathcal{M}$ such that every object in $\mathcal{M}$ is a λ-filtered colimit of objects in $\mathcal{G}$.

In particular, given a locally presentable category, the accessibly embedded, full, reflective subcategories of this category are also locally presentable. We first recall what a full, reflective subcategory is before proving this result.

Definition 1.2.3. Let $\mathcal{M}$ be a category. A **full, reflective subcategory** $\mathcal{N}$ of $\mathcal{M}$ is a subcategory of $\mathcal{M}$ such that the inclusion functor $I\colon \mathcal{N} \to \mathcal{M}$ admits a left adjoint $L\colon \mathcal{M} \to \mathcal{N}$.

Remark 1.2.4. Equivalently, the inclusion $I\colon \mathcal{N} \to \mathcal{M}$ as above is full if and only if the counit $\epsilon\colon LI \Rightarrow \mathrm{id}_{\mathcal{N}}$ of the adjunction $L \dashv I$ is a natural isomorphism.

Proposition 1.2.5. *Let λ be a regular cardinal, and $\mathcal{M}$ be a locally presentable category with set $\mathcal{G}$ of λ-small objects such that every object in $\mathcal{M}$ is a λ-filtered colimit of objects in $\mathcal{G}$. Then, a full, reflective subcategory $\mathcal{N}$ of $\mathcal{M}$ such that the inclusion $I\colon \mathcal{N} \to \mathcal{M}$ preserves λ-filtered colimits is also locally presentable.*

Proof. Let $L\colon \mathcal{M} \to \mathcal{N}$ denote the left adjoint of the inclusion $I\colon \mathcal{N} \to \mathcal{M}$. We first prove that $\mathcal{N}$ is cocomplete. Let $F\colon \mathcal{A} \to \mathcal{N}$ be a diagram in $\mathcal{N}$. By post-composing F with the inclusion I, we get a diagram $IF\colon \mathcal{A} \to \mathcal{M}$. Since $\mathcal{M}$ is cocomplete, this diagram admits a colimit, denoted by $\operatorname{colim}_{\mathcal{A}} IF$, in $\mathcal{M}$. By applying the left adjoint L, we get an isomorphism in $\mathcal{N}$

$$L(\operatorname{colim}_{\mathcal{A}} IF) \cong \operatorname{colim}_{\mathcal{A}} LIF \cong \operatorname{colim}_{\mathcal{A}} F,$$

since L preserves colimits, and the counit $\epsilon\colon LI \cong \mathrm{id}_{\mathcal{N}}$ is a natural isomorphism. This shows that $L(\operatorname{colim}_{\mathcal{A}} IF)$ is a colimit for F and hence that $\mathcal{N}$ is cocomplete. Now, we show that $L\mathcal{G} = \{LG \mid G \in \mathcal{G}\}$ is a set of λ-small objects in $\mathcal{N}$ such that every object in $\mathcal{N}$ is a λ-filtered colimit of objects in $L\mathcal{G}$. First, note that every object LG is λ-small, for $G \in \mathcal{G}$. Indeed, we have

$$\begin{aligned}\mathcal{N}(LG, \operatorname{colim}_\mu X_\mu) &\cong \mathcal{M}(G, I(\operatorname{colim}_\mu X_\mu)) \cong \mathcal{M}(G, \operatorname{colim}_\mu IX_\mu)\\ &\cong \operatorname{colim}_\mu \mathcal{M}(G, IX_\mu) \cong \operatorname{colim}_\mu \mathcal{N}(LG, X_\mu),\end{aligned}$$

for every λ-filtered colimit of a diagram $\{X_\mu\}$ in $\mathcal{N}$, where we used the facts that $L \dashv I$ is an adjunction, that I preserves λ-filtered colimits, and that $G \in \mathcal{G}$ is λ-small in $\mathcal{M}$. Finally, given an object $X \in \mathcal{N}$, the object $IX \in \mathcal{M}$ is a λ-filtered colimit $IX \cong \operatorname{colim}_\mu G_\mu$ of objects $G_\mu \in \mathcal{G}$. Hence, we have an isomorphism in $\mathcal{N}$

$$X \cong LIX \cong L(\operatorname{colim}_\mu G_\mu) \cong \operatorname{colim}_\mu LG_\mu,$$

since the counit $\epsilon\colon LI \cong \mathrm{id}_{\mathcal{N}}$ is a natural isomorphism, and L preserves colimits. This shows that $\mathcal{N}$ is locally presentable. □

By results of [AR94], any locally presentable category is equivalent to a category of models for a limit-sketch. We introduce this terminology before stating this theorem, which will not be proven here since it involves more technicalities.

Definition 1.2.6. A **limit-sketch** is a tuple $\mathcal{L} = (\mathcal{A}, \mathbf{D}, \sigma)$ consisting of

(i) a small category $\mathcal{A}$,
(ii) a set $\mathbf{D}$ of diagrams in $\mathcal{A}$,
(iii) a map σ which assigns to each diagram $D \in \mathbf{D}$ a compatible cone $\sigma(D)$ under D in $\mathcal{A}$.

A **model** of the limit-sketch $\mathcal{L}$ is a functor $F\colon \mathcal{A}^{\mathrm{op}} \to \mathrm{Set}$ such that $F(\sigma(D))$ is the limit of the diagram $F(D)$ in Set. We denote by $\mathrm{Mod}(\mathcal{L})$ the full subcategory of the category $\mathrm{Set}^{\mathcal{A}^{\mathrm{op}}}$ of functors $\mathcal{A}^{\mathrm{op}} \to \mathrm{Set}$ and natural transformations between them.

Theorem 1.2.7. *A category is locally presentable if and only if it is equivalent to a category of models for a limit-sketch.*

Proof. This is [AR94, Corollary 1.52]. □

Using this description of locally presentable categories, it is straightforward to see that any category of presheaves in Set is locally presentable.

Example 1.2.8. Given a small category $\mathcal{A}$, the category $\mathrm{Set}^{\mathcal{A}^{\mathrm{op}}}$ of functors $\mathcal{A}^{\mathrm{op}} \to \mathrm{Set}$ and natural transformations is locally presentable. Indeed, it corresponds to the category of models for the limit-sketch $\mathcal{L} = (\mathcal{A}, \mathbf{D}, \sigma)$ with $\mathbf{D} = \emptyset$. In particular, all accessibly embedded, full, reflective subcategories of $\mathrm{Set}^{\mathcal{A}^{\mathrm{op}}}$ are also locally presentable by Proposition 1.2.5. Note that this also implies that the category Set itself is locally presentable.

Furthermore, from this description of locally presentable categories using limit-sketches, we can show that every locally presentable is also *complete*, i.e., it admits all small limits.

Proposition 1.2.9. *A locally presentable category is complete.*

Proof. We show that a category $\mathrm{Mod}(\mathcal{L})$ of models for a limit-sketch $\mathcal{L} = (\mathcal{A}, \mathbf{D}, \sigma)$ is complete, which implies the result since every locally presentable is equivalent to such a category by Theorem 1.2.7. Let $\{F_i\}_{i\in\mathcal{I}}$ be a diagram in $\mathrm{Mod}(\mathcal{L})$, where $\mathcal{I}$ is a small category. Then it is also a diagram in $\mathrm{Set}^{\mathcal{A}^{\mathrm{op}}}$ and, since $\mathrm{Set}^{\mathcal{A}^{\mathrm{op}}}$ is complete, the limit $\lim_{i\in\mathcal{I}} F_i$ in $\mathrm{Set}^{\mathcal{A}^{\mathrm{op}}}$ exists. We show that, for every diagram $D \in \mathbf{D}$, $(\lim_{i\in\mathcal{I}} F_i)(\sigma(D))$ is the limit of the diagram $(\lim_{i\in\mathcal{I}} F_i)(D)$. Since limits are computed point-wise in $\mathrm{Set}^{\mathcal{A}^{\mathrm{op}}}$, limits commute among each other, and $F_i \in \mathrm{Mod}(\mathcal{L})$ for all $i \in \mathcal{I}$, we have that

$$\begin{aligned}(\lim_{i\in\mathcal{I}} F_i)(\sigma(D)) &\cong \lim_{i\in\mathcal{I}} \left(F_i(\sigma(D))\right) \cong \lim_{i\in\mathcal{I}} \left(\lim_D F_i(D)\right) \\ &\cong \lim_D \left(\lim_{i\in\mathcal{I}} \left(F_i(D)\right)\right) \cong \lim_D \left(\lim_{i\in\mathcal{I}} F_i\right)(D),\end{aligned}$$

for every diagram $D \in \mathbf{D}$, which shows that $\lim_{i\in\mathcal{I}} F_i \in \mathrm{Mod}(\mathcal{L})$. This concludes the proof that $\mathrm{Mod}(\mathcal{L})$ is complete. ☐

Finally, we use this limit-sketch characterization of locally presentable categories to show that Cat is locally presentable. This proof can be adapted to the context of double categories, as we will see in Proposition 3.1.6.

Notation 1.2.10. For $n \geq 0$, we define the category $[n]$ to be the category induced by the poset $\{0 \leq 1 \leq \ldots \leq n\}$. In other words, the category $[n]$ is the category free on n composable morphisms, i.e., the category generated by the following data

$$0 \longrightarrow 1 \longrightarrow 2 \longrightarrow \cdots \longrightarrow n-1 \longrightarrow n\,.$$

In particular, the category $[0]$ is the terminal category, and the category $[1]$ is the category free on a morphism.

Proposition 1.2.11. *The category* Cat *of small categories and functors is locally presentable.*

Proof. We construct a limit-sketch $\mathcal{L} = (\mathcal{A}, \mathbf{D}, \sigma)$ such that the category $\mathrm{Mod}(\mathcal{L})$ of models for $\mathcal{L}$ is equivalent to Cat. Let $\mathcal{A} \coloneqq \Delta_3$ be the category generated by the following diagram

$$\begin{tikzcd}[column sep=large]
{[0]} \arrow[r, shift left=2, "d^1"] \arrow[r, shift right=2, "d^1"'] & {[1]} \arrow[l, "s^0" description] \arrow[r, shift left=4, "d^2"] \arrow[r, "d^1" description] \arrow[r, shift right=4, "d^0"'] & {[2]} \arrow[l, shift right=2, "s^0" description] \arrow[l, shift left=2, "s^1" description] \arrow[r, shift left=3, "d^3"] \arrow[r, shift left=1, "d^2" description] \arrow[r, shift right=1, "d^1" description] \arrow[r, shift right=3, "d^0"'] & {[3]}\,,
\end{tikzcd}$$

where the functors $d^i \colon [n-1] \to [n]$ are induced by the order-preserving maps which omit i, and the functors $s^j \colon [n] \to [n-1]$ are induced by the order-preserving maps which double j, for all $1 \leq n \leq 3$, $0 \leq i \leq n$ and $0 \leq j \leq n-1$. We define $\mathbf{D}$ to be the set containing the following two diagrams in Δ_3.

$$D_1 = \begin{tikzcd} {[0]} \arrow[r, "d^1"] \arrow[d, "d^0"'] & {[1]} \\ {[1]} & \end{tikzcd} \qquad D_2 = \begin{tikzcd} {[1]} \arrow[r, "d^2"] \arrow[d, "d^0"'] & {[2]} \\ {[2]} & \end{tikzcd}$$

Then we set $\sigma(D_1) = [2]$ and $\sigma(D_2) = [3]$ together with the following cones.

$$\begin{tikzcd} {[0]} \arrow[r, "d^1"] \arrow[d, "d^0"'] & {[1]} \arrow[d, "d^0"] \\ {[1]} \arrow[r, "d^2"'] & {[2]} \end{tikzcd} \qquad \begin{tikzcd} {[1]} \arrow[r, "d^2"] \arrow[d, "d^0"'] & {[2]} \arrow[d, "d^0"] \\ {[2]} \arrow[r, "d^3"'] & {[3]} \end{tikzcd}$$

Now, given a category $\mathcal{C}$, we can associate to it a functor $\mathrm{Cat}(-,\mathcal{C})\colon \Delta_3^{\mathrm{op}} \to \mathrm{Set}$. In particular, it is such that

$$\mathrm{Cat}([2],\mathcal{C}) \cong \mathrm{Cat}([1],\mathcal{C}) \underset{\mathrm{Cat}([0],\mathcal{C})}{\times} \mathrm{Cat}([1],\mathcal{C}) \text{ and } \mathrm{Cat}([3],\mathcal{C}) \cong \mathrm{Cat}([2],\mathcal{C}) \underset{\mathrm{Cat}([1],\mathcal{C})}{\times} \mathrm{Cat}([2],\mathcal{C})$$

since $[2] = [1] \bigsqcup_{[0]} [1]$, $[3] = [2] \bigsqcup_{[1]} [2]$, and $\mathrm{Cat}(-,\mathcal{C})$ sends colimits to limits. Hence $\mathrm{Cat}(-,\mathcal{C})$ is in $\mathrm{Mod}(\mathcal{L})$. This extends in an obvious way on morphisms, and gives a functor $F\colon \mathrm{Cat} \to \mathrm{Mod}(\mathcal{L})$.

Conversely, given a functor $X\colon \Delta_3^{\mathrm{op}} \to \mathrm{Set}$ in $\mathrm{Mod}(\mathcal{L})$, we can define a category GX as follows. Since a category can equivalently be defined as an internal category to Set, we describe the internal category to Set associated to GX. Its set of objects is given by the set $X[0]$ and its set of morphisms is given by the set $X[1]$, where the source map is $X(d^1)\colon X[1] \to X[0]$ and the target map is $X(d^0)\colon X[0] \to X[1]$. The identity map is given by $X(s^0)\colon X[0] \to X[1]$, and the composition map is given by

$$X[1] \times_{X[0]} X[1] \cong X[2] \xrightarrow{X(d^1)} X[1].$$

Associativity of the composition follows from the fact that

$$X[3] \cong X[2] \times_{X[1]} X[2] \cong X[1] \times_{X[0]} X[1] \times_{X[0]} X[1]$$

and that the below left diagram commutes in Δ_3, while unitality of the composition follows from the fact that the below right diagram commutes in Δ_3.

$$\begin{tikzcd} {[1]} \arrow[r, "d^1"] \arrow[d, "d^2"'] & {[2]} \arrow[d, "d^3"] \\ {[2]} \arrow[r, "d^1"'] & {[3]} \end{tikzcd} \qquad \begin{tikzcd} {[1]} \arrow[r, "s^0"] \arrow[d, "d^0"'] & {[2]} \arrow[d, "d^0"] \\ {[0]} \arrow[r, "s^1"'] & {[1]} \end{tikzcd}$$

Given a natural transformation $\alpha\colon X \Rightarrow Y$ in $\mathrm{Mod}(\mathcal{L})$, we can associate to it a functor $G\alpha\colon GX \to GY$ given by $\alpha_0\colon X[0] \to Y[0]$ on objects and $\alpha_1\colon X[1] \to Y[1]$. This defines a functor $G\colon \mathrm{Mod}(\mathcal{L}) \to \mathrm{Cat}$.

It is straightforward to see that $GF = \mathrm{id}_{\mathrm{Cat}}$. Moreover, there is a natural isomorphism $FG \cong \mathrm{id}_{\mathrm{Mod}(\mathcal{L})}$ given, at an object $X \in \mathrm{Mod}(\mathcal{L})$, by the isomorphism $FGX \cong X$ induced by the identities on $X[0]$ and $X[1]$, and by the comparison maps $X[1] \times_{X[0]} X[1] \cong X[2]$ and $X[2] \times_{X[1]} X[2] \cong X[3]$. This gives an equivalence of categories between Cat and $\mathrm{Mod}(\mathcal{L})$, and hence Cat is locally presentable by Theorem 1.2.7. $\square$

2. 2-categories

We now turn our attention to 2-categories, which can be seen as categories enriched in Cat, the category of (small) categories and functors. In particular, a 2-category also has morphisms between morphisms, called 2-*morphisms*, which are given by the morphisms of the hom-categories. In Section 2.1, we first introduce the category 2Cat of 2-categories and 2-functors, and show that it is locally presentable. Since every category can be seen as a category enriched in Cat with discrete categories of morphisms, there is a full embedding functor $D\colon \mathrm{Cat} \to 2\mathrm{Cat}$, and we show that this functor has both adjoints.

In Section 2.2, we show that 2Cat is cartesian closed, by introducing the notions of 2-*natural transformations* and *modifications*, which correspond to the morphisms and 2-morphisms of the internal hom 2-categories of 2Cat, respectively. In particular, a 2-natural transformation is an enriched version of a natural transformation between functors, which also satisfies the naturality conditions on the nose. However, since we also have a notion of 2-morphism between morphisms in a 2-category, we can relax this naturality condition by asking that it only holds up to 2-isomorphism. This gives rise to the notion of *pseudo-natural transformations*. In Section 2.3, we use this notion, as well as an adapted version of a *modification* to this pseudo-setting, to construct another internal hom 2-category, called the *pseudo-hom*. These pseudo-homs happen to be the internal homs of another symmetric monoidal structure on 2Cat, given by the *Gray tensor product*, introduced by Gray in [Gra74].

Finally, in Section 2.4, we define a notion of *equivalence* in a 2-category, which corresponds to a pair of opposite morphisms such that their composites are related by a 2-isomorphism to the identities. This generalizes the usual notion of equivalences of categories. An equivalence in a 2-category is further said to be *adjoint* when its 2-isomorphism components satisfy the *triangle identities*, analogous to the triangle identities of an adjunction. A classical 2-categorical result says that every equivalence can be promoted to such an adjoint equivalence, which is very useful in practice, as we will see throughout this thesis.

2.1. **The category** 2Cat**.** We recall that a category consists of a set of objects and a set of morphisms between each pair of objects. By also adding morphisms between morphisms, called 2-*morphisms*, we obtain the notion of a 2-*category*. In particular, a 2-category is a Cat-enriched category, where Cat is the category of categories and functors. In this section, we first introduce the category 2Cat of 2-categories and 2-functors. Since categories can be seen as 2-categories with only trivial 2-morphisms, there is an inclusion functor from Cat into 2Cat. We further introduce this functor, as well as its left and right adjoints.

Let us first give the definition of a 2-category.

Definition 2.1.1. A **2-category** $\mathcal{A}$ consists of

(i) objects $A, C, E, \ldots$,
(ii) morphisms $a\colon A \to C$ between objects A, C with an identity $\mathrm{id}_A\colon A \to A$ at each object A,
(iii) 2-morphisms $\alpha\colon a \Rightarrow c$ of the form

$$\begin{tikzcd} A \arrow[r, bend left, "a"] \arrow[r, bend right, "c"'] \arrow[r, phantom, "\Downarrow\alpha"] & C \end{tikzcd}$$

between morphisms $a\colon A \to C$ and $c\colon A \to C$ with an identity 2-morphism $\mathrm{id}_a\colon a \Rightarrow a$ at each morphism $a\colon A \to C$,
(iv) an associative and unital (horizontal) composition law for morphisms, and for 2-morphisms along shared object boundaries

$$\begin{tikzcd} A \arrow[r, bend left, "a"] \arrow[r, bend right, "c"'] \arrow[r, phantom, "\Downarrow\alpha"] & C \arrow[r, bend left, "a'"] \arrow[r, bend right, "c'"'] \arrow[r, phantom, "\Downarrow\alpha'"] & E \end{tikzcd} \quad = \quad \begin{tikzcd} A \arrow[r, bend left, "a'a"] \arrow[r, bend right, "c'c"'] \arrow[r, phantom, "\Downarrow\alpha' * \alpha"] & E \end{tikzcd},$$

(v) an associative and unital (vertical) composition law for 2-morphisms along shared morphism boundaries

$$\begin{tikzcd} A \arrow[r, "c"] \arrow[r, bend left=60, "a"] \arrow[r, bend right=60, "e"'] \arrow[r, phantom, bend left=30, "\Downarrow\alpha"] \arrow[r, phantom, bend right=30, "\Downarrow\gamma"] & C \end{tikzcd} \quad = \quad \begin{tikzcd} A \arrow[r, bend left, "a"] \arrow[r, bend right, "e"'] \arrow[r, phantom, "\Downarrow\gamma\alpha"] & C \end{tikzcd},$$

such that the horizontal and vertical compositions for 2-morphisms satisfy the interchange law.

Remark 2.1.2. Given a 2-category $\mathcal{A}$, for every pair of objects A and C in $\mathcal{A}$, we have a category $\mathcal{A}(A, C)$ of morphisms from A to C and 2-morphisms between them, with composition given by the vertical composition of 2-morphisms. Hence, a 2-category is a category enriched over Cat.

A morphism of 2-categories can then be defined as an assignment on objects, morphisms, and 2-morphisms which preserve all the 2-categorical structure.

Definition 2.1.3. Let $\mathcal{A}$ and $\mathcal{B}$ be 2-categories. A **2-functor** $F\colon \mathcal{A} \to \mathcal{B}$ sends

(i) each object $A \in \mathcal{A}$ to an object $FA \in \mathcal{B}$,
(ii) each morphism $a\colon A \to C$ in $\mathcal{A}$ to a morphism $Fa\colon FA \to FC$ in $\mathcal{B}$,
(iii) each 2-morphism $\alpha\colon a \Rightarrow c$ in $\mathcal{A}$ to a 2-morphism $F\alpha\colon Fa \Rightarrow Fc$ in $\mathcal{B}$,

in such a way that F preserves all compositions and identities.

Remark 2.1.4. A 2-functor $F\colon \mathcal{A} \to \mathcal{B}$ can equivalently be defined as an assignment on objects together with, for every pair of objects A and C in $\mathcal{A}$, a functor

$$F_{A,C}\colon \mathcal{A}(A, C) \to \mathcal{B}(FA, FC).$$

Hence, a 2-functor is a Cat-enriched functor.

Then, 2-categories with this notion of morphisms form a category.

Notation 2.1.5. We write 2Cat for the category of 2-categories and 2-functors. By Remarks 2.1.2 and 2.1.4, it is the same as the category Cat−Cat of Cat-enriched categories and Cat-enriched functors.

In particular, the category 2Cat satisfies the condition of "smallness" introduced in Definition 1.2.2. Since 2Cat is a full, reflective subcategory of the category of double categories, we do not prove here that 2Cat is locally presentable, but rather deduce it from the analogous result for double categories proved in Proposition 3.1.6 below.

Proposition 2.1.6. *The category* 2Cat *is locally presentable and, in particular, it is both complete and cocomplete.*

Proof. As we will see in Proposition 3.4.5, the category 2Cat is an accessibly embedded, full, reflective subcategory of the category DblCat of double categories and double functors, introduced in Section 3.1. Indeed, the horizontal embedding $\mathbb{H}\colon 2\mathrm{Cat} \to \mathrm{DblCat}$, introduced in Definition 3.4.1, is full and admits both adjoints by Proposition 3.4.5, and hence $\mathbb{H}$ preserves in particular all colimits. As we show in Proposition 3.1.6 that DblCat is locally presentable, it follows from Proposition 1.2.5 that 2Cat is also locally presentable. The second part of the statement follows from the definition of a locally presentable category and from Proposition 1.2.9. □

We describe below a set of objects which generate all 2-categories under colimits. For this, we first need to introduce the following notations.

Notation 2.1.7. Recall the categories $[n]$, for $n \geq 0$, introduced in Notation 1.2.10. We also denote by $[n]$ the corresponding locally discrete 2-category.

Notation 2.1.8. Given a category $\mathcal{C}$, we define the 2-category $\Sigma\mathcal{C}$ as the 2-category with two objects 0 and 1, and hom-categories given by $\Sigma\mathcal{C}(0,0) = [0] = \Sigma\mathcal{C}(1,1)$, $\Sigma\mathcal{C}(0,1) = \mathcal{C}$, and $\Sigma\mathcal{C}(1,0) = \emptyset$. It is sometimes called the *suspension* of $\mathcal{C}$, which explains the notation. In particular, the 2-category $\Sigma[1]$ is the 2-category free on a 2-morphism.

Remark 2.1.9. Every 2-category is a colimit of the following 2-categories: the terminal 2-category $[0]$, the 2-category $[1]$ free on a morphism, and the 2-category $\Sigma[1]$ free on a 2-morphism.

As mentioned in the introduction, a category can be seen as a 2-category with only trivial 2-morphisms. This gives a full embedding of Cat into 2Cat.

Definition 2.1.10. We define the **discrete embedding functor** $D\colon \mathrm{Cat} \to 2\mathrm{Cat}$. It sends a category $\mathcal{C}$ to the locally discrete 2-category $D\mathcal{C}$ with only trivial 2-morphisms, and a functor $F\colon \mathcal{C} \to \mathcal{D}$ to the 2-functor $DF\colon D\mathcal{C} \to D\mathcal{D}$ which acts as F does on the corresponding data.

This functor admits both adjoints. Its right adjoint, introduced below, is given by sending a 2-category to its underlying category.

Definition 2.1.11. We define the functor $U\colon 2\mathrm{Cat} \to \mathrm{Cat}$. It sends a 2-category $\mathcal{A}$ to its **underlying category** $U\mathcal{A}$ with the same objects as $\mathcal{A}$ and morphisms the morphisms of $\mathcal{A}$, and a 2-functor $F\colon \mathcal{A} \to \mathcal{B}$ to the functor $UF\colon U\mathcal{A} \to U\mathcal{B}$ which acts as F does on the corresponding data.

The left adjoint of $D\colon \mathrm{Cat} \to 2\mathrm{Cat}$ is given by sending a 2-category to a category whose hom-sets are given by taking the path components of each hom-category. In other words, there is a functors $\pi_0\colon \mathrm{Cat} \to \mathrm{Set}$ which sends a category to its set of objects, quotiented by the following equivalence relation: two objects are in the same equivalence class if and only if there is a zig-zag of morphisms between them. Then, the left adjoint of D is obtained by applying this functor π_0 locally.

Definition 2.1.12. We define the functor $P\colon 2\mathrm{Cat} \to \mathrm{Cat}$. It sends a 2-category $\mathcal{A}$ to the category $P\mathcal{A}$ with the same objects as $\mathcal{A}$ and with hom-sets $P\mathcal{A}(A,C) = \pi_0\mathcal{A}(A,C)$, for every pair of objects $A, C \in \mathcal{A}$, where $\pi_0\colon \mathrm{Cat} \to \mathrm{Set}$ is the functor which sends a category to its set of connected components. It sends a 2-functor $F\colon \mathcal{A} \to \mathcal{B}$ to the functor $PF\colon P\mathcal{A} \to P\mathcal{B}$ which acts as F on objects and, for every pair of objects $A, C \in \mathcal{A}$, is given by $\pi_0 F\colon \pi_0\mathcal{A}(A,C) \to \pi_0\mathcal{B}(FA,FC)$ on hom-sets.

We now show that, as promised, the functors P and U are indeed the left and right adjoints of D, respectively.

Proposition 2.1.13. *The functors P, D, and U form adjunctions*

$$\begin{tikzcd} \mathrm{Cat} \arrow[rr, "D" description] \arrow[rr, phantom, bend left=20, "\perp"] \arrow[rr, phantom, bend right=20, "\perp"] & & 2\mathrm{Cat} \arrow[ll, bend right=40, "P"'] \arrow[ll, bend left=40, "U"] \end{tikzcd} .$$

Moreover, the counit of the adjunction $P \dashv D$ and the unit of the adjunction $D \dashv U$ are identities. In particular, the functor $D\colon \mathrm{Cat} \to 2\mathrm{Cat}$ is a full embedding.

Proof. We first show that, for every category $\mathcal{C}$ and every 2-category $\mathcal{A}$, there is an isomorphism

$$2\mathrm{Cat}(\mathcal{A}, D\mathcal{C}) \cong \mathrm{Cat}(P\mathcal{A}, \mathcal{C})$$

natural in $\mathcal{C}$ and $\mathcal{A}$. Since every 2-morphism in $D\mathcal{C}$ is trivial, a 2-functor $F\colon \mathcal{A} \to D\mathcal{C}$ sends every 2-morphism of $\mathcal{A}$ to an identity 2-morphism. Hence it induces a functor $\hat{F}\colon P\mathcal{A} \to \mathcal{C}$

acting as F on objects and morphisms, since two morphisms in the same path component must be sent to the same morphism of $\mathcal{C}$ by F. Conversely, if $G\colon P\mathcal{A} \to \mathcal{C}$ is a functor, then it induces a 2-functor $\hat{G}\colon \mathcal{A} \to D\mathcal{C}$ which acts as G on objects and morphisms, and sends every 2-morphism to the identity 2-morphism of the image under G of its boundaries, which are in the same path component. These constructions are clearly inverse to each other and natural in $\mathcal{C}$ and $\mathcal{A}$. Hence $P \dashv D$ is an adjunction. Moreover, we have that $PD\mathcal{C} = \mathcal{C}$ for every category $\mathcal{C}$, since the category $D\mathcal{C}$ has only trivial 2-morphisms, and hence applying P does not identify morphisms in $\mathcal{C}$. This shows that the counit of $P \dashv D$ is an identity.

We now prove that, for every category $\mathcal{C}$ and every 2-category $\mathcal{A}$, there is an isomorphism

$$2\mathrm{Cat}(D\mathcal{C}, \mathcal{A}) \cong \mathrm{Cat}(\mathcal{C}, U\mathcal{A})$$

natural in $\mathcal{C}$ and $\mathcal{A}$. Since every 2-morphism in $D\mathcal{C}$ is trivial, the image of a 2-functor $F\colon D\mathcal{C} \to \mathcal{A}$ is included in the underlying category of $\mathcal{A}$ and hence it restricts to a functor $\hat{F}\colon \mathcal{C} \to U\mathcal{A}$. Conversely, every functor $G\colon \mathcal{C} \to U\mathcal{A}$ induces a 2-functor $\hat{G}\colon D\mathcal{C} \to \mathcal{A}$ which acts as G on objects and morphisms, and sends the trivial 2-morphisms of $D\mathcal{C}$ to the corresponding trivial 2-morphisms of $\mathcal{A}$. These constructions are clearly inverse to each other and natural in $\mathcal{C}$ and $\mathcal{A}$. Hence $D \dashv U$ is an adjunction. Moreover, we clearly have that $UD\mathcal{C} = \mathcal{C}$, for every category $\mathcal{C}$, and hence the unit of $D \dashv U$ is an identity. $\square$

2.2. **Cartesian closeness of** 2Cat**.** As mentioned in Proposition 2.1.6, the category 2Cat is complete. Then, by Remark 1.1.4, it admits a symmetric monoidal structure given by the product $\times\colon 2\mathrm{Cat} \times 2\mathrm{Cat} \to 2\mathrm{Cat}$. In this section, we show that 2Cat is cartesian closed, by constructing its internal hom 2-category. In other words, given 2-categories $\mathcal{A}$ and $\mathcal{B}$, we need to define a 2-category whose objects are the 2-functors from $\mathcal{A}$ to $\mathcal{B}$. The morphisms in this 2-category are defined as follows.

Definition 2.2.1. Let $F, G\colon \mathcal{A} \to \mathcal{B}$ be 2-functors. A **2-natural transformation** $\varphi\colon F \Rightarrow G$ consists of a morphism $\varphi_A\colon FA \to GA$ in $\mathcal{B}$, for each object $A \in \mathcal{A}$, such that

(n1) for every 2-morphism $\alpha\colon a \Rightarrow c$ in $\mathcal{A}$, the following pasting equality holds.

$$\begin{tikzcd} FA \arrow[r, bend left, "Fa"] \arrow[r, bend right, "Fc"'] \arrow[r, phantom, "\Downarrow F\alpha"] & FC \arrow[r, "\varphi_C"] & GC \end{tikzcd} = \begin{tikzcd} FA \arrow[r, "\varphi_A"] & GA \arrow[r, bend left, "Ga"] \arrow[r, bend right, "Gc"'] \arrow[r, phantom, "\Downarrow G\alpha"] & GC \end{tikzcd}$$

Then, given two such 2-natural transformations, we can define a notion of morphisms between these, which will be the 2-morphisms of the internal hom 2-category.

Definition 2.2.2. Let $F, G\colon \mathcal{A} \to \mathcal{B}$ be 2-functors, and let $\varphi, \psi\colon F \Rightarrow G$ be 2-natural transformations. A **modification** $\mu\colon \varphi \Rrightarrow \psi$ consists of a 2-morphism $\mu_A\colon \varphi_A \Rightarrow \psi_A$ in $\mathcal{B}$, for each object $A \in \mathcal{A}$, such that

(m1) for every morphism $a\colon A \to C$ in $\mathcal{A}$, the following pasting equality holds.

$$\begin{tikzcd} FA \arrow[r, "Fa"] & FC \arrow[r, bend left, "\varphi_C"] \arrow[r, bend right, "\psi_C"'] \arrow[r, phantom, "\Downarrow \mu_C"] & GC \end{tikzcd} = \begin{tikzcd} FA \arrow[r, bend left, "\varphi_A"] \arrow[r, bend right, "\psi_A"'] \arrow[r, phantom, "\Downarrow \mu_A"] & GA \arrow[r, "Ga"] & GC \end{tikzcd}$$

With these notions of morphisms and 2-morphisms between 2-functors, we can introduce a 2-category of 2-functors.

Definition 2.2.3. Let $\mathcal{A}$ and $\mathcal{B}$ be 2-categories. The **hom 2-category** $[\mathcal{A}, \mathcal{B}]_2$ is the 2-category whose

(i) objects are 2-functors from $\mathcal{A}$ to $\mathcal{B}$,
(ii) morphisms are 2-natural transformations, and
(iii) 2-morphisms are modifications.

We show that this gives an internal hom in 2Cat with respect to the cartesian structure.

Proposition 2.2.4. *The category* 2Cat *is cartesian closed with internal hom given by* $[-,-]_2$*, i.e., for every tuple of* 2*-categories* $\mathcal{A}$*,* $\mathcal{B}$*, and* $\mathcal{C}$*, we have an isomorphism*

$$\mathrm{2Cat}(\mathcal{A},[\mathcal{B},\mathcal{C}]_2)\cong \mathrm{2Cat}(\mathcal{A}\times\mathcal{B},\mathcal{C})$$

natural in $\mathcal{A}$*,* $\mathcal{B}$*, and* $\mathcal{C}$*. Furthermore, this isomorphism extends to an isomorphism of* 2*-categories*

$$[\mathcal{A},[\mathcal{B},\mathcal{C}]_2]_2\cong[\mathcal{A}\times\mathcal{B},\mathcal{C}]_2$$

natural in $\mathcal{A}$*,* $\mathcal{B}$*, and* $\mathcal{C}$*.*

Proof. By taking $\mathcal{A}$ to be the 2-category $[0]$ free on an object, $[1]$ free on a morphism, and $\Sigma[1]$ free on a 2-morphism, a 2-functor $\mathcal{A}\times\mathcal{B}\to\mathcal{C}$ is precisely a 2-functor $\mathcal{B}\to\mathcal{C}$, a 2-natural transformation, and a modification, respectively. This shows that there is a canonical isomorphism $\mathrm{2Cat}(\mathcal{A},[\mathcal{B},\mathcal{C}]_2)\cong\mathrm{2Cat}(\mathcal{A}\times\mathcal{B},\mathcal{C})$ when $\mathcal{A}\in\{[0],[1],\Sigma[1]\}$. Since every 2-category can be obtained as colimit of the 2-categories $[0]$, $[1]$, and $\Sigma[1]$ by Remark 2.1.9, products in 2Cat commutes with colimits, and $\mathrm{2Cat}(-,-)$ sends colimits in the first variable to limits, the result for a general 2-category $\mathcal{A}$ follows. The second statement follows from Proposition 1.1.3. □

2.3. **Gray tensor product for 2-categories.** The notion of 2-natural transformations can be weakened by requiring the naturality conditions to hold only up to 2-isomorphisms, rather than strictly. This yields the notion of a *pseudo-natural transformation*, and modifications extend to this setting. Hence, we also have a 2-category of 2-functors, whose morphisms are pseudo-natural transformations. This corresponds to the internal hom for another symmetric monoidal structure on 2Cat, called the *Gray tensor product.* In this section, we introduce these pseudo-homs and then show that there is a closed symmetric monoidal structure on 2Cat with respect to these.

Let us first introduce the "pseudo" notion of transformation between 2-functors.

Definition 2.3.1. Let $F,G\colon\mathcal{A}\to\mathcal{B}$ be 2-functors. A **pseudo-natural transformation** $\varphi\colon F\Rightarrow G$ consists of

(i) a morphism $\varphi_A\colon FA\to GA$ in $\mathcal{B}$, for each object $A\in\mathcal{A}$,
(ii) a 2-isomorphism φ_a in $\mathcal{B}$

$$\begin{tikzcd} FA \arrow[r, "\varphi_A"] \arrow[d, "Fa"'] \arrow[dr, phantom, "\cong\swarrow\varphi_a"] & GA \arrow[d, "Ga"] \\ FC \arrow[r, "\varphi_C"'] & GC \end{tikzcd},$$

for each morphism $a\colon A\to C$ in $\mathcal{A}$,

such that

(pn1) for every object $A\in\mathcal{A}$, we have $\varphi_{\mathrm{id}_A}=\mathrm{id}_{\varphi_A}$,
(pn2) for every pair of composable morphisms $a\colon A\to C$ and $c\colon C\to E$ in $\mathcal{A}$, the following pasting equality holds,

$$
\begin{tikzcd}
FA \arrow[r, "\varphi_A"] \arrow[d, "Fa"'] \arrow[dr, phantom, "\cong \swArrow \varphi_a"] & GA \arrow[d, "Ga"] \\
FC \arrow[r, "\varphi_C"'] \arrow[d, "Fa"'] \arrow[dr, phantom, "\cong \swArrow \varphi_c"] & GC \arrow[d, "Ga"] \\
FE \arrow[r, "\varphi_E"'] & GE
\end{tikzcd}
=
\begin{tikzcd}
FA \arrow[r, "\varphi_A"] \arrow[d, "F(ca)"'] \arrow[dr, phantom, "\cong \swArrow \varphi_{ca}"] & GA \arrow[d, "G(ca)"] \\
FE \arrow[r, "\varphi_E"'] & GE
\end{tikzcd}
$$

(pn3) for every 2-morphism $\alpha\colon a \Rightarrow c$ in $\mathcal{A}$, the following pasting equality holds.

$$
\begin{tikzcd}
FA \arrow[r, "\varphi_A"] \arrow[d, bend right=50, "Fc"'] \arrow[d, bend left=50, "Fa"] \arrow[d, phantom, "\overset{F\alpha}{\Leftarrow}"] \arrow[dr, phantom, "\cong \swArrow \varphi_a"] & GA \arrow[d, "Ga"] \\
FC \arrow[r, "\varphi_C"'] & GC
\end{tikzcd}
=
\begin{tikzcd}
FA \arrow[r, "\varphi_A"] \arrow[d, "Fc"'] \arrow[dr, phantom, "\cong \swArrow \varphi_c"] & GA \arrow[d, bend right=50, "Gc"'] \arrow[d, bend left=50, "Ga"] \arrow[d, phantom, "\overset{G\alpha}{\Leftarrow}"] \\
FC \arrow[r, "\varphi_C"'] & GC
\end{tikzcd}
$$

As in the case of 2-natural transformations, there is a notion of morphisms between pseudo-natural transformations, defined as follows.

Definition 2.3.2. Let $F, G\colon \mathcal{A} \to \mathcal{B}$ be 2-functors, and let $\varphi, \psi\colon F \Rightarrow G$ be pseudo-natural transformations. Then, a **modification** $\mu\colon \varphi \Rrightarrow \psi$ consists of a 2-morphism $\mu_A\colon \varphi_A \Rightarrow \psi_A$ in $\mathcal{B}$, for each object $A \in \mathcal{A}$, such that

(mp1) for every morphism $a\colon A \to C$ in $\mathcal{A}$, the following pasting equality holds.

$$
\begin{tikzcd}
FA \arrow[r, "\varphi_A"] \arrow[d, "Fa"'] \arrow[dr, phantom, "\cong \swArrow \varphi_a"] & GA \arrow[d, "Ga"] \\
FC \arrow[r, bend left, "\varphi_C"] \arrow[r, bend right, "\psi_C"'] \arrow[r, phantom, "\Downarrow \mu_C"] & GC
\end{tikzcd}
=
\begin{tikzcd}
FA \arrow[r, bend left, "\varphi_A"] \arrow[r, bend right, "\psi_A"'] \arrow[r, phantom, "\Downarrow \mu_A"] \arrow[d, "Fa"'] \arrow[dr, phantom, "\cong \swArrow \psi_a"] & GA \arrow[d, "Ga"] \\
FC \arrow[r, "\psi_C"'] & GC
\end{tikzcd}
$$

The 2-functors, pseudo-natural transformations, and modifications form a 2-category.

Definition 2.3.3. Let $\mathcal{A}$ and $\mathcal{B}$ be 2-categories. The **pseudo-hom 2-category** $[\mathcal{A}, \mathcal{B}]_{2,\mathrm{ps}}$ is the 2-category whose

(i) objects are double functors from $\mathcal{A}$ to $\mathcal{B}$,
(ii) morphisms are pseudo-natural transformations, and
(iii) 2-morphisms are modifications.

Using these pseudo-homs, we introduce a closed symmetric monoidal structure on 2Cat, called the *Gray tensor product*. A description of the Gray tensor product of two 2-categories can be found in Description 6.3.2.

Proposition 2.3.4. *There is a closed symmetric monoidal product* $\otimes_2$ *on* 2Cat*, called the* ***Gray tensor product****, such that, for every tuple of* 2*-categories* $\mathcal{A}$*,* $\mathcal{B}$*, and* $\mathcal{C}$*, we have an isomorphism*

$$\mathrm{2Cat}(\mathcal{A}, [\mathcal{B}, \mathcal{C}]_{2,\mathrm{ps}}) \cong \mathrm{2Cat}(\mathcal{A} \otimes_2 \mathcal{B}, \mathcal{C})$$

natural in $\mathcal{A}$*,* $\mathcal{B}$*, and* $\mathcal{C}$*. Furthermore, this isomorphism extends to an isomorphism of* 2*-categories*

$$[\mathcal{A}, [\mathcal{B}, \mathcal{C}]_{2,\mathrm{ps}}]_{2,\mathrm{ps}} \cong [\mathcal{A} \otimes_2 \mathcal{B}, \mathcal{C}]_{2,\mathrm{ps}}$$

natural in $\mathcal{A}$*,* $\mathcal{B}$*, and* $\mathcal{C}$*.*

Proof. The first part of the result follows from [Gra74]. The second statement follows from Proposition 1.1.3. □

2.4. **Equivalences in a 2-category.** Finally, in this last section on 2-categories, we introduce the notion of equivalences in a 2-category. This gives a weaker notion of invertibility for morphisms. Indeed, while an isomorphism has an inverse such that their composites are equal to the identity, an equivalence has an inverse such that their composites are only related by a 2-isomorphism to the identity.

Definition 2.4.1. Let $\mathcal{A}$ be a 2-category. A morphism $a\colon A \to C$ in $\mathcal{A}$ is an **equivalence** if there is a tuple (a, c, η, ϵ) with $c\colon C \to A$ a morphism in $\mathcal{A}$, and $\eta\colon \mathrm{id}_A \Rightarrow ca$, $\epsilon\colon ac \Rightarrow \mathrm{id}_C$ two 2-isomorphisms in $\mathcal{A}$.

We can further require that the 2-isomorphisms in the data of an equivalence satisfy the *triangle identities*, which are a generalization of the triangle identities satisfied by an adjunction between categories.

Definition 2.4.2. Let $\mathcal{A}$ be a 2-category. A morphism $a\colon A \to C$ in $\mathcal{A}$ is an **adjoint equivalence** if it is an equivalence (a, c, η, ϵ) and the 2-isomorphisms η and ϵ further satisfy the following triangle identities.

$$
\begin{tikzcd}
A \arrow[d, "a"'] \arrow[dr, equal] \arrow[dr, phantom, shift right=2, "\overset{\cong}{\swarrow}\eta"] & \\
C \arrow[r, "c"] \arrow[dr, equal] \arrow[dr, phantom, shift left=2, "\overset{\cong}{\swarrow}\epsilon"] & A \arrow[d, "a"] \\
& C
\end{tikzcd}
=
\begin{tikzcd}
A \arrow[d, "a"] \\
C
\end{tikzcd}
\qquad
\begin{tikzcd}
C \arrow[r, "c"] \arrow[dr, equal] \arrow[dr, phantom, shift left=2, "\overset{\cong}{\swarrow}\epsilon"] & A \arrow[d, "a"] \arrow[dr, equal] \arrow[dr, phantom, shift right=2, "\overset{\cong}{\swarrow}\eta"] & \\
& C \arrow[r, "c"'] & A
\end{tikzcd}
=
C \xrightarrow{c} A
$$

Notation 2.4.3. We often write $a\colon A \xrightarrow{\simeq} C$ to highlight the fact that a morphism a is an (adjoint) equivalence.

This "adjoint" version of equivalence is very useful in practice to make computations. Luckily, every equivalence can be promoted to an adjoint one, and hence we can always assume that a morphism in a 2-category which is an equivalence comes with adjoint equivalence data.

Proposition 2.4.4. *Every equivalence (a, c, η, ϵ) in a 2-category $\mathcal{A}$ can be promoted to an adjoint equivalence (a, c, η, ϵ').*

Proof. See, for example, [RV19, Proposition 2.1.12]. □

Finally, we show that an equivalence in a pseudo-hom 2-category is precisely a pseudo-natural transformation whose components are equivalences. We do not prove this result here, since it is a direct consequence of the analogous result holding for double categories (see Proposition 3.6.10).

Proposition 2.4.5. *Let $F, G\colon \mathcal{A} \to \mathcal{B}$ be 2-functors. A pseudo-natural transformation $\varphi\colon F \Rightarrow G$ is an equivalence in $[\mathcal{A}, \mathcal{B}]_{2,\mathrm{ps}}$ if and only if, for every object $A \in \mathcal{A}$, its morphism component $\varphi_A\colon FA \to GA$ is an equivalence in $\mathcal{B}$.*

Proof. This can be obtained from Proposition 3.6.10 by considering φ as a horizontal pseudo-natural transformation between the horizontal double functors $\mathbb{H}F$ and $\mathbb{H}G$. In particular, since all vertical morphisms in $\mathbb{H}\mathcal{A}$ are trivial, the square components of φ are vertical identity squares, by (hn1) of Definition 3.2.1. □

Definition 2.4.6. Let $F, G\colon \mathcal{A} \to \mathcal{B}$ be 2-functors. A pseudo-natural transformation $\varphi\colon F \Rightarrow G$ which satisfies the conditions of Proposition 2.4.5 is called a **pseudo-natural equivalence**. We say that φ is a **pseudo-natural adjoint equivalence** if it is an adjoint equivalence in the pseudo-hom 2-category $[\mathcal{A}, \mathcal{B}]_{2,\mathrm{ps}}$, or equivalently, if, for every object $A \in \mathcal{A}$, its morphism component $\varphi_A\colon FA \to GA$ is an adjoint equivalence in $\mathcal{B}$.

3. Double categories

We now introduce the other kind of 2-dimensional categories of interest in this thesis. While 2-categories have morphisms between the objects and 2-morphisms between the morphisms, a double category has two kinds of morphisms between the objects, called *horizontal* and *vertical* morphisms, and its 2-morphisms sit in a square with two horizontal boundaries and two vertical boundaries, and are therefore called *squares*. In Section 3.1, we first introduce the category DblCat of double categories and double functors. We also explain how double categories can equivalently be understood as internal categories to Cat, which allows us to adapt the proof of local presentability for Cat to DblCat. Furthermore, every 2-category can be seen as a horizontal double category with only trivial vertical morphisms, and hence it corresponds to an internal category to Cat with discrete category of objects. This point of view on 2-categories and double categories will be useful to get ∞-analogues of these notions.

In Section 3.2, we show that DblCat is cartesian closed by constructing internal hom double categories. The construction is analogous to the one for 2Cat, except that we now have two kinds of natural transformations between double functors: *horizontal* and *vertical natural transformations*. By relaxing the naturality conditions, as it was done in the 2-categorical case, we obtain *pseudo*-versions of these horizontal and vertical natural transformations, which allow us to construct a *pseudo-hom* double category. These correspond to the internal homs for another symmetric monoidal structure on DblCat, also called the *Gray tensor product*, introduced by Böhm in [Böh19].

As mentioned above, every 2-category $\mathcal{A}$ can be seen as a horizontal double category $\mathbb{H}\mathcal{A}$ and this defines a full embedding functor $\mathbb{H}\colon 2\mathrm{Cat} \to \mathrm{DblCat}$. In Section 3.4, we introduce this horizontal embedding and show that it has both adjoints. In particular, its right adjoint is given by extracting from a double category $\mathbb{A}$ its *underlying horizontal 2-category* $\mathbf{H}\mathbb{A}$ with the same objects as $\mathbb{A}$, morphisms the horizontal morphisms of $\mathbb{A}$, and 2-morphisms the squares in $\mathbb{A}$ with trivial vertical boundaries. Since this functor forgets the whole vertical structure of $\mathbb{A}$, we define another functor $\mathcal{V}\colon \mathrm{DblCat} \to 2\mathrm{Cat}$ which extracts from a double category $\mathbb{A}$ a 2-category $\mathcal{V}\mathbb{A}$, whose objects are the vertical morphisms of $\mathbb{A}$ and whose morphisms are the squares of $\mathbb{A}$. Put together, the two functors $\mathbf{H}$ and $\mathcal{V}$ recover most of the structure of a double category (except for the vertical composition of vertical morphisms), and they are used in Section 7 to construct the first model structure on DblCat from Lack's model structure on 2Cat. Note that there are also dual versions of these functors given by interchanging the horizontal and vertical directions. Finally, we also introduce a more homotopical version of the horizontal embedding $\mathbb{H}$, given by the functor $\mathbb{H}^{\simeq}\colon 2\mathrm{Cat} \to \mathrm{DblCat}$, which sends a 2-category $\mathcal{A}$ to the double category $\mathbb{H}^{\simeq}\mathcal{A}$, whose underlying horizontal category is still $\mathcal{A}$, and whose vertical morphisms are now given by the adjoint equivalences in $\mathcal{A}$. This functor will be the functor considered when looking at the homotopical inclusion of 2Cat into DblCat.

Using the adjunction $\mathbb{H} \dashv \mathbf{H}$, we induce in Section 3.5 a 2Cat-enrichment on DblCat from its monoidal structure given by the Gray tensor product. The hom 2-categories of DblCat are defined to be the underlying horizontal 2-categories of the pseudo-hom double categories, and this enrichment is tensored and cotensored with tensoring functor given by restricting the Gray tensor product on DblCat in one variable along the horizontal embedding $\mathbb{H}\colon 2\mathrm{Cat} \to \mathrm{DblCat}$. This enrichment is used to show that the two model structures on DblCat constructed in this thesis are enriched over 2Cat.

Finally, in Section 3.6, we introduce notions of weak invertibility for horizontal morphisms and squares. These are analogous to the notion of equivalences in a 2-category and can actually be defined as follows. A horizontal morphism in a double category $\mathbb{A}$ is a *horizontal equivalence* if it is an equivalence in the underlying horizontal 2-category $\mathbf{H}\mathbb{A}$, and a square is *weakly horizontally invertible* if it is an equivalence in the 2-category $\mathcal{V}\mathbb{A}$ of vertical morphisms, squares, and 2-morphisms as given in Definition 3.4.9. These notions were recently introduced independently by the author, Sarazola, and Verdugo in [MSV20a], and by Grandis and Paré in [GP19], whose terminology for the weakly horizontally squares is that of *equivalence cells*. With this terminology, we introduce *weakly horizontally invariant* double categories, which are such that every vertical morphism in this double category can be transferred along horizontal equivalences through a weakly horizontally invertible square. This notion was first introduced in by the author, Sarazola, and Verdugo in [MSV20b], and is a weaker version of the horizontally invariant double categories defined by Grandis and Paré in [GP99, §2.4]. We also prove here some technical results about weakly horizontally invertible squares, which will be useful in the rest of the thesis. These results appear in the appendix of the paper [Mos20] by the author.

3.1. **The category** DblCat**.**

As mentioned above, a double category has two kinds of morphisms between objects – called *horizontal* and *vertical* morphisms – and its 2-morphisms are called *squares*. It corresponds to an internal category to Cat, the category of categories and functors. It generalizes the concept of 2-categories, in the sense that every 2-category can be seen as a *horizontal* double category, with only trivial vertical morphisms. In this section, we introduce the category DblCat of double categories and double functors, and we explain how this category corresponds to the category of internal categories and internal functors to Cat. We also show that every 2-category gives rise to a horizontal category, and hence corresponds to an internal category to Cat with discrete category of objects.

Let us first introduce double categories.

Definition 3.1.1. A **double category** $\mathbb{A}$ consists of

(i) objects $A, A', C, C', \ldots$,
(ii) horizontal morphisms $a\colon A \to C$ between objects A, C with a horizontal identity morphism $\mathrm{id}_A\colon A \to A$ at each object A,
(iii) vertical morphisms $u\colon A \to\!\!\!\!\!\!\bullet\;\; A'$ between objects A, A' with a vertical identity morphism $e_A\colon A \to\!\!\!\!\!\!\bullet\;\; A$ at each object A,
(iv) squares $\alpha\colon (u \,{}^{a}_{a'}\, w)$ of the form

$$\begin{tikzcd} A \arrow[r, "a"] \arrow[d, "u"', "\bullet" marking] \arrow[dr, phantom, "\alpha"] & C \arrow[d, "w", "\bullet" marking] \\ A' \arrow[r, "a'"'] & C' \end{tikzcd}$$

between horizontal morphisms $a\colon A \to C$ and $a'\colon A' \to C'$ and vertical morphisms $u\colon A \to\!\!\!\!\!\!\bullet\;\; A'$ and $w\colon C \to\!\!\!\!\!\!\bullet\;\; C'$ with a vertical identity square $e_a\colon (e_A \,{}^{a}_{a}\, e_C)$ for each horizontal morphism $a\colon A \to C$, and a horizontal identity square $\mathrm{id}_u\colon (u \,{}^{e_A}_{e_{A'}}\, u)$ for each vertical morphism $u\colon A \to\!\!\!\!\!\!\bullet\;\; A'$, such that $\square_A = \mathrm{id}_{e_A} = e_{\mathrm{id}_A}$ for all objects A,

$$
\begin{tikzcd}
A \arrow[r, "a"] \arrow[d, equal, "\bullet" marking] \arrow[dr, phantom, "e_a"] & C \arrow[d, equal, "\bullet" marking] \\
A \arrow[r, "a"'] & C
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[r, equal] \arrow[d, "u"', "\bullet" marking] \arrow[dr, phantom, "\mathrm{id}_u"] & A \arrow[d, "u", "\bullet" marking] \\
A' \arrow[r, equal] & A'
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[r, equal] \arrow[d, equal, "\bullet" marking] \arrow[dr, phantom, "\square_A"] & A \arrow[d, equal, "\bullet" marking] \\
A \arrow[r, equal] & A
\end{tikzcd}
$$

(v) an associative and unital horizontal composition law for horizontal morphisms, and for squares along their vertical boundaries,
(vi) an associative and unital vertical composition law for vertical morphisms, and for squares along their horizontal boundaries,

such that horizontal and vertical compositions for squares satisfy the interchange law.

Remark 3.1.2. A double category $\mathbb{A}$ can equivalently be defined as an internal category to Cat, i.e., as a diagram in Cat of the form

$$
\begin{tikzcd}
\mathbb{A}_0 \arrow[r, "i" description] & \mathbb{A}_1 \arrow[l, shift right=2, "s"'] \arrow[l, shift left=2, "t"] & \mathbb{A}_1 \times_{\mathbb{A}_0} \mathbb{A}_1 \arrow[l, shift right=2, "\pi_0"'] \arrow[l, "c" description] \arrow[l, shift left=2, "\pi_1"]
\end{tikzcd}
$$

satisfying the relations of an internal category. Our convention is to look at the category $\mathbb{A}_0$ as the category of objects and vertical morphisms, and at the category $\mathbb{A}_1$ as the category of horizontal morphisms and squares.

Since a category is itself an internal category to Set, we can also consider a double category as a diagram in Set of the form

$$
\begin{tikzcd}
\mathbb{A}_{0,0} \arrow[r] \arrow[d] & \mathbb{A}_{1,0} \arrow[l, shift right=2] \arrow[l, shift left=2] \arrow[d] & \mathbb{A}_{1,0} \times_{\mathbb{A}_{0,0}} \mathbb{A}_{1,0} \arrow[l, shift right=2] \arrow[l] \arrow[l, shift left=2] \arrow[d] \\
\mathbb{A}_{0,1} \arrow[u, shift left=2] \arrow[u, shift right=2] \arrow[r] & \mathbb{A}_{1,1} \arrow[u, shift left=2] \arrow[u, shift right=2] \arrow[l, shift right=2] \arrow[l, shift left=2] & \mathbb{A}_{1,1} \times_{\mathbb{A}_{0,1}} \mathbb{A}_{1,1} \arrow[u, shift left=2] \arrow[u, shift right=2] \arrow[l, shift right=2] \arrow[l] \arrow[l, shift left=2] \\
\mathbb{A}_{0,1} \times_{\mathbb{A}_{0,0}} \mathbb{A}_{0,1} \arrow[u, shift left=2] \arrow[u] \arrow[u, shift right=2] \arrow[r] & \mathbb{A}_{1,1} \times_{\mathbb{A}_{1,0}} \mathbb{A}_{1,1} \arrow[u, shift left=2] \arrow[u] \arrow[u, shift right=2] \arrow[l, shift right=2] \arrow[l, shift left=2] & \mathrm{Grid}_{2\times 2}(\mathbb{A}_{1,1}) \arrow[u, shift left=2] \arrow[u] \arrow[u, shift right=2] \arrow[l, shift right=2] \arrow[l] \arrow[l, shift left=2]
\end{tikzcd} ,
$$

where $\mathbb{A}_{0,0}$ is the set of objects, $\mathbb{A}_{0,1}$ the set of vertical morphisms, $\mathbb{A}_{1,0}$ the set of horizontal morphisms, and $\mathbb{A}_{1,1}$ the set of squares.

There is a notion of morphism between double categories which consists of assignments on objects, horizontal morphisms, vertical morphisms, and squares that preserve all the double categorical structure.

Definition 3.1.3. Let $\mathbb{A}$ and $\mathbb{B}$ be double categories. A **double functor** $F\colon \mathbb{A} \to \mathbb{B}$ sends

(i) each object $A \in \mathbb{A}$ to an object $FA \in \mathbb{B}$,
(ii) each horizontal morphism $a\colon A \to C$ in $\mathbb{A}$ to a horizontal morphism $Fa\colon FA \to FC$ in $\mathbb{B}$,
(iii) each vertical morphism $u\colon A \twoheadrightarrow A'$ in $\mathbb{A}$ to a vertical morphism $Fu\colon FA \twoheadrightarrow FA'$ in $\mathbb{B}$,
(iv) each square $\alpha\colon (u \,{}^{a}_{a'}\, w)$ in $\mathbb{A}$ to a square $F\alpha\colon (Fu \,{}^{Fa}_{Fa'}\, Fw)$ in $\mathbb{B}$,

in such a way that F preserves horizontal and vertical compositions and identities.

Remark 3.1.4. A double functor $F\colon \mathbb{A} \to \mathbb{B}$ can equivalently be defined as an internal functor to Cat, i.e., it consists of two functors $F_0\colon \mathbb{A}_0 \to \mathbb{B}_0$ and $F_1\colon \mathbb{A}_1 \to \mathbb{B}_1$ making the following diagram commute.

$$
\begin{tikzcd}
\mathbb{A}_0 \arrow[r, "i" description] \arrow[d, "F_0"'] & \mathbb{A}_1 \arrow[l, shift right=2, "s"'] \arrow[l, shift left=2, "t"] \arrow[d, "F_1"'] & \mathbb{A}_1 \times_{\mathbb{A}_0} \mathbb{A}_1 \arrow[l, "c" description] \arrow[l, shift right=2, "\pi_0"'] \arrow[l, shift left=2, "\pi_1"] \arrow[d, "F_1 \times_{F_0} F_1"] \\
\mathbb{B}_0 \arrow[r, "i" description] & \mathbb{B}_1 \arrow[l, shift right=2, "s"'] \arrow[l, shift left=2, "t"] & \mathbb{B}_1 \times_{\mathbb{B}_0} \mathbb{B}_1 \arrow[l, "c" description] \arrow[l, shift right=2, "\pi_0"'] \arrow[l, shift left=2, "\pi_1"]
\end{tikzcd}
$$

Double categories and double functors form a category.

Notation 3.1.5. We write DblCat for the category of double categories and double functors. By Remarks 3.1.2 and 3.1.4, it is the same as the category Cat(Cat) of internal categories and internal functors to Cat.

The category DblCat also satisfies the conditions of "smallness" introduced in Definition 1.2.2. The proof works as in Proposition 1.2.11, where we proved that Cat is locally presentable, with the changes mentioned below.

Proposition 3.1.6. *The category* DblCat *is locally presentable and, in particular, it is both complete and cocomplete.*

Proof. Recall the category Δ_3 from Proposition 1.2.11. In the case of DblCat, we can construct a limit-sketch $\mathcal{L}' = (\mathcal{A}', \mathbf{D}', \sigma')$, where $\mathcal{A}' = \Delta_3 \times \Delta_3$. The diagrams in $\mathbf{D}'$ are given by the diagrams $D_1 \times [j]$, $[i] \times D_1$, $D_2 \times [j]$, and $[i] \times D_2$, for all $0 \leq i, j \leq 3$, where D_1 and D_2 are the diagrams in Δ_3 as given in the proof of Proposition 1.2.11. Similarly, the assignment σ' is induced by the assignment σ of Proposition 1.2.11. Then the proof that DblCat is equivalent to $\mathrm{Mod}(\mathcal{L}')$ works as in Proposition 1.2.11, using the characterization of double categories presented in Remark 3.1.2. The second part of the statement follows from the definition of a locally presentable category and from Proposition 1.2.9. ☐

As mentioned in the introduction, every 2-category can be seen as a horizontal double category. Dually, every 2-category also gives rise to a *vertical* double category with only trivial horizontal morphisms. In particular, since every category is a locally discrete 2-category, it can also be seen as a horizontal or vertical double category. As we explain below, the horizontal embedding of Cat into DblCat is more natural since it corresponds to the embedding of internal categories to Set into internal categories to Cat, induced by the inclusion Set $\to$ Cat, which sends a set to the corresponding discrete category.

Definition 3.1.7. Let $\mathcal{A}$ be a 2-category. We define its **associated horizontal double category** $\mathbb{H}\mathcal{A}$ to be the double category with the same objects as $\mathcal{A}$, horizontal morphisms the morphisms of $\mathcal{A}$, only trivial vertical morphisms, and squares

$$
\begin{tikzcd}
A \arrow[r, "a"] \arrow[d, equal, "\bullet" marking] \arrow[dr, phantom, "\alpha"] & B \arrow[d, equal, "\bullet" marking] \\
A \arrow[r, "c"'] & B
\end{tikzcd}
$$

given by the 2-morphisms $\alpha \colon a \Rightarrow c$ of $\mathcal{A}$. Dually, its **associated vertical double category** $\mathbb{V}\mathcal{A}$ has the same objects as $\mathcal{A}$, only trivial horizontal morphisms, vertical morphisms the morphisms of $\mathcal{A}$, and squares

$$
\begin{tikzcd}
A \arrow[r, equal] \arrow[d, "a"', "\bullet" marking] \arrow[dr, phantom, "\alpha"] & A \arrow[d, "c", "\bullet" marking] \\
A' \arrow[r, equal] & A'
\end{tikzcd}
$$

given by the 2-morphisms $\alpha \colon a \Rightarrow c$ of $\mathcal{A}$.

Remark 3.1.8. In particular, by considering the double category $\mathbb{H}\mathcal{A}$ associated to a 2-category $\mathcal{A}$ and by Remark 3.1.2, we can interpret the 2-category $\mathcal{A}$ as an internal category to Cat of the form

$$\mathcal{A}_0 \underset{t}{\overset{s}{\leftleftarrows}} \xrightarrow{i} \mathcal{A}_1 \underset{\pi_1}{\overset{\pi_0}{\leftleftarrows}} \xleftarrow{c} \mathcal{A}_1 \times_{\mathcal{A}_0} \mathcal{A}_1 \,,$$

where $\mathcal{A}_0$ is the discrete category of objects in $\mathcal{A}$, and $\mathcal{A}_1$ is the category of morphisms and 2-morphisms in $\mathcal{A}$. Our convention will be to look at a 2-category as a horizontal double category.

Remark 3.1.9. Given a category $\mathcal{C}$, the associated vertical double category $\mathbb{V}D\mathcal{C}$ corresponds to the internal category to Cat constant at $\mathcal{C}$, while the associated horizontal double category $\mathbb{H}D\mathcal{C}$ corresponds to the internal category to Cat with discrete categories of objects and of horizontal morphisms.

We finally describe a set of objects which generate all double categories under colimits.

Remark 3.1.10. In Proposition 3.1.6, we have seen that DblCat is locally presentable. Recall from Notation 2.1.7 the 2-categories $[0]$ and $[1]$. We also denote by $[0] = \mathbb{H}[0]$ the terminal double category. Then, every double category is a colimit of the following double categories: the terminal double category $[0]$, the double category $\mathbb{H}[1]$ free on a horizontal morphism, the double category $\mathbb{V}[1]$ free on a vertical morphism, and the double category $\mathbb{H}[1] \times \mathbb{V}[1]$ free on a square.

Note that the product $\mathbb{H}[1] \times \mathbb{V}[1]$ indeed gives a non-trivial square even if this square is induced by two identity squares. If we denote by $f\colon 0 \to 1$ the non-trivial horizontal morphism in $\mathbb{H}[1]$ and by $u\colon 0 \,-\!\!\bullet\!\!\to 1$ the non-trivial vertical morphism in $\mathbb{V}[1]$, then the identity squares e_f at f and id_u at u induce a non-trivial square

$$\begin{tikzcd}
(0,0) \arrow[r, "{(f,0)}"] \arrow[d, "{(0,u)}"', "\bullet" marking] \arrow[dr, phantom, "{(e_f, \mathrm{id}_u)}"] & (0,1) \arrow[d, "{(1,u)}", "\bullet" marking] \\
(1,0) \arrow[r, "{(f,1)}"'] & (1,1)
\end{tikzcd}$$

in the product $\mathbb{H}[1] \times \mathbb{V}[1]$. Indeed, it can not be trivial since none of its boundaries are trivial.

3.2. **Cartesian closeness of** DblCat**.** As mentioned in Proposition 3.1.6, the category DblCat is complete. Hence, by Remark 1.1.4, it admits a symmetric monoidal structure given by the product $\times\colon \mathrm{DblCat} \times \mathrm{DblCat} \to \mathrm{DblCat}$. In this section, we show that DblCat is cartesian closed by constructing an internal hom double category, for every pair of double categories $\mathbb{A}$ and $\mathbb{B}$, whose objects are double functors from $\mathbb{A}$ to $\mathbb{B}$. The horizontal and vertical morphisms in this double category are called *horizontal* and *vertical natural transformations* and generalize the concept of 2-natural transformations. Similarly, the squares are generalization of the concept of modifications between 2-natural transformations.

We first introduce the horizontal morphisms of the internal hom double category.

Definition 3.2.1. Let $F, G\colon \mathbb{A} \to \mathbb{B}$ be double functors. A **horizontal natural transformation** $\varphi\colon F \Rightarrow G$ consists of

(i) a horizontal morphism $\varphi_A\colon FA \to GA$ in $\mathbb{B}$, for each object $A \in \mathbb{A}$,
(ii) a square φ_u in $\mathbb{B}$

$$\begin{tikzcd} FA \arrow[r, "\varphi_A"] \arrow[d, "Fu"', "\bullet" marking] \arrow[dr, phantom, "\varphi_u"] & GA \arrow[d, "Gu", "\bullet" marking] \\ FA' \arrow[r, "\varphi_{A'}"'] & GA' \end{tikzcd},$$

for each vertical morphism $u\colon A \to A'$ in $\mathbb{A}$,

such that

(hn1) for every object $A \in \mathbb{A}$, we have $\varphi_{e_A} = e_{\varphi_A} \colon (e_{FA} \,{}^{\varphi_A}_{\varphi_A}\, e_{GA})$,

(hn2) for every pair of composable vertical morphisms $u\colon A \to A'$ and $u'\colon A' \to A''$ in $\mathbb{A}$, the following pasting equality holds,

$$\begin{tikzcd} FA \arrow[r, "\varphi_A"] \arrow[d, "Fu"', "\bullet" marking] \arrow[dr, phantom, "\varphi_u"] & GA \arrow[d, "Gu", "\bullet" marking] \\ FA' \arrow[r, "\varphi_{A'}"'] \arrow[d, "Fu'"', "\bullet" marking] \arrow[dr, phantom, "\varphi_{u'}"] & GA' \arrow[d, "Gu'", "\bullet" marking] \\ FA'' \arrow[r, "\varphi_{A''}"'] & GA'' \end{tikzcd} = \begin{tikzcd} FA \arrow[r, "\varphi_A"] \arrow[d, "F(u'u)"', "\bullet" marking] \arrow[dr, phantom, "\varphi_{u'u}"] & GA \arrow[d, "G(u'u)", "\bullet" marking] \\ FA'' \arrow[r, "\varphi_{A''}"'] & GA'' \end{tikzcd}$$

(hn3) for every square $\alpha\colon (u \,{}^{a}_{a'}\, w)$ in $\mathbb{A}$, the following pasting equality holds.

$$\begin{tikzcd} FA \arrow[r, "Fa"] \arrow[d, "Fu"', "\bullet" marking] \arrow[dr, phantom, "F\alpha"] & FC \arrow[r, "\varphi_C"] \arrow[d, "Fw"', "\bullet" marking] \arrow[dr, phantom, "\varphi_w"] & GC \arrow[d, "Gw", "\bullet" marking] \\ FA' \arrow[r, "Fa'"'] & FC' \arrow[r, "\varphi_{C'}"'] & GC' \end{tikzcd} = \begin{tikzcd} FA \arrow[r, "\varphi_A"] \arrow[d, "Fu"', "\bullet" marking] \arrow[dr, phantom, "\varphi_u"] & GA \arrow[r, "Ga"] \arrow[d, "Gu", "\bullet" marking] \arrow[dr, phantom, "G\alpha"] & GC \arrow[d, "Gw", "\bullet" marking] \\ FA' \arrow[r, "\varphi_{A'}"'] & GA' \arrow[r, "Ga'"'] & GC' \end{tikzcd}$$

The vertical morphisms of the internal hom double category can be obtained by transposing all the notions above.

Definition 3.2.2. Let $F, F'\colon \mathbb{A} \to \mathbb{B}$ be double functors. A **vertical natural transformation** $\nu\colon F \Rightarrow F'$ consists of

(i) a vertical morphism $\nu_A\colon FA \to F'A$ in $\mathbb{B}$, for each object $A \in \mathbb{A}$,

(ii) a square ν_a in $\mathbb{B}$

$$\begin{tikzcd} FA \arrow[r, "Fa"] \arrow[d, "\nu_A"', "\bullet" marking] \arrow[dr, phantom, "\nu_a"] & FC \arrow[d, "\nu_C", "\bullet" marking] \\ F'A \arrow[r, "F'a"'] & F'C \end{tikzcd},$$

for each horizontal morphism $a\colon A \to C$ in $\mathbb{A}$,

such that

(vn1) for every object $A \in \mathbb{A}$, we have $\nu_{\mathrm{id}_A} = \mathrm{id}_{\nu_A} \colon (\nu_A \,{}^{\mathrm{id}_{FA}}_{\mathrm{id}_{F'A}}\, \nu_A)$,

(vn2) for every pair of composable horizontal morphisms $a\colon A \to C$ and $c\colon C \to E$ in $\mathbb{A}$, the following pasting equality holds,

$$\begin{tikzcd} FA \arrow[r, "Fa"] \arrow[d, "\nu_A"', "\bullet" marking] \arrow[dr, phantom, "\nu_a"] & FC \arrow[r, "Fc"] \arrow[d, "\nu_C"', "\bullet" marking] \arrow[dr, phantom, "\nu_c"] & FE \arrow[d, "\nu_E", "\bullet" marking] \\ F'A \arrow[r, "F'a"'] & F'C \arrow[r, "F'c"'] & F'E \end{tikzcd} = \begin{tikzcd} FA \arrow[r, "F(ca)"] \arrow[d, "\nu_A"', "\bullet" marking] \arrow[dr, phantom, "\nu_{ca}"] & FE \arrow[d, "\nu_E", "\bullet" marking] \\ F'A \arrow[r, "F'(ca)"'] & F'E \end{tikzcd}$$

(vn3) for every square $\alpha\colon (u \;{}^{a}_{a'}\; w)$ in $\mathbb{A}$, the following pasting equality holds.

$$
\begin{tikzcd}
FA \arrow[r, "Fa"] \arrow[d, "Fu"', "\bullet" marking] \arrow[dr, phantom, "F\alpha"] & FC \arrow[d, "Fw", "\bullet" marking] \\
FA' \arrow[r, "Fa'"] \arrow[d, "\nu_{A'}"', "\bullet" marking] \arrow[dr, phantom, "\nu_{a'}"] & FC' \arrow[d, "\nu_{C'}", "\bullet" marking] \\
F'A' \arrow[r, "F'a'"'] & F'C'
\end{tikzcd}
=
\begin{tikzcd}
FA \arrow[r, "Fa"] \arrow[d, "\nu_A"', "\bullet" marking] \arrow[dr, phantom, "\nu_a"] & FC \arrow[d, "\nu_C", "\bullet" marking] \\
F'A \arrow[r, "F'a"'] \arrow[d, "F'u"', "\bullet" marking] \arrow[dr, phantom, "F'\alpha"] & F'C \arrow[d, "F'w", "\bullet" marking] \\
F'A' \arrow[r, "F'a'"'] & F'C'
\end{tikzcd}
$$

Finally, a square whose horizontal boundaries are horizontal natural transformations and vertical boundaries are vertical natural transformations is called a *modification*, and is defined as follows.

Definition 3.2.3. Let $F, F', G, G'\colon \mathbb{A} \to \mathbb{B}$ be double functors, let $\varphi\colon F \Rightarrow G$ and $\varphi'\colon F' \Rightarrow G'$ be horizontal natural transformations, and let $\nu\colon F \Rightarrow\!\!\!\bullet\; F'$ and $\xi\colon G \Rightarrow\!\!\!\bullet\; G'$ be vertical natural transformations. A **modification** $\mu\colon (\nu \;{}^{\varphi}_{\varphi'}\; \xi)$ in a square as below left consists of a square $\mu_A\colon (\nu_A \;{}^{\varphi_A}_{\varphi'_A}\; \xi_A)$ in $\mathbb{B}$ as below right, for each object $A \in \mathbb{A}$,

$$
\begin{tikzcd}
F \arrow[r, Rightarrow, "\varphi"] \arrow[d, Rightarrow, "\nu"', "\bullet" marking] \arrow[dr, phantom, "\mu"] & G \arrow[d, Rightarrow, "\xi", "\bullet" marking] \\
F' \arrow[r, Rightarrow, "\varphi'"'] & G'
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
FA \arrow[r, "\varphi_A"] \arrow[d, "\nu_A"', "\bullet" marking] \arrow[dr, phantom, "\mu_A"] & GA \arrow[d, "\xi_A", "\bullet" marking] \\
F'A \arrow[r, "\varphi'_A"'] & G'A
\end{tikzcd}
$$

such that

(dm1) for every vertical morphism $u\colon A \to\!\!\!\bullet\; A'$ in $\mathbb{A}$, the following pasting equality holds,

$$
\begin{tikzcd}
FA \arrow[r, "\varphi_A"] \arrow[d, "Fu"', "\bullet" marking] \arrow[dr, phantom, "\varphi_u"] & GA \arrow[d, "Gu", "\bullet" marking] \\
FA' \arrow[r, "\varphi_{A'}"] \arrow[d, "\nu_{A'}"', "\bullet" marking] \arrow[dr, phantom, "\mu_{A'}"] & GA' \arrow[d, "\xi_{A'}"] \\
F'A' \arrow[r, "\varphi'_{A'}"'] & G'A'
\end{tikzcd}
=
\begin{tikzcd}
FA \arrow[r, "\varphi_A"] \arrow[d, "\nu_A"', "\bullet" marking] \arrow[dr, phantom, "\mu_A"] & GA \arrow[d, "\xi_A", "\bullet" marking] \\
F'A \arrow[r, "\varphi'_A"'] \arrow[d, "F'u"', "\bullet" marking] \arrow[dr, phantom, "\varphi'_u"] & G'A \arrow[d, "G'u", "\bullet" marking] \\
F'A' \arrow[r, "\varphi'_{A'}"'] & G'A'
\end{tikzcd}
$$

(dm2) for every horizontal morphism $a\colon A \to C$ in $\mathbb{A}$, the following pasting equality holds.

$$
\begin{tikzcd}
FA \arrow[r, "Fa"] \arrow[d, "\nu_A"', "\bullet" marking] \arrow[dr, phantom, "\nu_a"] & FC \arrow[r, "\varphi_C"] \arrow[d, "\nu_C"', "\bullet" marking] \arrow[dr, phantom, "\mu_C"] & GC \arrow[d, "\xi_C", "\bullet" marking] \\
F'A \arrow[r, "F'a"'] & F'C \arrow[r, "\varphi'_C"'] & G'C
\end{tikzcd}
=
\begin{tikzcd}
FA \arrow[r, "\varphi_A"] \arrow[d, "\nu_A"', "\bullet" marking] \arrow[dr, phantom, "\mu_A"] & GA \arrow[r, "Ga"] \arrow[d, "\xi_A", "\bullet" marking] \arrow[dr, phantom, "\xi_a"] & GC \arrow[d, "\xi_C", "\bullet" marking] \\
F'A \arrow[r, "\varphi'_A"'] & G'A \arrow[r, "G'a"'] & G'C
\end{tikzcd}
$$

All together, they form a double category of double functors.

Definition 3.2.4. Let $\mathbb{A}$ and $\mathbb{B}$ be double categories. The **hom double category** $[\mathbb{A}, \mathbb{B}]$ is the double category whose

(i) objects are double functors from $\mathbb{A}$ to $\mathbb{B}$,
(ii) horizontal morphisms are horizontal natural transformations,
(iii) vertical morphisms are vertical natural transformations, and
(iv) squares are modifications.

This hom double category is the internal hom for the cartesian structure on DblCat, which shows that DblCat is cartesian closed.

Proposition 3.2.5. *The category* DblCat *is cartesian closed with internal hom given by* $[-,-]$*, i.e., for every tuple of double categories* $\mathbb{A}$*,* $\mathbb{B}$*, and* $\mathbb{C}$*, we have an isomorphism*

$$\mathrm{DblCat}(\mathbb{A}, [\mathbb{B}, \mathbb{C}]) \cong \mathrm{DblCat}(\mathbb{A} \times \mathbb{B}, \mathbb{C})$$

natural in $\mathbb{A}$*,* $\mathbb{B}$*, and* $\mathbb{C}$*. Furthermore, this isomorphism extends to an isomorphism of double categories*

$$[\mathbb{A}, [\mathbb{B}, \mathbb{C}]] \cong [\mathbb{A} \times \mathbb{B}, \mathbb{C}]$$

natural in $\mathbb{A}$*,* $\mathbb{B}$*, and* $\mathbb{C}$*.*

Proof. By taking $\mathbb{A}$ to be the double category $[0]$ free on an object, $\mathbb{H}[1]$ free on a horizontal morphism, $\mathbb{V}[1]$ free on a vertical morphism, and $\mathbb{H}[1] \times \mathbb{V}[1]$ free on a square, a double functor $\mathbb{A} \times \mathbb{B} \to \mathbb{C}$ is precisely a double functor $\mathbb{B} \to \mathbb{C}$, a horizontal natural transformation, a vertical natural transformation, and a modification, respectively. This shows that there is a canonical isomorphism $\mathrm{DblCat}(\mathbb{A}, [\mathbb{B}, \mathbb{C}]) \cong \mathrm{DblCat}(\mathbb{A} \times \mathbb{B}, \mathbb{C})$ when $\mathbb{A} \in \{[0], \mathbb{H}[1], \mathbb{V}[1], \mathbb{H}[1] \times \mathbb{V}[1]\}$. Since every double category can be obtained as a colimit of the double categories $[0]$, $\mathbb{H}[1]$, $\mathbb{V}[1]$, and $\mathbb{H}[1] \times \mathbb{V}[1]$ by Remark 3.1.10, products in DblCat commutes with colimits, and $\mathrm{DblCat}(-,-)$ sends colimits in the first variable to limits, the result for a general double category $\mathbb{A}$ follows. The second statement follows from Proposition 1.1.3. $\square$

3.3. **Gray tensor product for double categories.** In analogy to how we obtained a pseudo-natural transformation from a 2-natural transformation, we can define notions of horizontal and vertical *pseudo-natural* transformations between double functors. The notion of modification also extends to this pseudo-setting and we obtain a double category of double functors, horizontal and vertical pseudo-natural transformations, and modifications. Similarly to the case of 2Cat, this new double category of double functors is also the internal hom for a symmetric monoidal structure on DblCat, also called the *Gray tensor product.* In this section, we introduce the pseudo-hom double categories and show that there is a closed symmetric monoidal structure on DblCat with respect to these.

Let us first introduce horizontal pseudo-natural transformations between double functors.

Definition 3.3.1. Let $F, G \colon \mathbb{A} \to \mathbb{B}$ be double functors. A **horizontal pseudo-natural transformation** $\varphi \colon F \Rightarrow G$ consists of

(i) a horizontal morphism $\varphi_A \colon FA \to GA$ in $\mathbb{B}$, for each object $A \in \mathbb{A}$,

(ii) a square $\varphi_u \colon (Fu \,{}^{\varphi_A}_{\varphi_{A'}}\, Gu)$ in $\mathbb{B}$, for each vertical morphism $u \colon A \to A'$ in $\mathbb{A}$,

(iii) a vertically invertible square φ_a in $\mathbb{B}$

$$\begin{tikzcd} FA \arrow[r, "\varphi_A"] \arrow[d, equal] & GA \arrow[r, "Ga"] \arrow[d, phantom, "\varphi_a \cong"] & GC \arrow[d, equal] \\ FA \arrow[r, "Fa"'] & FC \arrow[r, "\varphi_C"'] & GC \end{tikzcd},$$

for each horizontal morphism $a \colon A \to C$ in $\mathbb{A}$,

such that φ satisfies (hn1-2) of Definition 3.2.1 and

(hpn3) for every object $A \in \mathbb{A}$, we have $\varphi_{\mathrm{id}_A} = e_{\varphi_A} \colon (e_{FA} \,{}^{\varphi_A}_{\varphi_A}\, e_{GA})$,

(hpn4) for every pair of composable horizontal morphisms $a \colon A \to C$ and $c \colon C \to E$ in $\mathbb{A}$, the following pasting equality holds,

$$
\begin{tikzcd}
FA \arrow[r, "\varphi_A"] \arrow[d, equal] & GA \arrow[r, "Ga"] \arrow[dr, phantom, "\varphi_a \, \cong"] & GC \arrow[r, "Gc"] \arrow[d, equal] \arrow[dr, phantom, "e_{Gc}"] & GE \arrow[d, equal] \\
FA \arrow[r, "Fa"'] \arrow[d, equal] \arrow[dr, phantom, "e_{Fa}"] & FC \arrow[r, "\varphi_C"'] \arrow[d, equal] & GC \arrow[r, "Gc"'] \arrow[dr, phantom, "\varphi_c \, \cong"] & GE \arrow[d, equal] \\
FA \arrow[r, "Fa"'] & FC \arrow[r, "Fc"'] & FE \arrow[r, "\varphi_E"'] & GE
\end{tikzcd}
=
\begin{tikzcd}
FA \arrow[r, "\varphi_A"] \arrow[d, equal] & GA \arrow[r, "G(ca)"] \arrow[d, phantom, "\varphi_{ca} \, \cong"] & GE \arrow[d, equal] \\
FA \arrow[r, "F(ca)"'] & FE \arrow[r, "\varphi_E"'] & GE
\end{tikzcd}
$$

(hpn5) for every square $\alpha\colon (u \,{}^{a}_{a'}\, w)$ in $\mathbb{A}$, the following pasting equality holds.

$$
\begin{tikzcd}
FA \arrow[r, "\varphi_A"] \arrow[d, equal] & GA \arrow[r, "Ga"] \arrow[d, phantom, "\varphi_a \, \cong"] & GC \arrow[d, equal] \\
FA \arrow[r, "Fa"] \arrow[d, "Fu"'] \arrow[dr, phantom, "F\alpha"] & FC \arrow[r, "\varphi_C"] \arrow[d, "Fw"] \arrow[dr, phantom, "\varphi_w"] & GC \arrow[d, "Gw"] \\
FA' \arrow[r, "Fa'"'] & FC' \arrow[r, "\varphi_{C'}"'] & GC'
\end{tikzcd}
=
\begin{tikzcd}
FA \arrow[r, "\varphi_A"] \arrow[d, "Fu"'] \arrow[dr, phantom, "\varphi_u"] & GA \arrow[r, "Ga"] \arrow[d, "Gu"] \arrow[dr, phantom, "G\alpha"] & GC \arrow[d, "Gw"] \\
FA' \arrow[r, "\varphi_{A'}"'] \arrow[d, equal] & GA' \arrow[r, "Ga'"'] \arrow[d, phantom, "\varphi_{a'} \, \cong"] & GC' \arrow[d, equal] \\
FA' \arrow[r, "Fa'"'] & FC' \arrow[r, "\varphi_{C'}"'] & GC'
\end{tikzcd}
$$

Again, by transposing the above notions, we get the dual concept of vertical pseudo-natural transformations.

Definition 3.3.2. Let $F, F'\colon \mathbb{A} \to \mathbb{B}$ be double functors. A **vertical pseudo-natural transformation** $\nu\colon F \Rightarrow F'$ consists of

(i) a vertical morphism $\nu_A\colon FA \to F'A$ in $\mathbb{B}$, for each object $A \in \mathbb{A}$,
(ii) a square $\nu_a\colon (\nu_A \,{}^{Fa}_{F'a}\, \nu_C)$ in $\mathbb{B}$, for each horizontal morphism $a\colon A \to C$ in $\mathbb{A}$,
(iii) a horizontally invertible square ν_u in $\mathbb{B}$

$$
\begin{tikzcd}
FA \arrow[r, equal] \arrow[d, "\nu_A"'] & FA \arrow[d, "Fu"] \\
F'A \arrow[r, phantom, "\underset{\nu_u}{\cong}"] \arrow[d, "F'u"'] & FA' \arrow[d, "\nu_{A'}"] \\
F'A' \arrow[r, equal] & F'A' ,
\end{tikzcd}
$$

for every vertical morphism $u\colon A \to A'$ in $\mathbb{A}$,

such that ν satisfies (vn1-2) of Definition 3.2.2 and

(vpn3) for every object $A \in \mathbb{A}$, we have $\nu_{e_A} = \mathrm{id}_{\nu_A}\colon (\nu_A \,{}^{\mathrm{id}_{FA}}_{\mathrm{id}_{F'A}}\, \nu_A)$,
(vpn4) for every pair of composable vertical morphisms $u\colon A \to A'$ and $u'\colon A' \to A''$ in $\mathbb{A}$, the following pasting equality holds,

$$
\begin{tikzcd}
FA \arrow[r, equal] \arrow[d, "\nu_A"', "\bullet" marking] & FA \arrow[r, equal] \arrow[d, "Fu"', "\bullet" marking] \arrow[dr, phantom, "\mathrm{id}_{Fu}"] & FA \arrow[d, "Fu", "\bullet" marking] \\
F'A \arrow[r, phantom, "\cong \nu_u"] \arrow[d, "F'u"', "\bullet" marking] & FA' \arrow[r, equal] \arrow[d, "\nu_{A'}", "\bullet" marking] & FA' \arrow[d, "Fu'", "\bullet" marking] \\
F'A' \arrow[r, equal] \arrow[d, "F'u'"', "\bullet" marking] \arrow[dr, phantom, "\mathrm{id}_{F'u'}"] & F'A' \arrow[r, phantom, "\cong \nu_{u'}"] \arrow[d, "F'u'", "\bullet" marking] & FA'' \arrow[d, "\nu_{A''}", "\bullet" marking] \\
F'A'' \arrow[r, equal] & F'A'' \arrow[r, equal] & F'A''
\end{tikzcd}
=
\begin{tikzcd}
FA \arrow[r, equal] \arrow[d, "\nu_A"', "\bullet" marking] & FA \arrow[d, "F(u'u)", "\bullet" marking] \\
F'A \arrow[r, phantom, "\cong \nu_{u'u}"] \arrow[d, "F'(u'u)"', "\bullet" marking] & FA'' \arrow[d, "\nu_{A''}", "\bullet" marking] \\
F'A'' \arrow[r, equal] & F'A''
\end{tikzcd}
$$

(vpn5) for every square $\alpha\colon (u \,{}^{a}_{a'}\, w)$ in $\mathbb{A}$, the following pasting equality holds.

$$
\begin{tikzcd}
FA \arrow[r, equal] \arrow[d, "\nu_A"', "\bullet" marking] & FA \arrow[r, "Fa"] \arrow[d, "Fu"', "\bullet" marking] \arrow[dr, phantom, "F\alpha"] & FC \arrow[d, "Fw", "\bullet" marking] \\
F'A \arrow[r, phantom, "\cong \nu_u"] \arrow[d, "F'u"', "\bullet" marking] & FA' \arrow[r, "Fa'"] \arrow[d, "\nu_{A'}"', "\bullet" marking] \arrow[dr, phantom, "\nu_{a'}"] & FC' \arrow[d, "\nu_{C'}", "\bullet" marking] \\
F'A' \arrow[r, equal] & F'A' \arrow[r, "F'a'"'] & F'C'
\end{tikzcd}
=
\begin{tikzcd}
FA \arrow[r, "Fa"] \arrow[d, "\nu_A"', "\bullet" marking] \arrow[dr, phantom, "\nu_a"] & FC \arrow[r, equal] \arrow[d, "\nu_C", "\bullet" marking] & FC \arrow[d, "Fw", "\bullet" marking] \\
F'A \arrow[r, "F'a"'] \arrow[d, "F'u"', "\bullet" marking] \arrow[dr, phantom, "F'\alpha"] & F'C \arrow[r, phantom, "\cong \nu_w"] \arrow[d, "F'w", "\bullet" marking] & FC' \arrow[d, "\nu_{C'}", "\bullet" marking] \\
F'A' \arrow[r, "F'a'"'] & F'C' \arrow[r, equal] & F'C'
\end{tikzcd}
$$

Finally, modifications also generalize to this pseudo-setting as follows.

Definition 3.3.3. Let $F, F', G, G'\colon \mathbb{A} \to \mathbb{B}$ be double functors, let $\varphi\colon F \Rightarrow G$ and $\varphi'\colon F' \Rightarrow G'$ be horizontal pseudo-natural transformations, and let $\nu\colon F \overset{\bullet}{\Rightarrow} F'$ and $\xi\colon G \overset{\bullet}{\Rightarrow} G'$ be vertical pseudo-natural transformations. A **modification** $\mu\colon (\nu \,{}^{\varphi}_{\varphi'}\, \xi)$ consists of a square $\mu_A\colon (\nu_A \,{}^{\varphi_A}_{\varphi'_A}\, \xi_A)$ in $\mathbb{B}$, for each object $A \in \mathbb{A}$, such that

(dmp1) for every vertical morphism $u\colon A \overset{\bullet}{\to} A'$ in $\mathbb{A}$, the following pasting equality holds,

$$
\begin{tikzcd}
FA \arrow[r, equal] \arrow[d, "\nu_A"', "\bullet" marking] & FA \arrow[r, "\varphi_A"] \arrow[d, "Fu"', "\bullet" marking] \arrow[dr, phantom, "\varphi_u"] & GA \arrow[d, "Gu", "\bullet" marking] \\
F'A \arrow[r, phantom, "\cong \nu_u"] \arrow[d, "F'u"', "\bullet" marking] & FA' \arrow[r, "\varphi_{A'}"] \arrow[d, "\nu_{A'}"', "\bullet" marking] \arrow[dr, phantom, "\mu_{A'}"] & GA' \arrow[d, "\xi_{A'}", "\bullet" marking] \\
F'A' \arrow[r, equal] & F'A' \arrow[r, "\varphi'_{A'}"'] & G'A'
\end{tikzcd}
=
\begin{tikzcd}
FA \arrow[r, "\varphi_A"] \arrow[d, "\nu_A"', "\bullet" marking] \arrow[dr, phantom, "\mu_A"] & GA \arrow[r, equal] \arrow[d, "\xi_A", "\bullet" marking] & GA \arrow[d, "Gu", "\bullet" marking] \\
F'A \arrow[r, "\varphi'_A"'] \arrow[d, "F'u"', "\bullet" marking] \arrow[dr, phantom, "\varphi'_u"] & G'A \arrow[r, phantom, "\cong \xi_u"] \arrow[d, "G'u", "\bullet" marking] & GA' \arrow[d, "\xi_{A'}", "\bullet" marking] \\
F'A' \arrow[r, "\varphi'_{A'}"'] & G'A' \arrow[r, equal] & G'A'
\end{tikzcd}
$$

(dmp2) for every horizontal morphism $a\colon A \to C$ in $\mathbb{A}$, the following pasting equality holds.

$$
\begin{tikzcd}
FA \arrow[r, "\varphi_A"] \arrow[d, equal, "\bullet" marking] \arrow[drr, phantom, "\varphi_a \cong"] & GA \arrow[r, "Ga"] & GC \arrow[d, equal, "\bullet" marking] \\
FA \arrow[r, "Fa"] \arrow[d, "\nu_A"', "\bullet" marking] \arrow[dr, phantom, "\nu_a"] & FC \arrow[r, "\varphi_C"] \arrow[d, "\nu_C"', "\bullet" marking] \arrow[dr, phantom, "\mu_C"] & GC \arrow[d, "\xi_C", "\bullet" marking] \\
F'A \arrow[r, "F'a"'] & F'C \arrow[r, "\varphi'_C"'] & G'C
\end{tikzcd}
=
\begin{tikzcd}
FA \arrow[r, "\varphi_A"] \arrow[d, "\nu_A"', "\bullet" marking] \arrow[dr, phantom, "\mu_A"] & GA \arrow[r, "Ga"] \arrow[d, "\xi_A", "\bullet" marking] \arrow[dr, phantom, "\xi_a"] & GC \arrow[d, "\xi_C", "\bullet" marking] \\
F'A \arrow[r, "\varphi'_A"'] \arrow[d, equal, "\bullet" marking] \arrow[drr, phantom, "\varphi'_a \cong"] & G'A \arrow[r, "G'a"'] & G'C \arrow[d, equal, "\bullet" marking] \\
F'A \arrow[r, "F'a"'] & F'C \arrow[r, "\varphi'_C"'] & G'C
\end{tikzcd}
$$

All together, they form a double category whose objects are double functors between fixed double categories.

Definition 3.3.4. Let $\mathbb{A}$ and $\mathbb{B}$ be double categories. The **pseudo-hom double category** $[\mathbb{A}, \mathbb{B}]_{\mathrm{ps}}$ is the double category whose

(i) objects are double functors from $\mathbb{A}$ to $\mathbb{B}$,
(ii) horizontal morphisms are horizontal pseudo-natural transformations,
(iii) vertical morphisms are vertical pseudo-natural transformations, and
(iv) squares are modifications.

One can construct a symmetric monoidal product on DblCat, also called the *Gray tensor product*, which is closed with respect to the above pseudo-homs. A description of the Gray tensor product of two double categories can be found in Description 8.5.1.

Proposition 3.3.5. *There is a closed symmetric monoidal product* $\otimes_{\mathrm{Gr}}$ *on* DblCat*, called the* ***Gray tensor product****, such that, for every tuple of double categories* $\mathbb{A}$*,* $\mathbb{B}$*, and* $\mathbb{C}$*, we have an isomorphism*

$$\mathrm{DblCat}(\mathbb{A}, [\mathbb{B}, \mathbb{C}]_{\mathrm{ps}}) \cong \mathrm{DblCat}(\mathbb{A} \otimes_{\mathrm{Gr}} \mathbb{B}, \mathbb{C})$$

natural in $\mathbb{A}$*,* $\mathbb{B}$*, and* $\mathbb{C}$*. Furthermore, this isomorphism extends to an isomorphism of double categories*

$$[\mathbb{A}, [\mathbb{B}, \mathbb{C}]_{\mathrm{ps}}]_{\mathrm{ps}} \cong [\mathbb{A} \otimes_{\mathrm{Gr}} \mathbb{B}, \mathbb{C}]_{\mathrm{ps}}$$

natural in $\mathbb{A}$*,* $\mathbb{B}$*, and* $\mathbb{C}$*.*

Proof. The first part of the result follows from [Böh19, §3]. The second statement follows from Proposition 1.1.3. □

3.4. **Relations between 2-categories and double categories.** As we have seen in Definition 3.1.7, every 2-category induces a horizontal and a vertical double category. We extend these constructions to functors $\mathbb{H}\colon 2\mathrm{Cat} \to \mathrm{DblCat}$ and $\mathbb{V}\colon 2\mathrm{Cat} \to \mathrm{DblCat}$ and show that these embeddings admit both adjoints. In particular, the right adjoints are given by sending a double category $\mathbb{A}$ to its *underlying horizontal* 2*-category* $\mathbf{H}\mathbb{A}$ and its *underlying vertical* 2*-category* $\mathbf{V}\mathbb{A}$, respectively, which are constructed by forgetting the vertical and horizontal structure of the double category, respectively. The functor $\mathbf{H}$ will be used to construct one of the model structures on DblCat by inducing it from Lack's model structure on 2Cat. However, since it only remembers the horizontal structure of a double category, we also need another functor, which we introduce now, sending a double category $\mathbb{A}$ to the 2-category $\mathcal{V}\mathbb{A}$, whose objects are the vertical morphisms of $\mathbb{A}$ and whose morphisms are the squares of $\mathbb{A}$; the 2-morphisms are as described in Definition 3.4.9. We then use the pair of functors $(\mathbf{H}, \mathcal{V})\colon \mathrm{DblCat} \to 2\mathrm{Cat} \times 2\mathrm{Cat}$ to construct one of the model structures on DblCat in Section 7. Finally, we also introduce here another embedding $\mathbb{H}^{\simeq}\colon 2\mathrm{Cat} \to \mathrm{DblCat}$, which gives a more homotopical version of the horizontal embedding. Indeed, it sends a 2-category $\mathcal{A}$ to a double category $\mathbb{H}^{\simeq}\mathcal{A}$ whose underlying horizontal 2-category is still $\mathcal{A}$, but its vertical morphisms are now given by the adjoint equivalences in $\mathcal{A}$, instead of just the identities.

Let us first introduce the horizontal embedding of 2Cat into DblCat.

Definition 3.4.1. We define the **horizontal embedding functor** $\mathbb{H}\colon 2\mathrm{Cat} \to \mathrm{DblCat}$. It sends a 2-category $\mathcal{A}$ to its associated horizontal double category $\mathbb{H}\mathcal{A}$ as given in Definition 3.1.7, and a 2-functor $F\colon \mathcal{A} \to \mathcal{B}$ to the double functor $\mathbb{H}F\colon \mathbb{H}\mathcal{A} \to \mathbb{H}\mathcal{B}$ which acts as F does on the corresponding data.

Remark 3.4.2. When restricted to the category Cat of categories and functors, the horizontal embedding $\mathbb{H}U\colon \mathrm{Cat} \to \mathrm{DblCat}$ corresponds to the functor $\mathrm{Cat}(\mathrm{Set}) \to \mathrm{Cat}(\mathrm{Cat})$ induced by the inclusion $\mathrm{Set} \to \mathrm{Cat}$.

This horizontal embedding has both adjoints. Its right adjoint is given by taking the underlying horizontal 2-category of a double category.

Definition 3.4.3. We define the functor $\mathbf{H}\colon \mathrm{DblCat} \to 2\mathrm{Cat}$. It sends a double category $\mathbb{A}$ to its **underlying horizontal 2-category $\mathbf{H}\mathbb{A}$** with the same objects as $\mathbb{A}$, morphisms the horizontal morphisms of $\mathbb{A}$, and 2-morphisms $\alpha\colon a \Rightarrow c$ given by the squares in $\mathbb{A}$ of the form

$$\begin{tikzcd} A \arrow[r, "a"] \arrow[d, equal, "\bullet" marking] \arrow[dr, phantom, "\alpha"] & C \arrow[d, equal, "\bullet" marking] \\ A \arrow[r, "c"'] & C . \end{tikzcd}$$

It sends a double functor $F\colon \mathbb{A} \to \mathbb{B}$ to the 2-functor $\mathbf{H}F\colon \mathbf{H}\mathbb{A} \to \mathbf{H}\mathbb{B}$ which acts as F does on the corresponding data.

The left adjoint of the horizontal embedding is constructed by identifying objects which are related by a zig-zag of vertical morphisms. This is also an instance of applying the path component functor $\pi_0\colon \mathrm{Cat} \to \mathrm{Set}$ to the category of objects and vertical morphisms, and then adapting the extra structure to this identification.

Definition 3.4.4. We define the functor $L\colon \mathrm{DblCat} \to 2\mathrm{Cat}$. It sends a double category $\mathbb{A}$ to the 2-category $L\mathbb{A}$ whose objects are given by equivalence classes of objects of $\mathbb{A}$ under the following relation: two objects are identified if and only if there is a zig-zag of vertical morphisms between them. Morphisms and 2-morphisms in $L\mathbb{A}$ are generated by the horizontal morphisms and squares of $\mathbb{A}$, respectively. Then the functor L sends a double functor $F\colon \mathbb{A} \to \mathbb{B}$ to the 2-functor $LF\colon L\mathbb{A} \to L\mathbb{B}$ which acts as F does on the corresponding data.

We now show that L and $\mathbf{H}$ are indeed the left and right adjoints of $\mathbb{H}$, respectively.

Proposition 3.4.5. *The functors L, $\mathbb{H}$, and $\mathbf{H}$ form adjunctions*

$$\begin{tikzcd} 2\mathrm{Cat} \arrow[rr, "\mathbb{H}" description] & & \mathrm{DblCat} . \arrow[ll, bend right, "L"'] \arrow[ll, bend left, "\mathbf{H}"] \arrow[ll, phantom, shift right=3, "\perp"] \arrow[ll, phantom, shift left=3, "\perp"] \end{tikzcd}$$

Moreover, the counit of the adjunction $L \dashv \mathbb{H}$ and the unit of the adjunction $\mathbb{H} \dashv \mathbf{H}$ are identities. In particular, the functor $\mathbb{H}\colon 2\mathrm{Cat} \to \mathrm{DblCat}$ is a full embedding.

Proof. We first show that, for every 2-category $\mathcal{C}$ and every double category $\mathbb{A}$, there is an isomorphism
$$\mathrm{DblCat}(\mathbb{A}, \mathbb{H}\mathcal{C}) \cong 2\mathrm{Cat}(L\mathbb{A}, \mathcal{C})$$
natural in $\mathcal{C}$ and $\mathbb{A}$. Since every vertical morphism in $\mathbb{H}\mathcal{C}$ is trivial, a double functor $F\colon \mathbb{A} \to \mathbb{H}\mathcal{C}$ sends every vertical morphism in $\mathbb{A}$ to a vertical identity. Hence it induces a 2-functor $\hat{F}\colon L\mathbb{A} \to \mathcal{C}$ acting as F on the corresponding data, since two objects in the same equivalence class must be sent to the same object of $\mathcal{C}$ by F. Conversely, if $G\colon L\mathbb{A} \to \mathcal{C}$ is a 2-functor, then it induces a double functor $\hat{G}\colon \mathbb{A} \to \mathbb{H}\mathcal{C}$ which acts as G on the corresponding data, and sends every vertical morphism to the vertical identity of the image under G of its boundaries, which are in the same equivalence class. These constructions are clearly inverse to each other and natural in $\mathcal{C}$ and $\mathbb{A}$. Hence $L \dashv \mathbb{H}$ is an adjunction. Moreover, we have that $L\mathbb{H}\mathcal{C} = \mathcal{C}$, for every 2-category $\mathcal{C}$, since the double category $\mathbb{H}\mathcal{C}$ has only trivial vertical morphisms, and hence applying L does not identify objects in $\mathcal{C}$. This shows that the counit of $L \dashv \mathbb{H}$ is an identity.

We now prove that, for every 2-category $\mathcal{C}$ and every double category $\mathbb{A}$, there is an isomorphism
$$\mathrm{DblCat}(\mathbb{H}\mathcal{C}, \mathbb{A}) \cong 2\mathrm{Cat}(\mathcal{C}, \mathbf{H}\mathbb{A})$$

natural in $\mathcal{C}$ and $\mathbb{A}$. Since every vertical morphism in $\mathbb{H}\mathcal{C}$ is trivial, the image of a double functor $F\colon \mathbb{H}\mathcal{C} \to \mathbb{A}$ is included in the underlying horizontal 2-category of $\mathbb{A}$ and hence it restricts to a 2-functor $\hat{F}\colon \mathcal{C} \to \mathbf{H}\mathbb{A}$. Conversely, every 2-functor $G\colon \mathcal{C} \to \mathbf{H}\mathbb{A}$ induces a double functor $\hat{G}\colon \mathbb{H}\mathcal{C} \to \mathbb{A}$ which acts as G on the corresponding data, and sends the trivial vertical morphisms of $\mathbb{H}\mathcal{C}$ to the corresponding trivial vertical morphisms of $\mathbb{A}$. These constructions are clearly inverse to each other and natural in $\mathcal{C}$ and $\mathbb{A}$. Hence $\mathbb{H} \dashv \mathbf{H}$ is an adjunction. Moreover, we clearly have that $\mathbf{H}\mathbb{H}\mathcal{C} = \mathcal{C}$, for every 2-category $\mathcal{C}$, and hence the unit of $\mathbb{H} \dashv \mathbf{H}$ is an identity. $\square$

Dually, we can define the vertical embedding functor.

Definition 3.4.6. We define the **vertical embedding functor** $\mathbb{V}\colon 2\mathrm{Cat} \to \mathrm{DblCat}$. It sends a 2-category $\mathcal{A}$ to its associated vertical double category $\mathbb{V}\mathcal{A}$ as given in Definition 3.1.7, and a 2-functor $F\colon \mathcal{A} \to \mathcal{B}$ to the double functor $\mathbb{V}F\colon \mathbb{V}\mathcal{A} \to \mathbb{V}\mathcal{B}$ which acts as F does on the corresponding data.

This vertical embedding has both a left and a right adjoint, which are constructed as above, by transposing the horizontal and vertical data.

Definition 3.4.7. We define the functor $\mathbf{V}\colon \mathrm{DblCat} \to 2\mathrm{Cat}$. It sends a double category $\mathbb{A}$ to its **underlying vertical 2-category** $\mathbf{V}\mathbb{A}$ with the same objects as $\mathbb{A}$, morphisms the vertical morphisms of $\mathbb{A}$, and 2-morphisms $\alpha\colon u \Rightarrow w$ given by the squares in $\mathbb{A}$ of the form

$$\begin{tikzcd} A \arrow[r, equal] \arrow[d, "u"'] \arrow[dr, phantom, "\alpha"] & A \arrow[d, "w"] \\ A' \arrow[r, equal] & A' . \end{tikzcd}$$

It sends a double functor $F\colon \mathbb{A} \to \mathbb{B}$ to the 2-functor $\mathbf{V}F\colon \mathbf{V}\mathbb{A} \to \mathbf{V}\mathbb{B}$ which acts as F does on the corresponding data.

Remark 3.4.8. By transposing the results in Proposition 3.4.5, one can show that the functor $\mathbf{V}$ is right adjoint to $\mathbb{V}$, and that this latter also admits a left adjoint.

We now introduce a functor $\mathcal{V}\colon \mathrm{DblCat} \to 2\mathrm{Cat}$ which extracts from a double category a 2-category whose objects are the vertical morphisms and whose morphisms are the squares. For this, recall that $\mathbb{V}[1]$ is the double category free on a vertical morphism.

Definition 3.4.9. We define the functor $\mathcal{V}\colon \mathrm{DblCat} \to 2\mathrm{Cat}$ to be

$$\mathcal{V} \coloneqq \mathbf{H}[\mathbb{V}[1], -]\colon \mathrm{DblCat} \longrightarrow 2\mathrm{Cat}.$$

More explicitly, it sends a double category $\mathbb{A}$ to the 2-category $\mathcal{V}\mathbb{A}$ whose objects are the vertical morphisms of $\mathbb{A}$, and whose morphisms are the squares of $\mathbb{A}$. A 2-morphism in $\mathcal{V}\mathbb{A}$ between parallel morphisms $\alpha\colon (u \; {}^{a}_{a'} \; w)$ and $\gamma\colon (u \; {}^{c}_{c'} \; w)$ consists of squares $\sigma_0\colon (e_A \; {}^{a}_{c} \; e_C)$ and $\sigma_1\colon (e_{A'} \; {}^{a'}_{c'} \; e_{C'})$ satisfying the following pasting equality in $\mathbb{A}$.

$$\begin{tikzcd} A \arrow[r, "a"] \arrow[d, equal] \arrow[dr, phantom, "\sigma_0"] & C \arrow[d, equal] \\ A \arrow[r, "c"] \arrow[d, "u"'] \arrow[dr, phantom, "\gamma"] & C \arrow[d, "w"] \\ A' \arrow[r, "c'"'] & C' \end{tikzcd} \quad = \quad \begin{tikzcd} A \arrow[r, "a"] \arrow[d, "u"'] \arrow[dr, phantom, "\alpha"] & C \arrow[d, "w"] \\ A' \arrow[r, "a'"'] \arrow[d, equal] \arrow[dr, phantom, "\sigma_1"] & C' \arrow[d, equal] \\ A' \arrow[r, "c'"'] & C' \end{tikzcd}$$

In particular, since $\mathcal{V}$ is a composite of two right adjoints, it is also a right adjoint.

Proposition 3.4.10. *The functor* $\mathcal{V}\colon \mathrm{DblCat} \to 2\mathrm{Cat}$ *has a left adjoint given by the functor* $\mathbb{L} \coloneqq \mathbb{H}(-) \times \mathbb{V}[1]\colon 2\mathrm{Cat} \to \mathrm{DblCat}$*, i.e., we have an adjunction*

$$\begin{tikzcd} \mathrm{DblCat} \arrow[r, bend right, "\mathcal{V}"'] \arrow[r, phantom, "\perp"] & 2\mathrm{Cat} \arrow[l, bend right, "\mathbb{L}"'] \end{tikzcd}.$$

Proof. Since $\mathcal{V}$ is given by the composite

$$\mathrm{DblCat} \xrightarrow{[\mathbb{V}[1],-]} \mathrm{DblCat} \xrightarrow{\mathbf{H}} 2\mathrm{Cat},$$

where the functor $[\mathbb{V}[1],-]$ is right adjoint to $-\times\mathbb{V}[1]$ by Proposition 3.2.5 and Remark 1.1.2, and the functor $\mathbf{H}$ is right adjoint to $\mathbb{H}$ by Proposition 3.4.5, it follows that $\mathcal{V}$ is right adjoint to the composite $\mathbb{L} \coloneqq \mathbb{H}(-)\times\mathbb{V}[1]$ of the two left adjoints. $\square$

Finally, we introduce the homotopical version of the horizontal embedding.

Definition 3.4.11. We define the functor $\mathbb{H}^{\simeq}\colon 2\mathrm{Cat} \to \mathrm{DblCat}$. It sends a 2-category $\mathcal{A}$ to the double category $\mathbb{H}^{\simeq}\mathcal{A}$ with the same objects as $\mathcal{A}$, horizontal morphisms the morphisms of $\mathcal{A}$, vertical morphisms the adjoint equivalences in $\mathcal{A}$, and squares

$$\begin{tikzcd} A \arrow[r, "a"] \arrow[d, "\underline{u}=(u,u',\eta_u,\epsilon_u)"'] \arrow[dr, phantom, "\alpha"] & B \arrow[d, "\underline{w}=(w,w',\eta_w,\epsilon_w)"] \\ A' \arrow[r, "c"'] & B' \end{tikzcd}$$

given by the 2-morphisms $\alpha\colon wa \Rightarrow cu$ in $\mathcal{A}$, where (u,u',η_u,ϵ_u) and (w,w',η_w,ϵ_w) are adjoint equivalences in $\mathcal{A}$. Compositions of horizontal and vertical morphisms are induced by the composition of morphisms and adjoint equivalences in $\mathcal{A}$, and compositions of squares are induced by the composition of 2-morphisms in $\mathcal{A}$. Then the functor $\mathbb{H}^{\simeq}$ sends a 2-functor $F\colon \mathcal{A}\to\mathcal{B}$ to the double functor $\mathbb{H}^{\simeq}F\colon \mathbb{H}^{\simeq}\mathcal{A}\to\mathbb{H}^{\simeq}\mathcal{B}$ which acts as F does on the corresponding data.

In particular, this functor admits a left adjoint, which we now define.

Definition 3.4.12. We define the functor $L^{\simeq}\colon \mathrm{DblCat}\to 2\mathrm{Cat}$ which sends a double category $\mathbb{A}$ to the 2-category $L^{\simeq}\mathbb{A}$ whose

(i) objects are the objects of $\mathbb{A}$,
(ii) morphisms are generated by
  - a morphism $a\colon A\to C$, for each horizontal morphisms $a\colon A\to C$ in $\mathbb{A}$,
  - two morphisms $\overline{u}\colon A\to A'$ and $\overline{u}'\colon A'\to A$, for each vertical morphism $u\colon A \mathrel{\mathrlap{\bullet}\to} A'$ in $\mathbb{A}$,

(iii) 2-morphisms are generated by
  - a 2-morphism $\alpha\colon \overline{w}a\Rightarrow a'\overline{u}$ for each square $\alpha\colon (u\,{}^{a}_{a'}\,w)$ in $\mathbb{A}$,
  - 2-isomorphisms $\eta_{\overline{u}}\colon \mathrm{id}_A\cong\overline{u}'\overline{u}$ and $\epsilon_{\overline{u}}\colon \overline{u}\,\overline{u}'\cong\mathrm{id}_{A'}$ satisfying the triangle identities, for each vertical morphism $u\colon A\mathrel{\mathrlap{\bullet}\to} A'$ in $\mathbb{A}$,

submitted to minimal relations making the inclusion $\mathbb{A}\to\mathbb{H}^{\simeq}L^{\simeq}\mathbb{A}$ into a double functor. The functor $L^{\simeq}$ sends a double functor $F\colon\mathbb{A}\to\mathbb{B}$ to the 2-functor $L^{\simeq}F\colon L^{\simeq}\mathbb{A}\to L^{\simeq}\mathbb{B}$ which acts as F does on the corresponding data. In particular, it sends an adjoint equivalence $(\overline{u},\overline{u}',\eta_{\overline{u}},\epsilon_{\overline{u}})$ in $L^{\simeq}\mathbb{A}$ associated to a vertical morphism u in $\mathbb{A}$ to the adjoint equivalence $(\overline{Fu},\overline{Fu}',\eta_{\overline{Fu}},\epsilon_{\overline{Fu}})$ in $L^{\simeq}\mathbb{B}$ associated to the vertical morphism Fu in $\mathbb{B}$.

Remark 3.4.13. The relations on the morphisms in $L^{\simeq}\mathbb{A}$ expressed by the fact that the inclusion $\mathbb{A}\to\mathbb{H}^{\simeq}L^{\simeq}\mathbb{A}$ is a double functor can be interpreted as follows. Two copies of morphisms coming from horizontal morphisms compose as in $\mathbb{A}$, and two copies of adjoint equivalences coming from vertical morphisms compose as in $\mathbb{A}$, while two copies of morphisms with one coming from a horizontal morphism and one coming from a vertical morphism compose freely.

Proposition 3.4.14. *The functors $\mathbb{H}^{\simeq}$ and $L^{\simeq}$ form an adjunction*

$$\begin{tikzcd} 2\mathrm{Cat} \arrow[r, bend right, "\mathbb{H}^{\simeq}"'] \arrow[r, phantom, "\perp"] & \mathrm{DblCat} \arrow[l, bend right, "L^{\simeq}"'] \end{tikzcd}.$$

Proof. We show that, for every 2-category $\mathcal{C}$ and every double category $\mathbb{A}$, there is an isomorphism

$$\mathrm{DblCat}(\mathbb{A}, \mathbb{H}^{\simeq}\mathcal{C}) \cong 2\mathrm{Cat}(L^{\simeq}\mathbb{A}, \mathcal{C})$$

natural in $\mathcal{C}$ and $\mathbb{A}$. A double functor $F\colon \mathbb{A} \to \mathbb{H}^{\simeq}\mathcal{C}$ induces a 2-functor $\hat{F}\colon L^{\simeq}\mathbb{A} \to \mathcal{C}$ which acts as F on objects and on the morphisms of $L^{\simeq}\mathbb{A}$ coming from horizontal morphisms in $\mathbb{A}$, sends an adjoint equivalence $(\overline{u}, \overline{u}', \eta_{\overline{u}}, \epsilon_{\overline{u}})$ in $L^{\simeq}\mathbb{A}$ associated to a vertical morphism u in $\mathbb{A}$ to the adjoint equivalence in $\mathcal{C}$ corresponding to the vertical morphism Fu in $\mathbb{H}^{\simeq}\mathcal{C}$, and sends a 2-morphism in $L^{\simeq}\mathbb{A}$ coming from a square α in $\mathbb{A}$ to the 2-morphism in $\mathcal{C}$ corresponding to the square $F\alpha$ in $\mathbb{H}^{\simeq}\mathcal{C}$. Conversely, if $G\colon L^{\simeq}\mathbb{A} \to \mathcal{C}$ is a 2-functor, it induces a double functor $\hat{G}\colon \mathbb{A} \to \mathbb{H}^{\simeq}\mathcal{C}$ which acts as G on objects and horizontal morphisms, sends a vertical morphism u in $\mathbb{A}$ to the vertical morphism in $\mathbb{H}^{\simeq}\mathbb{A}$ corresponding to the adjoint equivalence $(G\overline{u}, G\overline{u}', G\eta_{\overline{u}}, G\epsilon_{\overline{u}})$ in $\mathcal{C}$, and sends a square α in $\mathbb{A}$ to the square in $\mathbb{H}^{\simeq}\mathcal{C}$ corresponding to the 2-morphism $G\alpha$ in $\mathcal{C}$. These constructions are clearly inverse to each other and natural in $\mathcal{C}$ and $\mathbb{A}$. Hence $L^{\simeq} \dashv \mathbb{H}^{\simeq}$ is an adjunction. □

However, the functor $\mathbb{H}^{\simeq}$ does not have a right adjoint, since it does not preserve colimits, as we now show.

Remark 3.4.15. The functor $\mathbb{H}^{\simeq}$ does not have a right adjoint since it does not preserve colimits. To see this, consider the following span of 2-categories $\mathcal{B} \leftarrow \mathcal{A} \to \mathcal{C}$. We set $\mathcal{A}$ to be the 2-category with two objects 0 and 1, and freely generated by two morphisms $f\colon 0 \to 1$ and $g\colon 1 \to 0$ and two 2-morphisms $\eta\colon \mathrm{id}_0 \Rightarrow gf$ and $\epsilon\colon fg \Rightarrow \mathrm{id}_1$. Then let $\mathcal{B}$ be the category obtained from $\mathcal{A}$ by inverting the 2-morphism η, and $\mathcal{C}$ be the category obtained from $\mathcal{A}$ by inverting the 2-morphism ϵ. Then the pushout $\mathcal{B} \sqcup_{\mathcal{A}} \mathcal{C}$ contains an equivalence (f, g, η, ϵ) and hence the double category $\mathbb{H}^{\simeq}(\mathcal{B} \sqcup_{\mathcal{A}} \mathcal{C})$ contains a vertical morphism induced by this equivalence (or the corresponding adjoint equivalence given by Proposition 2.4.4). However, the double categories $\mathbb{H}^{\simeq}\mathcal{A}$, $\mathbb{H}^{\simeq}\mathcal{B}$, and $\mathbb{H}^{\simeq}\mathcal{C}$ do not have non-trivial vertical morphisms, since there are no equivalences in $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$. This shows that $\mathbb{H}^{\simeq}$ does not preserve pushouts.

3.5. 2Cat-**enrichment of** DblCat.

By considering the underlying horizontal 2-category of the pseudo-hom double categories defined in Definition 3.3.4, we obtain an enrichment of DblCat over 2Cat endowed with the closed symmetric monoidal structure given by the Gray tensor product $\otimes_2$ (see Proposition 2.3.4). Furthermore, this enrichment is tensored and cotensored, where the tensoring functor is given by restricting the Gray tensor product along the horizontal embedding in one variable. We are considering this enrichment here, since we will show that both model structures that we construct on DblCat are 2Cat-enriched with respect to this enrichment. In this section, we first show that this induces a tensored and cotensored enrichment of DblCat over 2Cat, and then we compare the pseudo-hom 2-categories given by this enrichment with the pseudo-homs of 2Cat. These technical results are useful in Section 7 when constructing one of the model structures on DblCat.

We first define tensors of double categories over 2Cat.

Definition 3.5.1. We define the **tensoring functor** $\otimes\colon \mathrm{DblCat}\times 2\mathrm{Cat}\to \mathrm{DblCat}$ to be the composite

$$\mathrm{DblCat}\times 2\mathrm{Cat} \xrightarrow{\mathrm{id}\times \mathbb{H}} \mathrm{DblCat}\times \mathrm{DblCat} \xrightarrow{\otimes_{\mathrm{Gr}}} \mathrm{DblCat}\,,$$

where $\otimes_{\mathrm{Gr}}$ is the Gray tensor product on DblCat as defined in Proposition 3.3.5.

We show that this gives a 2Cat-enrichment of DblCat with hom 2-categories given by the underlying horizontal 2-categories of the pseudo-hom double category $[-,-]_{\mathrm{ps}}$ introduced in Definition 3.3.4.

Proposition 3.5.2. *The category* DblCat *is a tensored and cotensored* 2Cat*-enriched category with*

(i) hom 2-categories given by $\mathbf{H}[\mathbb{A},\mathbb{B}]_{\mathrm{ps}}$, for every pair of double categories $\mathbb{A}$ and $\mathbb{B}$,
(ii) tensors given by $\mathbb{A}\otimes\mathcal{C}\coloneqq \mathbb{A}\otimes_{\mathrm{Gr}}\mathbb{H}\mathcal{C}$, for every double category $\mathbb{A}$ and every 2-category $\mathcal{C}$,
(iii) cotensors given by $[\mathbb{H}\mathcal{C},\mathbb{B}]_{\mathrm{ps}}$, for every double category $\mathbb{B}$ and every 2-category $\mathcal{C}$.

Proof. Let $\mathbb{A}$ and $\mathbb{B}$ be double categories and $\mathcal{C}$ be a 2-category. We have isomorphisms

$$\begin{aligned}\mathrm{DblCat}(\mathbb{A}\otimes\mathcal{C},\mathbb{B}) &= \mathrm{DblCat}(\mathbb{A}\otimes_{\mathrm{Gr}}\mathbb{H}\mathcal{C},\mathbb{B})\cong \mathrm{DblCat}(\mathbb{H}\mathcal{C}\otimes_{\mathrm{Gr}}\mathbb{A},\mathbb{B})\\ &\cong \mathrm{DblCat}(\mathbb{H}\mathcal{C},[\mathbb{A},\mathbb{B}]_{\mathrm{ps}})\cong 2\mathrm{Cat}(\mathcal{C},\mathbf{H}[\mathbb{A},\mathbb{B}]_{\mathrm{ps}})\end{aligned}$$

natural in $\mathcal{C}$, $\mathbb{A}$, and $\mathbb{B}$, by definition of the Gray tensor product $\otimes_{\mathrm{Gr}}$ in Proposition 3.3.5, the fact that it is symmetric, and the adjunction $\mathbb{H}\dashv\mathbf{H}$ of Proposition 3.4.5. Furthermore, we also have an isomorphism

$$\mathrm{DblCat}(\mathbb{A}\otimes\mathcal{C},\mathbb{B}) = \mathrm{DblCat}(\mathbb{A}\otimes_{\mathrm{Gr}}\mathbb{H}\mathcal{C},\mathbb{B})\cong \mathrm{DblCat}(\mathbb{A},[\mathbb{H}\mathcal{C},\mathbb{B}]_{\mathrm{ps}})$$

natural in $\mathcal{C}$, $\mathbb{A}$, and $\mathbb{B}$, by Proposition 3.3.5. Then, by definition of $\otimes$, associativity of $\otimes_{\mathrm{Gr}}$, and Corollary 3.5.7 below, we have isomorphisms

$$(\mathbb{A}\otimes\mathcal{C})\otimes\mathcal{D} = (\mathbb{A}\otimes_{\mathrm{Gr}}\mathbb{H}\mathcal{C})\otimes\mathcal{D}\cong \mathbb{A}\otimes_{\mathrm{Gr}}(\mathbb{H}\mathcal{C}\otimes\mathcal{D})\cong \mathbb{A}\otimes_{\mathrm{Gr}}\mathbb{H}(\mathcal{C}\otimes_2\mathcal{D})\cong \mathbb{A}\otimes(\mathcal{C}\otimes_2\mathcal{D})$$

natural in $\mathcal{C}$, $\mathcal{D}$, and $\mathbb{A}$. Hence, it follows from Corollary 1.1.16 that DblCat is tensored and cotensored over 2Cat. $\square$

First, we note that the underlying horizontal 2-category of the pseudo-hom double category out of a vertical double category is the same as the underlying horizontal 2-category of the strict hom double category. Hence, this tells us that we could have equivalently defined the functors $\mathcal{V}\colon \mathrm{DblCat}\to 2\mathrm{Cat}$ of Definition 3.4.9 as the composite of the pseudo-hom $[\mathbb{V}[1],-]_{\mathrm{ps}}$ out of the vertical morphism with $\mathbf{H}$, instead of the strict hom $[\mathbb{V}[1],-]$ with $\mathbf{H}$.

Lemma 3.5.3. *For every 2-category $\mathcal{A}$ and every double category $\mathbb{B}$, the 2-categories $\mathbf{H}[\mathbb{V}\mathcal{A},\mathbb{B}]_{\mathrm{ps}}$ and $\mathbf{H}[\mathbb{V}\mathcal{A},\mathbb{B}]$ coincide.*

Proof. Given a 2-category $\mathcal{A}$ and a double category $\mathbb{B}$, we show that the underlying horizontal 2-categories $\mathbf{H}[\mathbb{V}\mathcal{A},\mathbb{B}]_{\mathrm{ps}}$ and $\mathbf{H}[\mathbb{V}\mathcal{A},\mathbb{B}]$ coincide. First note that they have the same objects by definition. Then, since every horizontal morphism in $\mathbb{V}\mathcal{A}$ is trivial, it is straightforward to see that a horizontal pseudo-natural transformation between double functors $\mathbb{V}\mathcal{A}\to\mathbb{B}$ is a strict one, as every vertically invertible square component associated to a horizontal identity is trivial by (hpn3) of Definition 3.3.1. It is then clear that the 2-morphisms – given by the adapted version of modifications – in both 2-categories are the same by comparing Definitions 3.2.3 and 3.3.3. $\square$

As a direct consequence, we can see that tensoring with a vertical double category is equivalently given by taking the product with this vertical double category.

Corollary 3.5.4. *For every pair of* 2*-categories* $\mathcal{A}$ *and* $\mathcal{C}$*, there is an isomorphism of double categories* $\mathbb{V}\mathcal{A} \otimes \mathcal{C} \cong \mathbb{V}\mathcal{A} \times \mathbb{H}\mathcal{C}$ *natural in* $\mathcal{A}$ *and* $\mathcal{C}$*.*

Proof. For every pair of 2-categories $\mathcal{A}$ and $\mathcal{C}$, and every double category $\mathbb{B}$, we have isomorphisms

$$\begin{aligned}\mathrm{DblCat}(\mathbb{V}\mathcal{A} \otimes \mathcal{C}, \mathbb{B}) &\cong 2\mathrm{Cat}(\mathcal{C}, \mathbf{H}[\mathbb{V}\mathcal{A}, \mathbb{B}]_{\mathrm{ps}}) = 2\mathrm{Cat}(\mathcal{C}, \mathbf{H}[\mathbb{V}\mathcal{A}, \mathbb{B}]) \\ &\cong \mathrm{DblCat}(\mathbb{H}\mathcal{C}, [\mathbb{V}\mathcal{A}, \mathbb{B}]) \cong \mathrm{DblCat}(\mathbb{H}\mathcal{C} \times \mathbb{V}\mathcal{A}, \mathbb{B}) \\ &\cong \mathrm{DblCat}(\mathbb{V}\mathcal{A} \times \mathbb{H}\mathcal{C}, \mathbb{B})\end{aligned}$$

natural in $\mathcal{A}$, $\mathcal{C}$, and $\mathbb{B}$, where the first isomorphism follows from Proposition 3.5.2, the second from Lemma 3.5.3, the third from the adjunction $\mathbb{H} \dashv \mathbf{H}$ of Proposition 3.4.5, the fourth from Proposition 3.2.5, and the last one holds by symmetry of the product. By the Yoneda Lemma, we get an isomorphism of double categories $\mathbb{V}\mathcal{A} \otimes \mathcal{C} \cong \mathbb{V}\mathcal{A} \times \mathbb{H}\mathcal{C}$ natural in $\mathcal{A}$ and $\mathcal{C}$, as desired. □

Remark 3.5.5. The above results implies that, for every 2-category $\mathcal{A}$, we have an isomorphism of double categories $\mathbb{L}\mathcal{A} = \mathbb{V}[1] \times \mathbb{H}\mathcal{A} \cong \mathbb{V}[1] \otimes \mathcal{A}$ natural in $\mathcal{A}$, where $\mathbb{L}$ is the left adjoint of the functor $\mathcal{V}\colon \mathrm{DblCat} \to 2\mathrm{Cat}$ given by Proposition 3.4.10.

We now compare the hom 2-categories of the 2Cat-enrichment of DblCat with the pseudo-hom 2-categories of 2Cat. The following lemma tells us that the adjunction $\mathbb{H} \dashv \mathbf{H}$ extends to an enriched adjunction between DblCat with the 2Cat-enrichment given by $\mathbf{H}[-,-]_{\mathrm{ps}}$ and the 2Cat-enrichment given by the Gray tensor product on 2Cat.

Lemma 3.5.6. *For every* 2*-category* $\mathcal{A}$ *and every double category* $\mathbb{B}$*, there is an isomorphism of* 2*-categories* $\mathbf{H}[\mathbb{H}\mathcal{A}, \mathbb{B}]_{\mathrm{ps}} \cong [\mathcal{A}, \mathbf{H}\mathbb{B}]_{2,\mathrm{ps}}$ *natural in* $\mathcal{A}$ *and* $\mathbb{B}$*.*

Proof. Given a 2-category $\mathcal{A}$ and a double category $\mathbb{B}$, we want to show that there is an isomorphism of 2-categories $\mathbf{H}[\mathbb{H}\mathcal{A}, \mathbb{B}]_{\mathrm{ps}} \cong [\mathcal{A}, \mathbf{H}\mathbb{B}]_{2,\mathrm{ps}}$ natural in $\mathcal{A}$ and $\mathbb{B}$. First note that, by the adjunction $\mathbb{H} \dashv \mathbf{H}$ of Proposition 3.4.5, there is an isomorphism between the underlying sets of objects $\mathrm{DblCat}(\mathbb{H}\mathcal{A}, \mathbb{B}) \cong 2\mathrm{Cat}(\mathcal{A}, \mathbf{H}\mathbb{B})$ natural in $\mathcal{A}$ and $\mathbb{B}$. Then, since $\mathbb{H}\mathcal{A}$ has no non trivial vertical morphisms, it is straightforward to see that a horizontal pseudo-natural transformation of double functors $\mathbb{H}\mathcal{A} \to \mathbb{B}$ corresponds to a pseudo-natural transformation of 2-functors $\mathcal{A} \to \mathbf{H}\mathbb{B}$ as every square component associated to a trivial vertical morphism is trivial by (hn1) of Definition 3.2.1. The rest of the data is the same and this can be seen by comparing Definitions 2.3.1 and 3.3.1. Similarly, one can check that modifications of double functors $\mathbb{H}\mathcal{A} \to \mathbb{B}$ correspond to modifications of 2-functors $\mathcal{A} \to \mathbf{H}\mathbb{B}$ by comparing Definitions 2.3.2 and 3.3.3. □

As a direct consequence, we can see that tensoring with a horizontal double category is given by the horizontal double category associated to the Gray tensor product of 2-categories. In other words, the functor $\mathbb{H}\colon 2\mathrm{Cat} \to \mathrm{DblCat}$ preserves the Gray tensor products.

Corollary 3.5.7. *For every pair of* 2*-categories* $\mathcal{A}$ *and* $\mathcal{C}$*, there is an isomorphism of double categories* $\mathbb{H}\mathcal{A} \otimes \mathcal{C} \cong \mathbb{H}(\mathcal{A} \otimes_2 \mathcal{C})$ *natural in* $\mathcal{A}$ *and* $\mathcal{C}$*.*

Proof. For every pair of 2-categories $\mathcal{A}$ and $\mathcal{C}$, and every double category $\mathbb{B}$, we have isomorphisms

$$\begin{aligned}\mathrm{DblCat}(\mathbb{H}\mathcal{A} \otimes \mathcal{C}, \mathbb{B}) &\cong 2\mathrm{Cat}(\mathcal{C}, \mathbf{H}[\mathbb{H}\mathcal{A}, \mathbb{B}]_{\mathrm{ps}}) \cong 2\mathrm{Cat}(\mathcal{C}, [\mathcal{A}, \mathbf{H}\mathbb{B}]_{2,\mathrm{ps}}) \\ &\cong 2\mathrm{Cat}(\mathcal{C} \otimes_2 \mathcal{A}, \mathbf{H}\mathbb{B}) \cong 2\mathrm{Cat}(\mathcal{A} \otimes_2 \mathcal{C}, \mathbf{H}\mathbb{B}) \\ &\cong \mathrm{DblCat}(\mathbb{H}(\mathcal{A} \otimes_2 \mathcal{C}), \mathbb{B})\end{aligned}$$

natural in $\mathcal{A}$, $\mathcal{C}$, and $\mathbb{B}$, where the first isomorphism follows from Proposition 3.5.2, the second from Lemma 3.5.6, the third from Proposition 2.3.4, the fourth from the fact that $\otimes_2$ is symmetric, and the last one from the adjunction $\mathbb{H} \dashv \mathbf{H}$ of Proposition 3.4.5. By the Yoneda Lemma, we get an isomorphism of double categories $\mathbb{H}\mathcal{A} \otimes \mathcal{C} \cong \mathbb{H}(\mathcal{A} \otimes_2 \mathcal{C})$ natural in $\mathcal{A}$ and $\mathcal{C}$, as desired. □

Finally, we show that the functor $\mathcal{V}$ of Definition 3.4.9 also commutes with pseudo-hom in the same way that $\mathbf{H}$ does.

Corollary 3.5.8. *For every 2-category $\mathcal{A}$ and every double category $\mathbb{B}$, there is an isomorphism of 2-categories $\mathcal{V}[\mathbb{H}\mathcal{A}, \mathbb{B}]_{\mathrm{ps}} \cong [\mathcal{A}, \mathcal{V}\mathbb{B}]_{2,\mathrm{ps}}$ natural in $\mathcal{A}$ and $\mathbb{B}$.*

Proof. Given a 2-category $\mathcal{A}$ and a double category $\mathbb{B}$, we have isomorphisms

$$\begin{aligned}
\mathcal{V}[\mathbb{H}\mathcal{A}, \mathbb{B}]_{\mathrm{ps}} &= \mathbf{H}[\mathbb{V}[1], [\mathbb{H}\mathcal{A}, \mathbb{B}]_{\mathrm{ps}}] = \mathbf{H}[\mathbb{V}[1], [\mathbb{H}\mathcal{A}, \mathbb{B}]_{\mathrm{ps}}]_{\mathrm{ps}} \\
&\cong \mathbf{H}[\mathbb{V}[1] \otimes_{\mathrm{Gr}} \mathbb{H}\mathcal{A}, \mathbb{B}]_{\mathrm{ps}} \cong \mathbf{H}[\mathbb{H}\mathcal{A}, [\mathbb{V}[1], \mathbb{B}]_{\mathrm{ps}}]_{\mathrm{ps}} \\
&\cong [\mathcal{A}, \mathbf{H}[\mathbb{V}[1], \mathbb{B}]_{\mathrm{ps}}]_{2,\mathrm{ps}} = [\mathcal{A}, \mathbf{H}[\mathbb{V}[1], \mathbb{B}]]_{2,\mathrm{ps}} = [\mathcal{A}, \mathcal{V}\mathbb{B}]_{2,\mathrm{ps}}
\end{aligned}$$

natural in $\mathcal{A}$ and $\mathbb{B}$, where the equalities hold by definition of $\mathcal{V}$ (see Definition 3.4.9) and Lemma 3.5.3, and the first isomorphism follows from Proposition 3.3.5, the second from the symmetry of the Gray tensor product and Proposition 3.3.5, and the last one from Lemma 3.5.6. □

3.6. **Weak horizontal invertibility in a double category.** In this last section on double categories, we introduce notions of weak invertibility for horizontal morphisms and squares in a double category $\mathbb{A}$. Since these correspond to the morphisms of the induced 2-categories $\mathbf{H}\mathbb{A}$ and $\mathcal{V}\mathbb{A}$, respectively, we can define a *horizontal equivalence* and a *weakly horizontally invertible square* to be an equivalence (see Definition 2.4.1) in the 2-category $\mathbf{H}\mathbb{A}$ and $\mathcal{V}\mathbb{A}$, respectively. We also introduce *weakly horizontally invariant double categories*, which will be the fibrant objects of the model structure on DblCat of Section 8. We study more carefully the behavior of weakly horizontally invertible squares. In particular, we show that weakly horizontally invertible squares in the double categories $\mathbb{H}\mathcal{A}$ and $\mathbb{H}^{\simeq}\mathcal{A}$ correspond to 2-isomorphisms in the 2-category $\mathcal{A}$. Finally, we introduce *horizontal pseudo-natural equivalences* as the horizontal pseudo-natural transformations whose square components are weakly horizontally invertible. In analogy to the case of pseudo-natural equivalences of Proposition 2.4.5, they correspond to the horizontal equivalences in pseudo-hom double category.

Let us first define horizontal (adjoint) equivalences.

Definition 3.6.1. Let $\mathbb{A}$ be a double category. A horizontal morphism $a\colon A \to C$ in $\mathbb{A}$ is a **horizontal equivalence** if it is an equivalence in the underlying horizontal 2-category $\mathbf{H}\mathbb{A}$. In other words, there is a tuple (a, c, η, ϵ) with $c\colon C \to A$ a horizontal morphism in $\mathbb{A}$ and η, ϵ two vertically invertible squares in $\mathbb{A}$ of the form

$$\begin{tikzcd}
A \arrow[rr, equal] \arrow[d, equal, "\bullet" marking] & \arrow[d, phantom, "\eta\ \cong"] & A \arrow[d, equal, "\bullet" marking] \\
A \arrow[r, "a"'] & C \arrow[r, "c"'] & A,
\end{tikzcd}
\qquad
\begin{tikzcd}
C \arrow[r, "c"] \arrow[d, equal, "\bullet" marking] & A \arrow[r, "a"] \arrow[d, phantom, "\epsilon\ \cong"] & C \arrow[d, equal, "\bullet" marking] \\
C \arrow[rr, equal] & & C.
\end{tikzcd}$$

Furthermore, the horizontal morphism $a\colon A \to B$ is a **horizontal adjoint equivalence** if it is an adjoint equivalence in $\mathbf{H}\mathbb{A}$, i.e., if the vertically invertible squares η and ϵ further satisfy the following triangle identities.

$$
\begin{tikzcd}
A \arrow[rr, equal] \arrow[d, equal, "\bullet" marking] & & A \arrow[r, "a"] \arrow[d, equal, "\bullet" marking] & C \arrow[d, equal, "\bullet" marking] \\
A \arrow[r, "a"] \arrow[d, equal, "\bullet" marking] \arrow[urr, phantom, "\eta\ \cong"] & C \arrow[r, "c"'] \arrow[d, equal, "\bullet" marking] & A \arrow[r, "a"'] \arrow[ur, phantom, "e_a"] & C \arrow[d, equal, "\bullet" marking] \\
A \arrow[r, "a"'] \arrow[ur, phantom, "e_a"] & C \arrow[rr, equal] \arrow[urr, phantom, "\epsilon\ \cong"] & & C
\end{tikzcd}
=
\begin{tikzcd}
A \arrow[r, "a"] \arrow[d, equal, "\bullet" marking] & C \arrow[d, equal, "\bullet" marking] \\
A \arrow[r, "a"'] \arrow[ur, phantom, "e_a"] & C
\end{tikzcd}
$$

$$
\begin{tikzcd}
C \arrow[r, "c"] \arrow[d, equal, "\bullet" marking] & A \arrow[rr, equal] \arrow[d, equal, "\bullet" marking] & & A \arrow[d, equal, "\bullet" marking] \\
C \arrow[r, "c"'] \arrow[d, equal, "\bullet" marking] \arrow[ur, phantom, "e_c"] & A \arrow[r, "a"'] \arrow[urr, phantom, "\eta\ \cong"] & C \arrow[r, "c"] \arrow[d, equal, "\bullet" marking] & A \arrow[d, equal, "\bullet" marking] \\
C \arrow[rr, equal] \arrow[urr, phantom, "\epsilon\ \cong"] & & C \arrow[r, "c"'] \arrow[ur, phantom, "e_c"] & A
\end{tikzcd}
=
\begin{tikzcd}
C \arrow[r, "c"] \arrow[d, equal, "\bullet" marking] & A \arrow[d, equal, "\bullet" marking] \\
C \arrow[r, "c"'] \arrow[ur, phantom, "e_c"] & A
\end{tikzcd}
$$

We often denote by $a\colon A \xrightarrow{\simeq} C$ a horizontal (adjoint) equivalence in $\mathbb{A}$.

By considering equivalences in the 2-category $\mathcal{V}\mathbb{A}$, we can define weakly horizontally invertible squares in $\mathbb{A}$.

Definition 3.6.2. Let $\mathbb{A}$ be a double category. A square $\alpha\colon (u\ {}^{a}_{a'}\ w)$ in $\mathbb{A}$ is **weakly horizontally invertible** if it is an equivalence in the 2-category $\mathcal{V}\mathbb{A}$. In other words, if there is a square $\gamma\colon (w\ {}^{c}_{c'}\ u)$ in $\mathbb{A}$ together with vertically invertible squares (η,ϵ) and (η',ϵ') satisfying the following pasting equalities.

$$
\begin{tikzcd}
A \arrow[rr, equal] \arrow[d, equal, "\bullet" marking] & & A \arrow[d, equal, "\bullet" marking] \\
A \arrow[r, "a"] \arrow[d, "u"', "\bullet" marking] \arrow[urr, phantom, "\eta\ \cong"] & C \arrow[r, "c"] \arrow[d, "w"', "\bullet" marking] & A \arrow[d, "u", "\bullet" marking] \\
A' \arrow[r, "a'"'] \arrow[ur, phantom, "\alpha"] & C' \arrow[r, "c'"'] \arrow[ur, phantom, "\gamma"] & A'
\end{tikzcd}
=
\begin{tikzcd}
A \arrow[rr, equal] \arrow[d, "u"', "\bullet" marking] & & A \arrow[d, "u", "\bullet" marking] \\
A' \arrow[rr, equal] \arrow[d, equal, "\bullet" marking] \arrow[urr, phantom, "\mathrm{id}_u"] & & A' \arrow[d, equal, "\bullet" marking] \\
A' \arrow[r, "a'"'] \arrow[urr, phantom, "\eta'\ \cong"] & C' \arrow[r, "c'"'] & A'
\end{tikzcd}
$$

$$
\begin{tikzcd}
C \arrow[r, "c"] \arrow[d, equal, "\bullet" marking] & A \arrow[r, "a"] & C \arrow[d, equal, "\bullet" marking] \\
C \arrow[rr, equal] \arrow[d, "w"', "\bullet" marking] \arrow[urr, phantom, "\epsilon\ \cong"] & & C \arrow[d, "w", "\bullet" marking] \\
C' \arrow[rr, equal] \arrow[urr, phantom, "\mathrm{id}_w"] & & C'
\end{tikzcd}
=
\begin{tikzcd}
C \arrow[r, "c"] \arrow[d, "w"', "\bullet" marking] & A \arrow[r, "a"] \arrow[d, "u"', "\bullet" marking] & C \arrow[d, "w", "\bullet" marking] \\
C' \arrow[r, "c'"'] \arrow[d, equal, "\bullet" marking] \arrow[ur, phantom, "\gamma"] & A' \arrow[r, "a'"'] \arrow[ur, phantom, "\alpha"] & C' \arrow[d, equal, "\bullet" marking] \\
C' \arrow[rr, equal] \arrow[urr, phantom, "\epsilon'\ \cong"] & & C'
\end{tikzcd}
$$

Note that the horizontal boundaries of the square α are horizontal equivalences (a,c,η,ϵ) and (a',c',η',ϵ'), called the **horizontal equivalence data** of α. We call γ a **weak inverse** of α with respect to the horizontal equivalence data (a,c,η,ϵ) and (a',c',η',ϵ').

We often write

$$
\begin{tikzcd}
A \arrow[r, "a"] \arrow[d, "u"', "\bullet" marking] & C \arrow[d, "w", "\bullet" marking] \\
A' \arrow[r, "a'"'] \arrow[ur, phantom, "\alpha\ \simeq"] & C'
\end{tikzcd}
$$

to denote that the square α is weakly horizontally invertible in $\mathbb{A}$.

Remark 3.6.3. Note that a square α is an adjoint equivalence in $\mathcal{V}\mathbb{A}$ if and only if it is weakly horizontally invertible and its horizontal equivalence data are horizontal adjoint equivalences. In this case, we call them the **horizontal adjoint equivalence data** of α.

As a direct consequence of the 2-categorical result saying that any equivalence in a 2-category can be promoted to an adjoint equivalence (see Proposition 2.4.4), we get that horizontal equivalences can be promoted to an adjoint one and, similarly, that a weakly horizontally invertible square can be promoted to one whose horizontal equivalence data is an adjoint one.

Lemma 3.6.4. *Let $\mathbb{A}$ be a double category. Every horizontal equivalence in $\mathbb{A}$ can be promoted to a horizontal adjoint equivalence. Moreover, every weakly horizontally invertible square in $\mathbb{A}$ can be promoted to one with horizontal adjoint equivalence data.*

Proof. Since horizontal equivalences and weakly horizontally invertible squares are equivalences in the 2-categories $\mathbf{H}\mathbb{A}$ and $\mathcal{V}\mathbb{A}$, respectively, this result is a direct consequence of Proposition 2.4.4. □

With this terminology, we can introduce a "weak" version of the horizontally invariant double categories introduced by Grandis and Paré in [GP99, §2.4]. These *weakly horizontally invariant double categories* will be the fibrant objects in the model structure on DblCat constructed in Section 8, and were first introduced in [MSV20b].

Definition 3.6.5. A double category $\mathbb{A}$ is **weakly horizontally invariant** if, for every pair of horizontal equivalences $a\colon A \xrightarrow{\simeq} C$ and $a'\colon A' \xrightarrow{\simeq} C'$ in $\mathbb{A}$ and every vertical morphism $w\colon C \twoheadrightarrow C'$ in $\mathbb{A}$, i.e., for every diagram in $\mathbb{A}$ as depicted below left, there is a vertical morphism $u\colon A \twoheadrightarrow A'$ in $\mathbb{A}$ together with a weakly horizontally invertible square α in $\mathbb{A}$ as depicted below right.

$$\begin{tikzcd} A \arrow[r, "a", "\simeq"'] & C \arrow[d, "w"] \\ A' \arrow[r, "\simeq", "a'"'] & C' \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "a", "\simeq"'] \arrow[d, "u"'] \arrow[dr, phantom, "\alpha\ \simeq"] & C \arrow[d, "w"] \\ A' \arrow[r, "\simeq", "a'"'] & C' \end{tikzcd}$$

We now prove some technical results about weakly horizontally invertible squares The first result says that, given a weakly horizontally invertible square α and horizontal adjoint equivalence data for its horizontal boundaries, we can define a weak inverse to α with respect to these horizontal adjoint equivalences, and that this weak inverse is unique.

Proposition 3.6.6. *Let $\mathbb{A}$ be a double category, and let α be a weakly horizontally invertible square in $\mathbb{A}$ of the form*

$$\begin{tikzcd} A \arrow[r, "a"] \arrow[d, "u"'] \arrow[dr, phantom, "\alpha\ \simeq"] & C \arrow[d, "w"] \\ A' \arrow[r, "a'"'] & C' \end{tikzcd} .$$

Let (a, c, η, ϵ) and $(a', c', \eta', \epsilon')$ be horizontal adjoint equivalence data in $\mathbb{A}$ for the horizontal equivalences a and a'. Then there is a unique weak inverse

$$\begin{tikzcd} C \arrow[r, "c"] \arrow[d, "w"'] \arrow[dr, phantom, "\gamma"] & A \arrow[d, "u"] \\ C' \arrow[r, "c'"'] & A' \end{tikzcd}$$

of α in $\mathbb{A}$ with respect to the horizontal adjoint equivalence data (a,c,η,ϵ) and (a',c',η',ϵ').

Proof. Since every weakly horizontally invertible square can be promoted to one with horizontal adjoint equivalence data by Lemma 3.6.4, there is a weak inverse $\beta\colon (w\ {}^{b}_{b'}\ u)$ of α in $\mathbb{A}$ with respect to horizontal adjoint equivalence data (a,b,μ,δ) and (a',b',μ',δ'). We define $\gamma\colon (w\ {}^{c}_{c'}\ u)$ to be the square given by the following pasting.

$$
\begin{tikzcd}
C \arrow[r, "c"] \arrow[d, "w"'] \arrow[dr, phantom, "\gamma"] & A \arrow[d, "u"] \\
C' \arrow[r, "c'"'] & A'
\end{tikzcd}
=
\begin{tikzcd}
C \arrow[rr, equal] \arrow[d, equal] \arrow[drr, phantom, "\delta^{-1}\cong"] & & C \arrow[r, "c"] \arrow[d, equal] \arrow[dr, phantom, "e_c"] & A \arrow[d, equal] \\
C \arrow[r, "b"] \arrow[d, equal] \arrow[dr, phantom, "e_b"] & A \arrow[r, "a"] \arrow[d, equal] \arrow[drr, phantom, "\eta^{-1}\cong"] & C \arrow[r, "c"'] & A \arrow[d, equal] \\
C \arrow[r, "b"] \arrow[d, "w"'] \arrow[dr, phantom, "\beta"] & A \arrow[rr, equal] \arrow[d, "u"] \arrow[drr, phantom, "\mathrm{id}_u"] & & A \arrow[d, "u"] \\
C' \arrow[r, "b'"'] \arrow[d, equal] \arrow[dr, phantom, "e_{b'}"] & A' \arrow[rr, equal] \arrow[d, equal] \arrow[drr, phantom, "\eta'\cong"] & & A' \arrow[d, equal] \\
C' \arrow[r, "b'"'] \arrow[d, equal] & A' \arrow[r, "a'"'] \arrow[dl, phantom, "\delta'\cong"] & C' \arrow[r, "c'"] \arrow[d, equal] \arrow[dr, phantom, "e_{c'}"] & A' \arrow[d, equal] \\
C' \arrow[rr, equal] & & C' \arrow[r, "c'"'] & A'
\end{tikzcd}
$$

We verify that γ is a weak inverse of α with respect to the horizontal adjoint equivalence data (a,c,η,ϵ) and (a',c',η',ϵ'). We have that

$$
\begin{tikzcd}
A \arrow[rr, equal] \arrow[d, equal] \arrow[drr, phantom, "\eta\cong"] & & A \arrow[d, equal] \\
A \arrow[r, "a"] \arrow[d, "u"'] \arrow[dr, phantom, "\alpha"] & C \arrow[r, "c"] \arrow[d, "w"] \arrow[dr, phantom, "\gamma"] & A \arrow[d, "u"] \\
A' \arrow[r, "a'"'] \arrow[d, equal] \arrow[drr, phantom, "(\eta')^{-1}\cong"] & C' \arrow[r, "c'"'] & A' \arrow[d, equal] \\
A' \arrow[rr, equal] & & A'
\end{tikzcd}
=
\begin{tikzcd}
A \arrow[rrrr, equal] \arrow[d, equal] \arrow[drrrr, phantom, "\eta\cong"] & & & & A \arrow[d, equal] \\
A \arrow[r, "a"] \arrow[d, equal] \arrow[dr, phantom, "e_a"] & C \arrow[rr, equal] \arrow[d, equal] \arrow[drr, phantom, "\delta^{-1}\cong"] & & C \arrow[r, "c"] \arrow[d, equal] \arrow[dr, phantom, "e_c"] & A \arrow[d, equal] \\
A \arrow[r, "a"] \arrow[d, equal] \arrow[dr, phantom, "e_a"] & C \arrow[r, "b"] \arrow[d, equal] \arrow[dr, phantom, "e_b"] & A \arrow[r, "a"] \arrow[d, equal] \arrow[drr, phantom, "\eta^{-1}\cong"] & C \arrow[r, "c"'] & A \arrow[d, equal] \\
A \arrow[r, "a"] \arrow[d, "u"'] \arrow[dr, phantom, "\alpha"] & C \arrow[r, "b"] \arrow[d, "w"] \arrow[dr, phantom, "\beta"] & A \arrow[rr, equal] \arrow[d, "u"] \arrow[drr, phantom, "\mathrm{id}_u"] & & A \arrow[d, "u"] \\
A' \arrow[r, "a'"'] \arrow[d, equal] \arrow[dr, phantom, "e_{a'}"] & C' \arrow[r, "b'"'] \arrow[d, equal] \arrow[dr, phantom, "e_{b'}"] & A' \arrow[rr, equal] \arrow[d, equal] \arrow[drr, phantom, "\eta'\cong"] & & A' \arrow[d, equal] \\
A' \arrow[r, "a'"'] \arrow[d, equal] \arrow[dr, phantom, "e_{a'}"] & C' \arrow[r, "b'"'] \arrow[d, equal] & A' \arrow[r, "a'"'] \arrow[d, phantom, "\delta'\cong"] & C' \arrow[r, "c'"] \arrow[d, equal] \arrow[dr, phantom, "e_{c'}"] & A' \arrow[d, equal] \\
A' \arrow[r, "a'"'] \arrow[d, equal] \arrow[drrrr, phantom, "(\eta')^{-1}\cong"] & C' \arrow[rr, equal] & & C' \arrow[r, "c'"'] & A' \arrow[d, equal] \\
A' \arrow[rrrr, equal] & & & & A'
\end{tikzcd}
$$

$$
= \begin{tikzcd}
A \arrow[rrrr, equal] \arrow[d, equal, "\bullet" marking] \arrow[drrrr, phantom, "\eta\ \cong"] & & & & A \arrow[d, equal, "\bullet" marking] \\
A \arrow[r, "a"] \arrow[d, equal, "\bullet" marking] \arrow[dr, phantom, "e_a"] & C \arrow[rr, equal] \arrow[d, equal, "\bullet" marking] \arrow[drr, phantom, "\delta^{-1}\cong"] & & C \arrow[r, "c"] \arrow[d, equal, "\bullet" marking] \arrow[dr, phantom, "e_c"] & A \arrow[d, equal, "\bullet" marking] \\
A \arrow[r, "a"'] \arrow[d, equal, "\bullet" marking] \arrow[drr, phantom, "\mu^{-1}\cong"] & C \arrow[r, "b"'] & A \arrow[r, "a"'] \arrow[d, equal, "\bullet" marking] \arrow[drr, phantom, "\eta^{-1}\cong"] & C \arrow[r, "c"'] & A \arrow[d, equal, "\bullet" marking] \\
A \arrow[rr, equal] \arrow[d, "u"', "\bullet" marking] \arrow[drr, phantom, "\mathrm{id}_u"] & & A \arrow[rr, equal] \arrow[d, "u", "\bullet" marking] \arrow[drr, phantom, "\mathrm{id}_u"] & & A \arrow[d, "u", "\bullet" marking] \\
A' \arrow[rr, equal] \arrow[d, equal, "\bullet" marking] \arrow[drr, phantom, "\mu'\ \cong"] & & A' \arrow[rr, equal] \arrow[d, equal, "\bullet" marking] \arrow[drr, phantom, "\eta'\ \cong"] & & A' \arrow[d, equal, "\bullet" marking] \\
A' \arrow[r, "a'"] \arrow[d, equal, "\bullet" marking] \arrow[dr, phantom, "e_{a'}"] & C' \arrow[r, "b'"] \arrow[d, equal, "\bullet" marking] \arrow[drr, phantom, "\delta'\ \cong"] & A' \arrow[r, "a'"] & C' \arrow[r, "c'"] \arrow[d, equal, "\bullet" marking] \arrow[dr, phantom, "e_{c'}"] & A' \arrow[d, equal, "\bullet" marking] \\
A' \arrow[r, "a'"'] \arrow[d, equal, "\bullet" marking] \arrow[drrrr, phantom, "(\eta')^{-1}\cong"] & C' \arrow[rr, equal] & & C' \arrow[r, "c'"'] & A' \arrow[d, equal, "\bullet" marking] \\
A' \arrow[rrrr, equal] & & & & A'
\end{tikzcd}
= \begin{tikzcd}
A \arrow[r, equal] \arrow[d, "u"', "\bullet" marking] \arrow[dr, phantom, "\mathrm{id}_u"] & A \arrow[d, "u", "\bullet" marking] \\
A' \arrow[r, equal] & A'
\end{tikzcd},
$$

where the first equality holds by definition of γ, the second since β is a weak inverse of α with respect to the horizontal adjoint equivalence data (a,b,μ,δ) and (a',b',μ',δ'), and the last one by the triangle identities for (μ,δ) and (μ',δ') and the fact that η^{-1} and $(\eta')^{-1}$ are the vertical inverses of η and η', respectively. Similarly, we have that

$$
\begin{tikzcd}
C \arrow[rr, equal] \arrow[d, equal, "\bullet" marking] \arrow[drr, phantom, "\epsilon^{-1}\cong"] & & C \arrow[d, equal, "\bullet" marking] \\
C \arrow[r, "c"] \arrow[d, "w"', "\bullet" marking] \arrow[dr, phantom, "\gamma"] & A \arrow[r, "a"] \arrow[d, "u"', "\bullet" marking] \arrow[dr, phantom, "\alpha"] & C \arrow[d, "w", "\bullet" marking] \\
C' \arrow[r, "c'"'] \arrow[d, equal, "\bullet" marking] \arrow[drr, phantom, "\epsilon'\ \cong"] & A' \arrow[r, "a'"'] & C' \arrow[d, equal, "\bullet" marking] \\
C' \arrow[rr, equal] & & C'
\end{tikzcd}
=
\begin{tikzcd}
C \arrow[rr, equal] \arrow[d, equal, "\bullet" marking] \arrow[drr, phantom, "\delta^{-1}\cong"] & & C \arrow[rr, equal] \arrow[d, equal, "\bullet" marking] \arrow[drr, phantom, "\epsilon^{-1}\cong"] & & C \arrow[d, equal, "\bullet" marking] \\
C \arrow[r, "b"] \arrow[d, equal, "\bullet" marking] \arrow[dr, phantom, "e_b"] & A \arrow[r, "a"] \arrow[d, equal, "\bullet" marking] \arrow[drr, phantom, "\eta^{-1}\cong"] & C \arrow[r, "c"] & A \arrow[r, "a"] \arrow[d, equal, "\bullet" marking] \arrow[dr, phantom, "e_a"] & C \arrow[d, equal, "\bullet" marking] \\
C \arrow[r, "b"] \arrow[d, "w"', "\bullet" marking] \arrow[dr, phantom, "\beta"] & A \arrow[rr, equal] \arrow[d, "u", "\bullet" marking] \arrow[drr, phantom, "\mathrm{id}_u"] & & A \arrow[r, "a"] \arrow[d, "u"', "\bullet" marking] \arrow[dr, phantom, "\alpha"] & C \arrow[d, "w", "\bullet" marking] \\
C' \arrow[r, "b'"'] \arrow[d, equal, "\bullet" marking] \arrow[dr, phantom, "e_{b'}"] & A' \arrow[rr, equal] \arrow[d, equal, "\bullet" marking] \arrow[drr, phantom, "\eta'\ \cong"] & & A' \arrow[r, "a'"'] \arrow[d, equal, "\bullet" marking] \arrow[dr, phantom, "e_{a'}"] & C' \arrow[d, equal, "\bullet" marking] \\
C' \arrow[r, "b'"'] \arrow[d, equal, "\bullet" marking] \arrow[drr, phantom, "\delta'\ \cong"] & A' \arrow[r, "a'"'] & C' \arrow[r, "c'"'] \arrow[d, equal, "\bullet" marking] \arrow[drr, phantom, "\epsilon'\ \cong"] & A' \arrow[r, "a'"'] & C' \arrow[d, equal, "\bullet" marking] \\
C' \arrow[rr, equal] & & C' \arrow[rr, equal] & & C'
\end{tikzcd}
$$

$$
= \begin{tikzcd}
C \arrow[rr, equal] \arrow[d, equal, "\bullet" marking] \arrow[drr, phantom, "\delta^{-1} \mathrel{\rotatebox{90}{$\cong$}}"] & & C \arrow[d, equal, "\bullet" marking] \\
C \arrow[r, "b"] \arrow[d, "w"', "\bullet" marking] \arrow[dr, phantom, "\beta"] & A \arrow[r, "a"] \arrow[d, "u"', "\bullet" marking] \arrow[dr, phantom, "\alpha"] & C \arrow[d, "w", "\bullet" marking] \\
C' \arrow[r, "b'"'] \arrow[d, equal, "\bullet" marking] \arrow[drr, phantom, "\delta' \mathrel{\rotatebox{90}{$\cong$}}"] & A' \arrow[r, "a'"'] & C' \arrow[d, equal, "\bullet" marking] \\
C' \arrow[rr, equal] & & C'
\end{tikzcd}
=
\begin{tikzcd}
C \arrow[r, equal] \arrow[d, "w"', "\bullet" marking] \arrow[dr, phantom, "\mathrm{id}_w"] & C \arrow[d, "w", "\bullet" marking] \\
C' \arrow[r, equal] & C'
\end{tikzcd},
$$

where the first equality holds by definition of γ, the second by the triangle identities for (η, ϵ) and (η', ϵ'), and the last one since β is a weak inverse of α with respect to the horizontal adjoint equivalence data (a, b, μ, δ) and (a', b', μ', δ'). This shows that γ is a weak inverse of α with respect to the horizontal adjoint equivalence data (a, c, η, ϵ) and $(a', c', \eta', \epsilon')$, and hence the existence of a weak inverse for α with respect to the horizontal adjoint equivalence data (a, c, η, ϵ) and $(a', c', \eta', \epsilon')$.

We now prove uniqueness. Suppose that $\gamma' \colon (w \, {}^{c}_{c'} \, u)$ is another weak inverse of α with respect to the horizontal adjoint equivalence data (a, c, η, ϵ) and $(a', c', \eta', \epsilon')$. Then we have that

$$
\begin{tikzcd}
C \arrow[r, "c"] \arrow[d, "w"', "\bullet" marking] \arrow[dr, phantom, "\gamma'"] & A \arrow[d, "u", "\bullet" marking] \\
C' \arrow[r, "c'"'] & A'
\end{tikzcd}
=
\begin{tikzcd}
C \arrow[r, "c"] \arrow[d, "w"', "\bullet" marking] \arrow[dr, phantom, "\gamma'"] & A \arrow[r, equal] \arrow[d, "u", "\bullet" marking] \arrow[dr, phantom, "\mathrm{id}_u"] & A \arrow[d, "u", "\bullet" marking] \\
C' \arrow[r, "c'"'] & A' \arrow[r, equal] & A'
\end{tikzcd}
=
\begin{tikzcd}
C \arrow[r, "c"] \arrow[d, equal, "\bullet" marking] \arrow[dr, phantom, "e_c"] & A \arrow[rr, equal] \arrow[d, equal, "\bullet" marking] \arrow[drr, phantom, "\eta \mathrel{\rotatebox{90}{$\cong$}}"] & & A \arrow[d, equal, "\bullet" marking] \\
C \arrow[r, "c"] \arrow[d, "w"', "\bullet" marking] \arrow[dr, phantom, "\gamma'"] & A \arrow[r, "a"] \arrow[d, "u"', "\bullet" marking] \arrow[dr, phantom, "\alpha"] & C \arrow[r, "c"] \arrow[d, "w"', "\bullet" marking] \arrow[dr, phantom, "\gamma"] & A \arrow[d, "u", "\bullet" marking] \\
C' \arrow[r, "c'"'] \arrow[d, equal, "\bullet" marking] \arrow[dr, phantom, "e_{c'}"] & A' \arrow[r, "a'"'] \arrow[d, equal, "\bullet" marking] \arrow[drr, phantom, "(\eta')^{-1} \mathrel{\rotatebox{90}{$\cong$}}"] & C' \arrow[r, "c'"'] & A' \arrow[d, equal, "\bullet" marking] \\
C' \arrow[r, "c'"'] & A' \arrow[rr, equal] & & A'
\end{tikzcd}
$$

$$
=
\begin{tikzcd}
C \arrow[rr, equal] \arrow[d, equal, "\bullet" marking] \arrow[drr, phantom, "\epsilon^{-1} \mathrel{\rotatebox{90}{$\cong$}}"] & & C \arrow[r, "c"] \arrow[d, equal, "\bullet" marking] \arrow[dr, phantom, "e_c"] & A \arrow[d, equal, "\bullet" marking] \\
C \arrow[r, "c"] \arrow[d, "w"', "\bullet" marking] \arrow[dr, phantom, "\gamma'"] & A \arrow[r, "a"] \arrow[d, "u"', "\bullet" marking] \arrow[dr, phantom, "\alpha"] & C \arrow[r, "c"] \arrow[d, "w"', "\bullet" marking] \arrow[dr, phantom, "\gamma"] & A \arrow[d, "u", "\bullet" marking] \\
C' \arrow[r, "c'"'] \arrow[d, equal, "\bullet" marking] \arrow[drr, phantom, "\epsilon' \mathrel{\rotatebox{90}{$\cong$}}"] & A' \arrow[r, "a'"'] & C' \arrow[r, "c'"'] \arrow[d, equal, "\bullet" marking] \arrow[dr, phantom, "e_{c'}"] & A' \arrow[d, equal, "\bullet" marking] \\
C' \arrow[rr, equal] & & C' \arrow[r, "c'"'] & A'
\end{tikzcd}
=
\begin{tikzcd}
C \arrow[r, equal] \arrow[d, "w"', "\bullet" marking] \arrow[dr, phantom, "\mathrm{id}_w"] & C \arrow[r, "c"] \arrow[d, "w"', "\bullet" marking] \arrow[dr, phantom, "\gamma"] & A \arrow[d, "u", "\bullet" marking] \\
C' \arrow[r, equal] & C' \arrow[r, "c'"'] & A'
\end{tikzcd}
=
\begin{tikzcd}
C \arrow[r, "c"] \arrow[d, "w"', "\bullet" marking] \arrow[dr, phantom, "\gamma"] & A \arrow[d, "u", "\bullet" marking] \\
C' \arrow[r, "c'"'] & A'
\end{tikzcd},
$$

where the second equality holds since γ is a weak inverse of α with respect to the horizontal adjoint equivalence data (a, c, η, ϵ) and $(a', c', \eta', \epsilon')$, the third one by the triangle identities for (η, ϵ) and (η', ϵ'), and the fourth one since γ' is a weak inverse of α with respect to the horizontal adjoint equivalence data (a, c, η, ϵ) and $(a', c', \eta', \epsilon)$. This shows that $\gamma' = \gamma$ and that the weak inverse of α with respect to the horizontal adjoint equivalence data (a, c, η, ϵ) and $(a', c', \eta', \epsilon')$ is unique. $\square$

We now show that a square with trivial vertical boundaries is weakly horizontally invertible if and only if it is vertically invertible. In particular, since the horizontal double category $\mathbb{H}\mathcal{A}$ associated to a 2-category $\mathcal{A}$ has only trivial vertical morphisms, its weakly horizontally invertible squares correspond to the 2-isomorphisms in $\mathcal{A}$.

Proposition 3.6.7. *Consider a square α in a double category $\mathbb{A}$ of the form*

$$\begin{tikzcd}
A \arrow[r, "a", "\simeq"'] \arrow[d, equal, "\bullet" marking] \arrow[dr, phantom, "\alpha"] & C \arrow[d, equal, "\bullet" marking] \\
A \arrow[r, "\simeq", "a'"'] & C
\end{tikzcd},$$

where a and a' are horizontal equivalences. Then the square α is weakly horizontally invertible if and only if it is vertically invertible.

Proof. Suppose first that α is weakly horizontally invertible. Let (a, c, η, ϵ) and $(a', c', \eta', \epsilon')$ be horizontal adjoint equivalence data for a and a', and let $\gamma\colon (e_C \ {}^{c}_{c'}\ e_A)$ be the unique weak inverse of α, given by Proposition 3.6.6, with respect the horizontal adjoint equivalence data (a, c, η, ϵ) and $(a', c', \eta', \epsilon')$. We define β to be the following pasting.

$$\begin{tikzcd}
A \arrow[r, "a'"] \arrow[d, equal, "\bullet" marking] \arrow[dr, phantom, "\beta"] & C \arrow[d, equal, "\bullet" marking] \\
A \arrow[r, "a"'] & C
\end{tikzcd}
=
\begin{tikzcd}
A \arrow[rr, equal] \arrow[d, equal, "\bullet" marking] \arrow[drr, phantom, "\eta\ \cong"] & & A \arrow[r, "a'"] \arrow[d, equal, "\bullet" marking] \arrow[dr, phantom, "e_{a'}"] & C \arrow[d, equal, "\bullet" marking] \\
A \arrow[r, "a"] \arrow[d, equal, "\bullet" marking] \arrow[dr, phantom, "e_a"] & C \arrow[r, "c"] \arrow[d, equal, "\bullet" marking] \arrow[dr, phantom, "\gamma"] & A \arrow[r, "a'"] \arrow[d, equal, "\bullet" marking] \arrow[dr, phantom, "e_{a'}"] & C \arrow[d, equal, "\bullet" marking] \\
A \arrow[r, "a"'] \arrow[d, equal, "\bullet" marking] \arrow[dr, phantom, "e_a"] & C \arrow[r, "c'"'] \arrow[d, equal, "\bullet" marking] \arrow[drr, phantom, "\epsilon'\ \cong"] & A \arrow[r, "a'"'] & C \arrow[d, equal, "\bullet" marking] \\
A \arrow[r, "a"'] & C \arrow[rr, equal] & & C
\end{tikzcd}$$

We now verify that β is the vertical inverse of α by showing that both vertical composites of α and β give the vertical identity square at a and a', respectively. We have that

$$\begin{tikzcd}
A \arrow[r, "a"] \arrow[d, equal, "\bullet" marking] \arrow[dr, phantom, "\alpha"] & C \arrow[d, equal, "\bullet" marking] \\
A \arrow[r, "a'"] \arrow[d, equal, "\bullet" marking] \arrow[dr, phantom, "\beta"] & C \arrow[d, equal, "\bullet" marking] \\
A \arrow[r, "a"'] & C
\end{tikzcd}
=
\begin{tikzcd}
A \arrow[rr, equal] \arrow[d, equal, "\bullet" marking] \arrow[drr, phantom, "\eta\ \cong"] & & A \arrow[r, "a"] \arrow[d, equal, "\bullet" marking] \arrow[dr, phantom, "e_a"] & C \arrow[d, equal, "\bullet" marking] \\
A \arrow[r, "a"] \arrow[d, equal, "\bullet" marking] \arrow[dr, phantom, "e_a"] & C \arrow[r, "c"] \arrow[d, equal, "\bullet" marking] \arrow[dr, phantom, "\gamma"] & A \arrow[r, "a"] \arrow[d, equal, "\bullet" marking] \arrow[dr, phantom, "\alpha"] & C \arrow[d, equal, "\bullet" marking] \\
A \arrow[r, "a"'] \arrow[d, equal, "\bullet" marking] \arrow[dr, phantom, "e_a"] & C \arrow[r, "c'"'] \arrow[d, equal, "\bullet" marking] \arrow[drr, phantom, "\epsilon'\ \cong"] & A \arrow[r, "a'"'] & C \arrow[d, equal, "\bullet" marking] \\
A \arrow[r, "a"'] & C \arrow[rr, equal] & & C
\end{tikzcd}$$

$$= \begin{tikzcd} A \arrow[rr, equal] \arrow[d, equal, "\bullet" marking] & & A \arrow[r, "a"] \arrow[d, equal, "\bullet" marking] & C \arrow[d, equal, "\bullet" marking] \\ A \arrow[r, "a"] \arrow[d, equal, "\bullet" marking] \arrow[urr, phantom, "\eta \ \cong"] & C \arrow[r, "c"'] \arrow[d, equal, "\bullet" marking] & A \arrow[r, "a"'] \arrow[ur, phantom, "e_a"] & C \arrow[d, equal, "\bullet" marking] \\ A \arrow[r, "a"'] \arrow[ur, phantom, "e_a"] & C \arrow[rr, equal] \arrow[urr, phantom, "\epsilon \ \cong"] & & C \end{tikzcd} = \begin{tikzcd} A \arrow[r, "a"] \arrow[d, equal, "\bullet" marking] & C \arrow[d, equal, "\bullet" marking] \\ A \arrow[r, "a"'] \arrow[ur, phantom, "e_a"] & C \end{tikzcd},$$

where the first equality holds by definition of β, the second since γ is the weak inverse of α with respect to the horizontal adjoint equivalence data (a, c, η, ϵ) and $(a', c', \eta', \epsilon')$, and the last one by the triangle identities for (η, ϵ). Similarly, one can show that the other vertical composite of β and α is $e_{a'}$, using the other relation saying that (α, γ) are weak inverses with respect to the horizontal adjoint equivalence data (a, c, η, ϵ) and $(a', c', \eta', \epsilon')$, and the triangle identities for (η', ϵ'). This shows that the square β is a vertical inverse of α and hence that the square α is vertically invertible.

Now suppose that α is vertically invertible with vertical inverse α^{-1}. Let (a, c, η, ϵ) be horizontal equivalence data for a. Then the data $(a', c, \eta', \epsilon')$ is a horizontal equivalence data for a', where the vertically invertible squares η' and ϵ' are given by the following pasting equalities.

$$\begin{tikzcd} A \arrow[rr, equal] \arrow[d, equal, "\bullet" marking] & & A \arrow[d, equal, "\bullet" marking] \\ A \arrow[r, "a'"'] \arrow[urr, phantom, "\eta' \ \cong"] & C \arrow[r, "c"'] & A \end{tikzcd} = \begin{tikzcd} A \arrow[rr, equal] \arrow[d, equal, "\bullet" marking] & & A \arrow[d, equal, "\bullet" marking] \\ A \arrow[r, "a"] \arrow[d, equal, "\bullet" marking] \arrow[urr, phantom, "\eta \ \cong"] & C \arrow[r, "c"] \arrow[d, equal, "\bullet" marking] & A \arrow[d, equal, "\bullet" marking] \\ A \arrow[r, "a'"'] \arrow[ur, phantom, "\alpha \ \cong"] & C \arrow[r, "c"'] \arrow[ur, phantom, "e_c"] & A \end{tikzcd}$$

$$\begin{tikzcd} C \arrow[r, "c"] \arrow[d, equal, "\bullet" marking] & A \arrow[r, "a'"] & C \arrow[d, equal, "\bullet" marking] \\ C \arrow[rr, equal] \arrow[urr, phantom, "\epsilon' \ \cong"] & & C \end{tikzcd} = \begin{tikzcd} C \arrow[r, "c"] \arrow[d, equal, "\bullet" marking] & A \arrow[r, "a'"] \arrow[d, equal, "\bullet" marking] & C \arrow[d, equal, "\bullet" marking] \\ C \arrow[r, "c"'] \arrow[d, equal, "\bullet" marking] \arrow[ur, phantom, "e_c"] & A \arrow[r, "a"'] \arrow[ur, phantom, "\alpha^{-1} \cong"] & C \arrow[d, equal, "\bullet" marking] \\ C \arrow[rr, equal] \arrow[urr, phantom, "\epsilon \ \cong"] & & C \end{tikzcd}$$

It is then straightforward to see that the vertical identity square e_c is a weak inverse of α with respect to the horizontal equivalence data (a, c, η, ϵ) and $(a', c, \eta', \epsilon')$. This shows that α is weakly horizontally invertible. $\square$

We now show that a square in the double category $\mathbb{H}^{\simeq}\mathcal{A}$ associated to a 2-category $\mathcal{A}$ is weakly horizontally invertible if and only if it is induced by a 2-isomorphism in $\mathcal{A}$.

Lemma 3.6.8. *Let $\mathcal{A}$ be a 2-category and let α be a square in $\mathbb{H}^{\simeq}\mathcal{A}$ of the form*

$$\begin{tikzcd} A \arrow[r, "a", "\simeq"'] \arrow[d, "\bullet" marking, "\underline{u} = (u, u', \eta_u, \epsilon_u)"'] & C \arrow[d, "\bullet" marking, "\underline{w} = (w, w', \eta_w, \epsilon_w)"] \\ A' \arrow[r, "\simeq", "a'"'] \arrow[ur, phantom, "\alpha"] & C' \end{tikzcd},$$

where the horizontal morphisms a and a' are equivalences in $\mathcal{A}$, and the vertical morphisms $\underline{u}$ and $\underline{w}$ in $\mathbb{H}^{\simeq}\mathcal{A}$ are induced by the adjoint equivalences $(u, u', \eta_u, \epsilon_u)$ and $(w, w', \eta_w, \epsilon_w)$ in $\mathcal{A}$. Then α is weakly horizontally invertible if and only if its associated 2-morphism $\alpha\colon wa \Rightarrow a'u$ is a 2-isomorphism.

Proof. First note that, by definition of $\mathbb{H}^{\simeq}\mathcal{A}$, the square α in $\mathbb{H}^{\simeq}\mathcal{A}$ corresponds to a 2-morphism $\alpha\colon wa \Rightarrow a'u$ in $\mathcal{A}$, which also gives rise to a square $\overline{\alpha}$ in $\mathbb{H}^{\simeq}\mathcal{A}$, defined as the following pasting

$$\begin{tikzcd}
A \arrow[r, "a"] \arrow[d, equal] & C \arrow[r, "w"] \arrow[d, phantom, "\overline{\alpha}"] & C' \arrow[d, equal] \\
A \arrow[r, "u"'] & A' \arrow[r, "a'"'] & C'
\end{tikzcd}
=
\begin{tikzcd}
A \arrow[r, equal] \arrow[d, equal] \arrow[dr, phantom, "\mathrm{id}_u"] & A \arrow[r, "a"] \arrow[d, "\underline{u}"'] \arrow[dr, phantom, "\alpha"] & C \arrow[r, "w"] \arrow[d, "\underline{w}"] \arrow[dr, phantom, "\mathrm{id}_w"] & C' \arrow[d, equal] \\
A \arrow[r, "u"'] & A' \arrow[r, "a'"'] & C' \arrow[r, equal] & C'
\end{tikzcd},$$

where the squares id_u and id_w are induced by the identity 2-morphisms in $\mathcal{A}$ at u and w, respectively. Note that the composites wa and $a'u$ are horizontal equivalences in $\mathbb{H}^{\simeq}\mathcal{A}$. We show that α is weakly horizontally invertible if and only if $\overline{\alpha}$ is weakly horizontally invertible, and then conclude by applying Proposition 3.6.7.

Let (a, c, η, ϵ) and $(a', c', \eta', \epsilon')$ be adjoint equivalence data in $\mathcal{A}$ for a and a'. Suppose that α is weakly horizontally invertible, and let γ be its weak inverse in $\mathbb{H}^{\simeq}\mathcal{A}$, given by Proposition 3.6.6,

$$\begin{tikzcd}
C \arrow[r, "c"] \arrow[d, "\underline{w}"'] \arrow[dr, phantom, "\gamma"] & A \arrow[d, "\underline{u}"] \\
C' \arrow[r, "c'"'] & A'
\end{tikzcd}$$

with respect to the horizontal adjoint equivalence data (a, c, η, ϵ) and $(a', c', \eta', \epsilon')$. We define $\overline{\gamma}$ to be the following pasting of squares in $\mathbb{H}^{\simeq}\mathcal{A}$

$$\begin{tikzcd}
C' \arrow[r, "w'"] \arrow[d, equal] & C \arrow[r, "c"] \arrow[d, phantom, "\overline{\gamma}"] & A \arrow[d, equal] \\
C' \arrow[r, "c'"'] & A' \arrow[r, "u'"'] & A
\end{tikzcd}
=
\begin{tikzcd}
C' \arrow[r, "w'"] \arrow[d, equal] \arrow[dr, phantom, "\epsilon_w"] & C \arrow[r, "c"] \arrow[d, "\underline{w}"'] \arrow[dr, phantom, "\gamma"] & A \arrow[r, equal] \arrow[d, "\underline{u}"] \arrow[dr, phantom, "\eta_u"] & A \arrow[d, equal] \\
C' \arrow[r, equal] & C' \arrow[r, "c'"'] & A' \arrow[r, "u'"'] & A
\end{tikzcd},$$

where the squares ϵ_w and η_u are induced by the 2-isomorphisms $\epsilon_w\colon ww' \Rightarrow \mathrm{id}_{C'}$ and $\eta_u\colon \mathrm{id}_A \Rightarrow u'u$ of $\mathcal{A}$. We show that $\overline{\gamma}$ is a weak inverse for the square $\overline{\alpha}$ with respect to the composite of the horizontal adjoint equivalence data of (a, c, η, ϵ) with $(w, w', \eta_w, \epsilon_w)$, and of $(u, u', \eta_u, \epsilon_u)$ with $(a', c', \eta', \epsilon')$. We have that

$$\begin{tikzcd}
A \arrow[rrrr, equal] \arrow[d, equal] & & \arrow[d, phantom, "\eta\ \cong"] & & A \arrow[d, equal] \\
A \arrow[r, "a"] \arrow[d, equal] \arrow[dr, phantom, "e_a"] & C \arrow[rr, equal] \arrow[d, equal] & \arrow[d, phantom, "\eta_w\ \cong"] & C \arrow[r, "c"] \arrow[d, equal] \arrow[dr, phantom, "e_c"] & A \arrow[d, equal] \\
A \arrow[r, "a"'] \arrow[d, equal] & C \arrow[r, "w"'] \arrow[d, phantom, "\overline{\alpha}"] & C' \arrow[r, "w'"'] \arrow[d, equal] & C \arrow[r, "c"'] \arrow[d, phantom, "\overline{\gamma}"] & A \arrow[d, equal] \\
A \arrow[r, "u"'] & A' \arrow[r, "a'"'] & C' \arrow[r, "c'"'] & A' \arrow[r, "u'"'] & A
\end{tikzcd}$$

$$
= \quad
\begin{tikzcd}
A \arrow[r, equal] \arrow[d, equal, "\bullet" marking] \arrow[dr, phantom, "\square_A"] & A \arrow[rrrr, equal] \arrow[d, equal, "\bullet" marking] \arrow[drrrr, phantom, "\eta \ \cong"] & & & & A \arrow[r, equal] \arrow[d, equal, "\bullet" marking] \arrow[dr, phantom, "\square_A"] & A \arrow[d, equal, "\bullet" marking] \\
A \arrow[r, equal] \arrow[d, equal, "\bullet" marking] \arrow[dr, phantom, "\square_A"] & A \arrow[r, "a"] \arrow[d, equal, "\bullet" marking] \arrow[dr, phantom, "e_a"] & C \arrow[rr, equal] \arrow[d, equal, "\bullet" marking] \arrow[drr, phantom, "\eta_w \ \cong"] & & C \arrow[r, "c"] \arrow[d, equal, "\bullet" marking] \arrow[dr, phantom, "e_c"] & A \arrow[r, equal] \arrow[d, equal, "\bullet" marking] \arrow[dr, phantom, "\square_A"] & A \arrow[d, equal, "\bullet" marking] \\
A \arrow[r, equal] \arrow[d, equal, "\bullet" marking] \arrow[dr, phantom, "\mathrm{id}_u"] & A \arrow[r, "a"] \arrow[d, "\underline{u}"', "\bullet" marking] \arrow[dr, phantom, "\alpha"] & C \arrow[r, "w"] \arrow[d, "\underline{w}", "\bullet" marking] \arrow[dr, phantom, "\mathrm{id}_w"] & C' \arrow[r, "w'"] \arrow[d, equal, "\bullet" marking] \arrow[dr, phantom, "\epsilon_w"] & C \arrow[r, "c"] \arrow[d, "\underline{w}"', "\bullet" marking] \arrow[dr, phantom, "\gamma"] & A \arrow[r, equal] \arrow[d, "\underline{u}", "\bullet" marking] \arrow[dr, phantom, "\eta_u"] & A \arrow[d, equal, "\bullet" marking] \\
A \arrow[r, "u"'] & A' \arrow[r, "a'"'] & C' \arrow[r, equal] & C' \arrow[r, equal] & C' \arrow[r, "c'"'] & A' \arrow[r, "u'"'] & A
\end{tikzcd}
$$

$$
= \quad
\begin{tikzcd}
A \arrow[r, equal] \arrow[d, equal, "\bullet" marking] \arrow[dr, phantom, "\square_A"] & A \arrow[rr, equal] \arrow[d, equal, "\bullet" marking] \arrow[drr, phantom, "\eta \ \cong"] & & A \arrow[r, equal] \arrow[d, equal, "\bullet" marking] \arrow[dr, phantom, "\square_A"] & A \arrow[d, equal, "\bullet" marking] \\
A \arrow[r, equal] \arrow[d, equal, "\bullet" marking] \arrow[dr, phantom, "\mathrm{id}_u"] & A \arrow[r, "a"] \arrow[d, "\underline{u}"', "\bullet" marking] \arrow[dr, phantom, "\alpha"] & C \arrow[r, "c"] \arrow[d, "\underline{w}"', "\bullet" marking] \arrow[dr, phantom, "\gamma"] & A \arrow[r, equal] \arrow[d, "\underline{u}", "\bullet" marking] \arrow[dr, phantom, "\eta_u"] & A \arrow[d, equal, "\bullet" marking] \\
A \arrow[r, "u"'] & A' \arrow[r, "a'"'] & C' \arrow[r, "c'"'] & A' \arrow[r, "u'"'] & A
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
A \arrow[rrrr, equal] \arrow[d, equal, "\bullet" marking] \arrow[drrrr, phantom, "\eta_u \ \cong"] & & & & A \arrow[d, equal, "\bullet" marking] \\
A \arrow[r, "u"] \arrow[d, equal, "\bullet" marking] \arrow[dr, phantom, "e_u"] & C \arrow[rr, equal] \arrow[d, equal, "\bullet" marking] \arrow[drr, phantom, "\eta' \ \cong"] & & C \arrow[r, "u'"] \arrow[d, equal, "\bullet" marking] \arrow[dr, phantom, "e_{u'}"] & A \arrow[d, equal, "\bullet" marking] \\
A \arrow[r, "u"'] & C \arrow[r, "a'"'] & C' \arrow[r, "c'"'] & C \arrow[r, "u'"'] & A \, ,
\end{tikzcd}
$$

where the first equality holds by definition of $\overline{\alpha}$ and $\overline{\gamma}$, the second by the triangle identities for (η_w, ϵ_w), and the third since γ is the weak inverse of α with respect to the horizontal adjoint equivalence data (a, c, η, ϵ) and $(a', c', \eta', \epsilon')$ and by rearranging the square. The other relation involving the counits can be shown similarly. Hence $\overline{\gamma}$ is a weak inverse of $\overline{\alpha}$, and this shows that $\overline{\alpha}$ is weakly horizontally invertible.

Now suppose that the square $\overline{\alpha}$ is weakly horizontally invertible, and let $\overline{\gamma}$ be its weak inverse in $\mathbb{H}^{\simeq}\mathcal{A}$, given by Proposition 3.6.6,

$$
\begin{tikzcd}
C' \arrow[r, "w'"] \arrow[d, equal, "\bullet" marking] \arrow[drr, phantom, "\overline{\gamma}"] & C \arrow[r, "c"] & A \arrow[d, equal, "\bullet" marking] \\
C' \arrow[r, "c'"'] & A' \arrow[r, "u'"'] & A
\end{tikzcd}
$$

with respect to the composite of the horizontal adjoint equivalence data (a, c, η, ϵ) and $(w, w', \eta_w, \epsilon_w)$, and the composite of the horizontal adjoint equivalence data $(u, u', \eta_u, \epsilon_u)$ and $(a', c', \eta', \epsilon')$. We define γ to be the following pasting of squares in $\mathbb{H}^{\simeq}\mathcal{A}$

$$
\begin{tikzcd}
C \arrow[r, "c"] \arrow[d, "\underline{w}"', "\bullet" marking] \arrow[dr, phantom, "\gamma"] & A \arrow[d, "\underline{u}", "\bullet" marking] \\
C' \arrow[r, "c'"'] & A'
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
C \arrow[r, equal] \arrow[d, "\underline{w}"', "\bullet" marking] \arrow[dr, phantom, "\eta_w"] & C \arrow[r, "c"] \arrow[d, equal, "\bullet" marking] \arrow[dr, phantom, "e_c"] & A \arrow[d, equal, "\bullet" marking] \\
C' \arrow[r, "w'"'] \arrow[d, equal, "\bullet" marking] \arrow[drr, phantom, "\overline{\gamma}"] & C \arrow[r, "c"'] & A \arrow[d, equal, "\bullet" marking] \\
C' \arrow[r, "c'"] \arrow[d, equal, "\bullet" marking] \arrow[dr, phantom, "e_{c'}"] & A' \arrow[r, "u'"] \arrow[d, equal, "\bullet" marking] \arrow[dr, phantom, "\epsilon_u"] & A \arrow[d, "\underline{u}", "\bullet" marking] \\
C' \arrow[r, "c'"'] & A' \arrow[r, equal] & A' \, ,
\end{tikzcd}
$$

where the squares η_w and ϵ_u are induced by the 2-isomorphisms $\eta_w\colon \mathrm{id}_C \Rightarrow w'w$ and $\epsilon_u\colon uu' \Rightarrow \mathrm{id}_{A'}$ of $\mathcal{A}$. We show that γ is a weak inverse for the square α with respect to the horizontal adjoint equivalence data (a,c,η,ϵ) and (a',c',η',ϵ'). We have that

$$
\begin{tikzcd}
A \arrow[rr, equal] \arrow[d, equal, "\bullet" marking] \arrow[rr, phantom, shift right=4ex, "\eta\ \cong"] & & A \arrow[d, equal, "\bullet" marking] \\
A \arrow[r, "a"] \arrow[d, "\underline{u}"', "\bullet" marking] \arrow[r, phantom, shift right=4ex, "\alpha"] & C \arrow[r, "c"] \arrow[d, "\underline{w}"', "\bullet" marking] \arrow[r, phantom, shift right=4ex, "\gamma"] & A \arrow[d, "\underline{u}", "\bullet" marking] \\
A' \arrow[r, "a'"'] & C' \arrow[r, "c'"'] & A'
\end{tikzcd}
=
\begin{tikzcd}
A \arrow[rrrr, equal] \arrow[d, equal, "\bullet" marking] \arrow[rrrr, phantom, shift right=4ex, "\eta\ \cong"] & & & & A \arrow[d, equal, "\bullet" marking] \\
A \arrow[r, "a"] \arrow[d, equal, "\bullet" marking] \arrow[r, phantom, shift right=4ex, "e_a"] & C \arrow[r, equal] \arrow[d, equal, "\bullet" marking] \arrow[r, phantom, shift right=4ex, "\mathrm{id}_w"] & C \arrow[r, equal] \arrow[d, "\underline{w}"', "\bullet" marking] \arrow[r, phantom, shift right=4ex, "\eta_w"] & C \arrow[r, "c"] \arrow[d, equal, "\bullet" marking] \arrow[r, phantom, shift right=4ex, "e_c"] & A \arrow[d, equal, "\bullet" marking] \\
A \arrow[r, "a"'] \arrow[d, equal, "\bullet" marking] \arrow[rr, phantom, shift right=4ex, "\overline{\alpha}"] & C \arrow[r, "w"'] & C' \arrow[r, "w'"'] \arrow[d, equal, "\bullet" marking] \arrow[rr, phantom, shift right=4ex, "\overline{\gamma}"] & C \arrow[r, "c"'] & A \arrow[d, equal, "\bullet" marking] \\
A \arrow[r, "u"] \arrow[d, "\underline{u}"', "\bullet" marking] \arrow[r, phantom, shift right=4ex, "\mathrm{id}_u"] & A' \arrow[r, "a'"] \arrow[d, equal, "\bullet" marking] \arrow[r, phantom, shift right=4ex, "e_{a'}"] & C' \arrow[r, "c'"] \arrow[d, equal, "\bullet" marking] \arrow[r, phantom, shift right=4ex, "e_{c'}"] & A' \arrow[r, "u'"] \arrow[d, equal, "\bullet" marking] \arrow[r, phantom, shift right=4ex, "\epsilon_u"] & A \arrow[d, "\underline{u}", "\bullet" marking] \\
A' \arrow[r, equal] & A' \arrow[r, "a'"'] & C' \arrow[r, "c'"'] & A' \arrow[r, equal] & A'
\end{tikzcd}
$$

$$
=
\begin{tikzcd}
A \arrow[rrrr, equal] \arrow[d, equal, "\bullet" marking] \arrow[rrrr, phantom, shift right=4ex, "\eta_u\ \cong"] & & & & A \arrow[d, equal, "\bullet" marking] \\
A \arrow[r, "u"] \arrow[d, "\underline{u}"', "\bullet" marking] \arrow[r, phantom, shift right=4ex, "\mathrm{id}_u"] & A' \arrow[rr, equal] \arrow[d, equal, "\bullet" marking] \arrow[rr, phantom, shift right=4ex, "\eta'\ \cong"] & & A' \arrow[r, "u'"] \arrow[d, equal, "\bullet" marking] \arrow[r, phantom, shift right=4ex, "\epsilon_u"] & A \arrow[d, "\underline{u}", "\bullet" marking] \\
A' \arrow[r, equal] & A' \arrow[r, "a'"'] & C' \arrow[r, "c'"'] & A' \arrow[r, equal] & A'
\end{tikzcd}
=
\begin{tikzcd}
A \arrow[rr, equal] \arrow[d, "\underline{u}"', "\bullet" marking] \arrow[rr, phantom, shift right=4ex, "\mathrm{id}_{\underline{u}}"] & & A \arrow[d, "\underline{u}", "\bullet" marking] \\
A' \arrow[rr, equal] \arrow[d, equal, "\bullet" marking] \arrow[rr, phantom, shift right=4ex, "\eta'\ \cong"] & & A' \arrow[d, equal, "\bullet" marking] \\
A' \arrow[r, "a'"'] & C' \arrow[r, "c'"'] & A'
\end{tikzcd},
$$

where the first equality holds by definition of γ and by the relation between α and $\overline{\alpha}$, the second since $\overline{\gamma}$ is a weak inverse of $\overline{\alpha}$ with respect to the composites of the horizontal adjoint equivalence data of (a,c,η,ϵ) with (w,w',η_w,ϵ_w), and of (u,u',η_u,ϵ_u) with (a',c',η',ϵ'), and the third by the triangle identities for (η_u,ϵ_u). The other relation involving the counits can be shown similarly. Hence γ is a weak inverse of α, and this shows that α is weakly horizontally invertible.

We have seen that the square α is weakly horizontally invertible if and only if $\overline{\alpha}$ is weakly horizontally invertible. By Proposition 3.6.7, since the vertical boundaries of $\overline{\alpha}$ are trivial, this holds if and only if $\overline{\alpha}$ is vertically invertible, which in turns holds if and only if the corresponding 2-morphism $\alpha\colon wa \Rightarrow a'u$ is a 2-isomorphism in $\mathcal{A}$. This concludes the proof. □

As a consequence of this result, we can see that the weakly horizontally invertible squares in a double category $\mathbb{A}$ induce 2-isomorphisms in the induced 2-category $L^{\simeq}\mathbb{A}$, where $L^{\simeq}$ is the left adjoint of the homotopical horizontal embedding $\mathbb{H}^{\simeq}$ (see Definition 3.4.12 and Proposition 3.4.14).

Lemma 3.6.9. *Let $\mathbb{A}$ be a double category.*

(i) If $a\colon A \xrightarrow{\simeq} C$ is a horizontal equivalence in $\mathbb{A}$, then the corresponding morphism $a\colon A \xrightarrow{\simeq} C$ in $L^{\simeq}\mathbb{A}$ is an equivalence.

(ii) If $\alpha\colon (u \,{}^{a}_{a'}\, w)$ is a weakly horizontally invertible square in $\mathbb{A}$, then the corresponding 2-morphism $\alpha\colon \overline{w}a \Rightarrow a'\overline{u}$ in $L^{\simeq}\mathbb{A}$ is a 2-isomorphism, where $\overline{u}$ and $\overline{w}$ are the equivalences in $L^{\simeq}\mathbb{A}$ induced by the vertical morphisms u and w of $\mathbb{A}$.

Proof. We first prove (i). Let (a, c, η, ϵ) be a horizontal equivalence in $\mathbb{A}$. Then, by construction of $L^{\simeq}\mathbb{A}$, there are corresponding morphisms a and c, and corresponding 2-isomorphisms $\eta\colon \mathrm{id}_A \Rightarrow ca$ and $\epsilon\colon ac \Rightarrow \mathrm{id}_C$ in $L^{\simeq}\mathbb{A}$. Hence this shows that (a, c, η, ϵ) is an equivalence in $L^{\simeq}\mathbb{A}$.

Now suppose that we have a weakly horizontally invertible square α in $\mathbb{A}$ as in (ii). Its horizontal boundaries a and a' are horizontal equivalences in $\mathbb{A}$, and hence they are also equivalences in $L^{\simeq}\mathbb{A}$ by (i). Let $(\overline{u}, \overline{u}', \eta_{\overline{u}}, \epsilon_{\overline{u}})$ and $(\overline{w}, \overline{w}', \eta_{\overline{w}}, \epsilon_{\overline{w}})$ be the adjoint equivalence data in $L^{\simeq}\mathbb{A}$ induced by the vertical morphisms u and w of $\mathbb{A}$. Then, the square α induces a 2-morphism $\alpha\colon \overline{w}a \Rightarrow a'\overline{u}$ in $L^{\simeq}\mathbb{A}$, which itself induces a square $\overline{\alpha}$ in $\mathbb{H}^{\simeq}L^{\simeq}\mathbb{A}$ of the form

$$\begin{tikzcd}
A \arrow[r, "a"] \arrow[d, "{(\overline{u},\overline{u}',\eta_{\overline{u}},\epsilon_{\overline{u}})}"', "\bullet" marking] \arrow[dr, phantom, "\overline{\alpha}\ \simeq"] & C \arrow[d, "{(\overline{w},\overline{w}',\eta_{\overline{w}},\epsilon_{\overline{w}})}", "\bullet" marking] \\
A' \arrow[r, "a'"'] & C'
\end{tikzcd} ,$$

where a and a' are equivalences in $L^{\simeq}\mathcal{A}$. The relations expressing the fact that α is weakly horizontally invertible in $\mathbb{A}$ translate to relations in $\mathbb{H}^{\simeq}L^{\simeq}\mathbb{A}$, which imply that the corresponding square $\overline{\alpha}$ is weakly horizontally invertible in $\mathbb{H}^{\simeq}L^{\simeq}\mathbb{A}$. Hence, by Lemma 3.6.8, we get that the associated 2-morphism $\alpha\colon \overline{w}a \Rightarrow a'\overline{u}$ is a 2-isomorphism in $L^{\simeq}\mathbb{A}$. □

Finally, we prove that a horizontal equivalence in the pseudo-hom double category of Definition 3.3.4 is precisely a horizontal pseudo-natural transformation whose square components are weakly horizontally invertible squares.

Proposition 3.6.10. *Let $F, G\colon \mathbb{A} \to \mathbb{B}$ be double functors. A horizontal pseudo-natural transformation $\varphi\colon F \Rightarrow G$ is a horizontal equivalence in $[\mathbb{A}, \mathbb{B}]_{\mathrm{ps}}$ if and only if, for every vertical morphism $u\colon A \mathrel{\bullet\!\!\!\rightarrow} A'$ in $\mathbb{A}$, its square component $\varphi_u\colon (Fu\ {}^{\varphi_A}_{\varphi_{A'}}\ Gu)$ is a weakly horizontally invertible square in $\mathbb{B}$. In particular, for every object $A \in \mathbb{A}$, its horizontal morphism component $\varphi_A\colon FA \to GA$ is a horizontal equivalence in $\mathbb{B}$.*

Proof. Suppose first that $\varphi\colon F \Rightarrow G$ is a horizontal equivalence in $[\mathbb{A}, \mathbb{B}]_{\mathrm{ps}}$. Then there is a horizontal equivalence data $(\varphi, \psi, \eta, \epsilon)$ in $[\mathbb{A}, \mathbb{B}]_{\mathrm{ps}}$ for φ. By evaluating this data at an object $A \in \mathbb{A}$, we get a horizontal equivalence $(\varphi_A, \psi_A, \eta_A, \epsilon_A)$ in $\mathbb{B}$. Moreover, the relations satisfied by the vertically invertible modifications η and ϵ, as given in (dmp1) of Definition 3.3.3 – or similarly by (dm1) of Definition 3.2.3 since the vertical boundaries of η and ϵ are trivial –, tell us that the squares (φ_u, ψ_u) are weak inverses with respect to the horizontal equivalence data $(\varphi_A, \psi_A, \eta_A, \epsilon_A)$ and $(\varphi_{A'}, \psi_{A'}, \eta_{A'}, \epsilon_{A'})$, for every vertical morphism $u\colon A \mathrel{\bullet\!\!\!\rightarrow} A'$ in $\mathbb{A}$. This shows that φ is such that its square component φ_u is weakly horizontally invertible in $\mathbb{B}$, for every vertical morphism $u\colon A \mathrel{\bullet\!\!\!\rightarrow} A'$ in $\mathbb{A}$.

Now suppose that $\varphi\colon F \Rightarrow G$ is such that its square component $\varphi_u\colon (Fu\ {}^{\varphi_A}_{\varphi_{A'}}\ Gu)$ is weakly horizontally invertible in $\mathbb{B}$, for every vertical morphism $u\colon A \mathrel{\bullet\!\!\!\rightarrow} A'$ in $\mathbb{A}$. Then, its horizontal morphism component $\varphi_A\colon FA \to GA$ is a horizontal equivalence in $\mathbb{B}$, for every object $A \in \mathbb{A}$, since it is the horizontal boundary of a weakly horizontally invertible square. Let us fix horizontal adjoint equivalence data $(\varphi_A, \psi_A, \eta_A, \epsilon_A)$ in $\mathbb{B}$, for each object $A \in \mathbb{A}$. We define a horizontal pseudo-natural transformation $\psi\colon G \Rightarrow F$ and two modifications η and ϵ in $[\mathbb{A}, \mathbb{B}]_{\mathrm{ps}}$, such that $(\varphi, \psi, \eta, \epsilon)$ is a horizontal adjoint equivalence data for φ in $[\mathbb{A}, \mathbb{B}]_{\mathrm{ps}}$. Given a vertical morphism $u\colon A \mathrel{\bullet\!\!\!\rightarrow} A'$ in $\mathbb{B}$, we define $\psi_u\colon (Gu\ {}^{\psi_A}_{\psi_{A'}}\ Fu)$ to be the unique weak inverse of φ_u in $\mathbb{B}$, given by Proposition 3.6.6, with respect to the horizontal adjoint equivalence data $(\varphi_A, \psi_A, \eta_A, \epsilon_A)$ and $(\varphi_{A'}, \psi_{A'}, \eta_{A'}, \epsilon_{A'})$. It is straightforward to see that ψ satisfies (hn1-2) of Definition 3.2.1, since φ satisfies these conditions and by the fact that the weak inverse of φ_u with respect to the horizontal adjoint equivalence data $(\varphi_A, \psi_A, \eta_A, \epsilon_A)$ and $(\varphi_{A'}, \psi_{A'}, \eta_{A'}, \epsilon_{A'})$ is unique by Proposition 3.6.6. Given a horizontal morphism $a\colon A \to C$ in $\mathbb{A}$, we define ψ_a to be the vertically invertible square in $\mathbb{B}$ given by the following pasting.

$$
\begin{tikzcd}
GA \arrow[r, "\psi_A"] \arrow[d, equal, "\bullet" marking] \arrow[drr, phantom, "\psi_a \cong"] & FA \arrow[r, "Fa"] & FC \arrow[d, equal, "\bullet" marking] \\
GA \arrow[r, "Ga"'] & GC \arrow[r, "\psi_C"'] & FC
\end{tikzcd}
=
\begin{tikzcd}
GA \arrow[r, "\psi_A"] \arrow[d, equal, "\bullet" marking] \arrow[dr, phantom, "e_{\psi_A}"] & FA \arrow[r, "Fa"] \arrow[d, equal, "\bullet" marking] \arrow[dr, phantom, "e_{Fa}"] & FC \arrow[rr, equal] \arrow[d, equal, "\bullet" marking] \arrow[drr, phantom, "\eta_C \cong"] & & FC \arrow[d, equal, "\bullet" marking] \\
GA \arrow[r, "\psi_A"] \arrow[d, equal, "\bullet" marking] \arrow[dr, phantom, "e_{\psi_A}"] & FA \arrow[r, "Fa"'] \arrow[d, equal, "\bullet" marking] \arrow[drr, phantom, "\varphi_a^{-1} \cong"] & FC \arrow[r, "\varphi_C"] & GC \arrow[r, "\psi_C"] \arrow[d, equal, "\bullet" marking] \arrow[dr, phantom, "e_{\psi_C}"] & FC \arrow[d, equal, "\bullet" marking] \\
GA \arrow[r, "\psi_A"'] \arrow[d, equal, "\bullet" marking] \arrow[drr, phantom, "\epsilon_A \cong"] & FA \arrow[r, "\varphi_A"'] & GA \arrow[r, "Ga"] \arrow[d, equal, "\bullet" marking] \arrow[dr, phantom, "e_{Ga}"] & GC \arrow[r, "\psi_C"'] \arrow[d, equal, "\bullet" marking] \arrow[dr, phantom, "e_{\psi_C}"] & FC \arrow[d, equal, "\bullet" marking] \\
GA \arrow[rr, equal] & & GA \arrow[r, "Ga"'] & GC \arrow[r, "\psi_A"'] & FC
\end{tikzcd}
$$

The fact that ψ satisfies (hpn3-4) of Definition 3.3.1 follows from the triangles identities for (η_A, ϵ_A), for all objects $A \in \mathbb{A}$, and the fact that φ satisfies these conditions. Finally, (hpn5) of Definition 3.3.1 for ψ follows from the fact that (φ_u, ψ_u) are weak inverses with respect to the horizontal adjoint equivalence data $(\varphi_A, \psi_A, \eta_A, \epsilon_A)$ and $(\varphi_{A'}, \psi_{A'}, \eta_{A'}, \epsilon_{A'})$, for all vertical morphisms $u\colon A \to\!\!\!\!\bullet\ A'$, and the fact that φ satisfies this condition. Moreover, the squares η_A and ϵ_A, for all objects $A \in \mathbb{A}$, assemble into vertically invertible modifications η and ϵ in $[\mathbb{A}, \mathbb{B}]_{\mathrm{ps}}$, as desired. This shows that $(\varphi, \psi, \eta, \epsilon)$ is a horizontal adjoint equivalence data for φ in $[\mathbb{A}, \mathbb{B}]_{\mathrm{ps}}$, and this concludes the proof. $\square$

Definition 3.6.11. Let $F, G\colon \mathbb{A} \to \mathbb{B}$ be double functors. A horizontal pseudo-natural transformation $\varphi\colon F \Rightarrow G$ which satisfies the conditions of Proposition 3.6.10 is called a **horizontal pseudo-natural equivalence**. We say that φ is a **horizontal pseudo-natural adjoint equivalence** if it is a horizontal adjoint equivalence in the pseudo-hom double category $[\mathbb{A}, \mathbb{B}]_{\mathrm{ps}}$, or equivalently, if, for every object $A \in \mathbb{A}$, its horizontal morphism component $\varphi_A\colon FA \to GA$ is a horizontal adjoint equivalence in $\mathbb{B}$.

PART II.

BACKGROUND ON MODEL CATEGORIES

Model categories provide a good environment to do homotopy theory. While in a category, two objects are considered to be "the same" if they are isomorphic, model categories allow us to relax this notion of sameness. Indeed, in a model category, there is a class of morphisms, called *weak equivalences*, which is used to compare two objects of the ambient category. For example, a good notion of a weak equivalence between categories is that of an equivalence of categories – defined as a functor which has an inverse up to natural isomorphisms –, rather than that of an isomorphism of categories. A more geometric example is given by that of topological spaces, where a continuous map is a *weak equivalence* if it induces isomorphisms between homotopy groups in all dimensions.

In particular, from a model category, we can construct its *homotopy category*, which is given by localizing at the class of weak equivalences. Hence, in this homotopy category, two objects are isomorphic if and only if they are weakly equivalent in the model category we started with. In particular, the homotopy category of a model category is obtained by restricting to the *cofibrant-fibrant* objects, and by taking homotopy classes of morphisms between these objects. We therefore often call a model category after its class of (cofibrant-)fibrant objects. Furthermore, there is a notion of a *Quillen equivalence* between model categories which allows us to interpret the homotopy theories of two different model categories as being the same. In particular, a Quillen equivalence induces an equivalence between homotopy categories, which motivates the fact that it gives a homotopical version of an equivalence.

Model categories were actually introduced by Quillen [Qui67] to axiomatize the homotopy theory of topological spaces mentioned above. In particular, Quillen constructs a model structure on the category of simplicial sets which models the homotopy theory of spaces, in the sense that there is a Quillen equivalence between the model structure on topological spaces and Quillen's model structure on simplicial sets. Moreover, the fibrant objects in this model structure on simplicial sets are the *Kan complexes* and, since the homotopy category whose objects are Kan complexes is equivalent to that of topological spaces, we often refer to the Kan complexes as "spaces".

Model categories provide the language we use in this thesis to prove the desired theorems. In order to compare 2-categories, double categories, and their ∞-analogues, we construct model categories in which each of these notions corresponds to the fibrant objects, and we compare the model categories using homotopical analogues of adjunctions between categories.

In Section 4, we first recall the basic definition of the theory of model categories. We also explain how to construct the homotopy category of a model category, and give the homotopical version of adjunctions between model categories, called *Quillen pairs*. In particular, a Quillen pair induces an adjunction between the homotopy categories, which shows that it models the correct homotopical notion. We also introduce at the end of Section 4, enriched and monoidal categories, which are model structures on the underlying category of a tensored, cotensored, and enriched (or closed monoidal) category satisfying the *pushout-product axiom*. In particular, all model structures considered in this thesis are enriched and/or monoidal.

Then, in Section 5, we recall two constructions of model categories. The first one is given by inducing a model structure on a category from another model category along an adjunction. These are called *left-* and *right-induced model structures*, depending on whether we induce the model structure along a left or a right adjoint, and the results are based on work by Garner, Hess, Kędziorek, Riehl, and Shipley in [HKRS17, GKR20]. The second one is given by localizing a model category at a set of morphisms, and is called a *left Bousfield localization*. The results of this section are based on Hirschhorn's book [Hir03] and Werndli's thesis [Wer16]. However, we restrict ourselves here to the case of a model category enriched over the model structure on simplicial sets for Kan complexes mentioned above, since we will only apply this theorem in such a context. This construction is very useful to restrict the class of fibrant objects of a model structure to a smaller class, and will be used in Part IV. to build model structures whose fibrant objects are the $(\infty, 2)$-categories and the double $(\infty, 1)$-categories, respectively.

4. Model categories

A model category contains three classes of morphisms, called *weak equivalences*, *cofibrations*, and *fibrations*. A cofibration which is also a weak equivalence is then called a *trivial cofibration* and, similarly, a fibration which is also a weak equivalence is a *trivial fibration*. Cofibrations and trivial fibrations, as well as trivial cofibrations and fibrations, form *weak factorization systems*. In Section 4.1, we first introduce model categories, using this terminology. In particular, we prove some classical results about weak factorization systems, which will be used throughout this thesis. We also introduce *cofibrant* and *fibrant objects* which are such that the unique morphism from the initial object is a cofibration and the unique morphism to the terminal object is a fibration, respectively.

Often, the weak factorization systems of a model category are *cofibrantly generated*, in the sense that there is a set of cofibrations and a set of trivial cofibrations which generate the whole class of such. In Section 4.2, we introduce this property, which can also be defined for any weak factorization system. We then prove the *small object argument*, which says that the weak factorization system generated by any set of morphisms in a locally presentable category always exists, based on results by [Hir03]. Most of the model structures considered in this thesis are cofibrantly generated, which is a very useful condition in practice since we can often restrict our attention to the sets of generating (trivial) cofibrations to prove a result for all (trivial) cofibrations.

Then, in Section 4.3, we construct the homotopy category of a model category. This homotopy category is given by restricting to the cofibrant-fibrant objects, and taking homotopy classes of morphisms between such objects. In particular, a Whitehead Theorem for model categories tells us that the weak equivalences between cofibrant-fibrant objects are precisely the *homotopy equivalences*, i.e., the morphisms which have an inverse up to homotopy. This ensures that they correspond to the invertible morphisms in the homotopy category.

After having introduced the main definitions about model categories, we define in Section 4.4 the notion of a *Quillen pair*, which gives a homotopical version of an adjunction. Indeed, we show that a Quillen pair induces an adjunction between homotopy categories. We also introduce *Quillen reflections*, *co-reflections*, and *equivalences*, and show that they induce the corresponding 1-categorical notion between homotopy categories, which justifies their definitions. In particular, a Quillen equivalence gives the right type of comparison to identify two homotopy theories. The results of Sections 4.3 and 4.4 are based on [DS95].

Finally, in Section 4.5, we study *enriched model categories*. These are model structures on the underlying category of an enriched, tensored, and cotensored category satisfying an additional axiom, called the *pushout-product axiom*. In particular, when we have a model structure on a closed monoidal category which is enriched over itself, we say that the model structure is *monoidal.* In Part I., we have seen that the categories 2Cat of 2-categories and DblCat double categories both admit a closed symmetric monoidal structure given by the Gray tensor product, and that DblCat is also tensored, cotensored, and enriched over 2Cat. In Part III., we will see that the model structure on 2Cat is monoidal with respect to the Gray tensor product, and that both model structures on DblCat are enriched over 2Cat. The second-constructed model structure on DblCat is further monoidal with respect to the Gray tensor product for double categories.

4.1. **Model categories via weak factorization systems.** Traditionally, a model category is defined to be a category equipped with three classes of morphisms – called *cofibrations*, *fibrations*, and *weak equivalences* – which satisfy several axioms. In particular, the class of cofibrations and the class of fibrations which are also weak equivalences – called *trivial fibrations* – form a *weak factorization system*: every morphism of the ambient category can be factored into a cofibration followed by a trivial fibration, and cofibrations are precisely the morphisms which have the left lifting property with respect to trivial fibrations, and dually. Similarly, the class of trivial cofibrations and the class of fibrations also form a weak factorization system. Therefore, weak factorization systems allow us to give a more concise definition of model categories and this is the approach that we are taking here.

Let us first recall what we mean by a lifting property.

Definition 4.1.1. Let $\mathcal{M}$ be a category and let $l\colon A \to B$ and $r\colon X \to Y$ be morphisms in $\mathcal{M}$. If, in every commutative square in $\mathcal{M}$ of the form

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "l"'] & X \arrow[d, "r"] \\ B \arrow[r, "g"'] \arrow[ur, dashed, "h"] & Y \end{tikzcd},$$

there is a lift $h\colon B \to X$ with $hl = f$ and $rh = g$, we say that l has the **left lifting property** with respect to r, or equivalently, that r has the **right lifting property** with respect to l.

Given a class $\mathcal{C}$ of morphisms in $\mathcal{M}$, we say that a morphism f in $\mathcal{M}$ has the **left** (resp. **right**) **lifting property** with respect to $\mathcal{C}$ if f has the left (resp. right) lifting property with respect to each morphism in $\mathcal{C}$.

Notation 4.1.2. Let $\mathcal{C}$ be a class of morphisms in a category $\mathcal{M}$. We denote by ${}^{\boxslash}\mathcal{C}$ the class of morphisms in $\mathcal{M}$ which have the left lifting property with respect to $\mathcal{C}$, and by $\mathcal{C}^{\boxslash}$ the class of morphisms in $\mathcal{M}$ which have the right lifting property with respect to $\mathcal{C}$.

Using these notations, a weak factorization system on a category is defined as follows.

Definition 4.1.3. Let $\mathcal{M}$ be a category. A **weak factorization system** $(\mathcal{L}, \mathcal{R})$ on $\mathcal{M}$ consists of two classes $\mathcal{L}$ and $\mathcal{R}$ of morphisms in $\mathcal{M}$ satisfying the following conditions.

(wfs1) Every morphism f in $\mathcal{M}$ can be factored as $f = rl$ with $l \in \mathcal{L}$ and $r \in \mathcal{R}$.
(wfs2) We have that $\mathcal{L} = {}^{\boxslash}\mathcal{R}$ and $\mathcal{R} = \mathcal{L}^{\boxslash}$.

Since the left and right classes of a weak factorization system are defined by lifting properties against each other, we can show that they satisfy the following properties. Before stating these properties, we first recall the notion of a transfinite composition of morphisms.

Definition 4.1.4. Let $\mathcal{M}$ be a category and λ be an ordinal. A **transfinite composition** of morphisms in $\mathcal{M}$ is a sequence of composable morphisms

$$A_0 \xrightarrow{l_0} A_1 \xrightarrow{l_1} A_2 \longrightarrow \cdots \longrightarrow A_\mu \xrightarrow{l_\mu} A_{\mu+1} \longrightarrow \cdots \longrightarrow A_\kappa \longrightarrow \cdots ,$$

for all ordinals $\mu < \lambda$, such that, for every limit ordinal $\kappa < \lambda$, the unique morphism $\mathrm{colim}_{\mu<\kappa} A_\mu \to A_\kappa$ is an isomorphism. Let us denote by $\iota_\mu \colon A_\mu \to \mathrm{colim}_{\mu<\lambda} A_\mu$ the leg of the colimit cone, for $\mu < \lambda$. Then the morphism $\iota_0 \colon A_0 \to \mathrm{colim}_{\mu<\lambda} A_\mu$ is the **composite** of the transfinite composition.

Proposition 4.1.5. *Let $(\mathcal{L}, \mathcal{R})$ be a weak factorization system on a category $\mathcal{M}$. Then the classes $\mathcal{L}$ and $\mathcal{R}$ contain isomorphisms, and are closed under compositions and retracts. Furthermore,*

(i) the left class $\mathcal{L}$ is closed under coproducts, pushouts, and transfinite compositions,
(ii) the right class $\mathcal{R}$ is closed under products and pullbacks.

Proof. We prove the results for the left class $\mathcal{L}$, and those for the right class $\mathcal{R}$ can be proven dually. In the proof, to show that a certain morphism is contained in $\mathcal{L}$, we always show that it has the left lifting property with respect to every morphism in $\mathcal{R}$, i.e., that it is in ${}^\boxslash\mathcal{R} = \mathcal{L}$.

We first prove that $\mathcal{L}$ contains isomorphisms. Let $l \colon A \xrightarrow{\cong} B$ be an isomorphism in $\mathcal{M}$ and write $l^{-1} \colon B \to A$ for its inverse. Let $r \colon X \to Y$ be a morphism in $\mathcal{R}$ and consider a commutative square in $\mathcal{M}$ of the form

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "l"'] & X \arrow[d, "r"] \\ B \arrow[r, "g"'] \arrow[ur, dashed, "fl^{-1}"] & Y . \end{tikzcd}$$

Then the composite $fl^{-1} \colon B \to X$ is a lift in this diagram since we have $fl^{-1}l = f$ and $rfl^{-1} = gll^{-1} = g$. Hence $l \in \mathcal{L}$.

We now show that $\mathcal{L}$ is closed under compositions. Let $l \colon A \to B$ and $k \colon B \to C$ be morphisms in $\mathcal{L}$. Let $r \colon X \to Y$ be a morphism in $\mathcal{R}$ and consider a commutative diagram in $\mathcal{M}$ of the form

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "l"'] & X \arrow[dd, "r"] \\ B \arrow[d, "k"'] \arrow[ur, dashed, "h"] & \\ C \arrow[r, "g"'] \arrow[uur, dashed, "h'"'] & Y . \end{tikzcd}$$

Since l has the left lifting property with respect to r, there is a lift $h \colon B \to X$ such that $hl = f$ and $rh = gk$. Then, since k has the left lifting property with respect to r, there is a lift $h' \colon C \to X$ such that $h'k = h$ and $rh' = g$. Then h' is the desired lift for kl. Hence $kl \in \mathcal{L}$.

We now show that $\mathcal{L}$ is closed under retracts. Let $l \colon A \to B$ be a morphism in $\mathcal{L}$ and suppose that $k \colon C \to D$ is a retract of l through the following commutative diagram.

$$\begin{tikzcd}
C \arrow[r, "i"] \arrow[d, "k"'] \arrow[rr, bend left, equal] & A \arrow[r, "s"] \arrow[d, "l"] & C \arrow[d, "k"] \\
D \arrow[r, "j"'] \arrow[rr, bend right, equal] & B \arrow[r, "t"'] & D
\end{tikzcd}$$

Let $r\colon X \to Y$ be a morphism in $\mathcal{R}$ and consider a commutative square in $\mathcal{M}$ as below left. We compose it with the right-hand commutative square of the above diagram, as depicted below right.

$$\begin{tikzcd}
C \arrow[r, "f"] \arrow[d, "k"'] & X \arrow[d, "r"] \\
D \arrow[r, "g"'] \arrow[ur, dashed, "hj"] & Y
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
A \arrow[r, "s"] \arrow[d, "l"'] & C \arrow[r, "f"] \arrow[d, "k"] & X \arrow[d, "r"] \\
B \arrow[r, "t"'] \arrow[urr, dashed, "h"] & D \arrow[r, "g"'] & Y
\end{tikzcd}$$

Since l has the left lifting property with respect to r, there is a lift $h\colon B \to X$ in the above right diagram such that $hl = fs$ and $rh = gt$. Then, the composite $hj\colon D \to X$ is a lift in the above left diagram, since $hjk = hli = fsi = f$ and $rhj = gtj = g$. Hence $k \in \mathcal{L}$.

We now show that $\mathcal{L}$ is closed under coproducts. Let $\{l_i\colon A_i \to B_i \mid i \in I\}$ be a collection of morphisms in $\mathcal{L}$ such that their coproduct $\bigsqcup_{i\in I} l_i\colon \bigsqcup_{i\in I} A_i \to \bigsqcup_{i\in I} B_i$ exists. Let $r\colon X \to Y$ be a morphism in $\mathcal{R}$ and consider a commutative square in $\mathcal{M}$ as below left. We compose it with the inclusion morphisms of A_i and B_i into the coproducts, for every $i \in I$, as depicted below right.

$$\begin{tikzcd}
\bigsqcup_{i\in I} A_i \arrow[r, "f"] \arrow[d, "\bigsqcup_{i\in I} l_i"'] & X \arrow[d, "r"] \\
\bigsqcup_{i\in I} B_i \arrow[r, "g"'] \arrow[ur, dashed, "h"] & Y
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
A_i \arrow[r, "\iota_i^A"] \arrow[d, "l_i"'] & \bigsqcup_{i\in I} A_i \arrow[r, "f"] \arrow[d, "\bigsqcup_{i\in I} l_i"] & X \arrow[d, "r"] \\
B_i \arrow[r, "\iota_i^B"'] \arrow[urr, dashed, "h_i"] & \bigsqcup_{i\in I} B_i \arrow[r, "g"'] & Y
\end{tikzcd}$$

Since, for every $i \in I$, the morphism l_i is in $\mathcal{L}$, there is a lift $h_i\colon B_i \to X$ in the above right diagram such that $h_i l_i = f\iota_i^A$ and $rh_i = g\iota_i^B$. By the universal property of the coproduct $\bigsqcup_{i\in I} B_i$, there is a unique morphism $h\colon \bigsqcup_{i\in I} B_i \to X$ such that $h\iota_i^B = h_i$, for all $i \in I$. Then $h(\bigsqcup_{i\in I} l_i)\iota_i^A = h\iota_i^B l_i = h_i l_i = f\iota_i^A$ and $rh\iota_i^B = rh_i = g\iota_i^B$ and this gives $h(\bigsqcup_{i\in I} l_i) = f$ and $rh = g$ by uniqueness of such morphisms. Hence h is a lift in the above left diagram and $\bigsqcup_{i\in I} l_i \in \mathcal{L}$.

We now show that $\mathcal{L}$ is closed under pushouts. Let $l\colon A \to B$ be a morphism in $\mathcal{L}$ and suppose that $k\colon C \to D$ is a pushout of l through the following pushout diagram.

$$\begin{tikzcd}
A \arrow[r, "p"] \arrow[d, "l"'] & C \arrow[d, "k"] \\
B \arrow[r, "q"'] & D \arrow[ul, phantom, very near start, "\ulcorner"]
\end{tikzcd}$$

Let $r\colon X \to Y$ be a morphism in $\mathcal{R}$ and consider a commutative square in $\mathcal{M}$ as below left. We compose it with the above pushout square, as depicted below right.

$$\begin{tikzcd} C \arrow[r, "f"] \arrow[d, "k"'] & X \arrow[d, "r"] \\ D \arrow[r, "g"'] \arrow[ur, dashed, "h'"] & Y \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "p"] \arrow[d, "l"'] & C \arrow[r, "f"] \arrow[d, "k"] & X \arrow[d, "r"] \\ B \arrow[r, "q"'] \arrow[urr, dashed, "h"] & D \arrow[r, "g"'] \arrow[ul, phantom, "\ulcorner", very near start] & Y \end{tikzcd}$$

Since l has the left lifting property with respect to r, there is a lift $h\colon B \to X$ in the above right diagram such that $hl = fp$ and $rh = gq$. By the universal property of the pushout, since $hl = fp$, there is a unique morphism $h'\colon D \to X$ such that $h'k = f$ and $h'q = h$. Then $rh'k = rf = gk$ and $rh'q = rh = gq$ and hence $rh' = g$ by uniqueness of such a morphism. Hence h' is a lift in the above left diagram and $k \in \mathcal{L}$.

We finally show that $\mathcal{L}$ is closed under transfinite compositions. Let λ be an ordinal and suppose that we have a transfinite composition of morphisms

$$A_0 \xrightarrow{l_0} A_1 \xrightarrow{l_1} A_2 \longrightarrow \cdots \longrightarrow A_\mu \xrightarrow{l_\mu} A_{\mu+1} \longrightarrow \cdots \longrightarrow A_\kappa \longrightarrow \cdots$$

such that $l_\mu\colon A_\mu \to A_{\mu+1}$ is in $\mathcal{L}$, for all ordinals $\mu < \lambda$. We show that the composite $\iota_0\colon A_0 \to \operatorname{colim}_{\mu<\lambda} A_\mu$ is in $\mathcal{L}$. Let $r\colon X \to Y$ be a morphism in $\mathcal{R}$ and consider a commutative square of the form

$$\begin{tikzcd} A_0 \arrow[r, "f"] \arrow[d, "\iota_0"'] & X \arrow[d, "r"] \\ \operatorname{colim}_{\mu<\lambda} A_\mu \arrow[r, "g"'] \arrow[ur, dashed, "h"] & Y \end{tikzcd} .$$

We build a lift $h\colon \operatorname{colim}_{\mu<\lambda} A_\mu \to X$ in this diagram by transfinite induction. In other words, we construct a morphism $h_\mu\colon A_\mu \to X$ such that $rh_\mu = g\iota_\mu$, for each $\mu < \lambda$. First, we set $h_0 = f\colon A_0 \to X$ and we indeed have $rf = g\iota_0$. Given $\mu + 1 < \lambda$ a successor ordinal, consider the following commutative square

$$\begin{tikzcd} A_\mu \arrow[r, "h_\mu"] \arrow[d, "l_\mu"'] & X \arrow[d, "r"] \\ A_{\mu+1} \arrow[r, "g\iota_{\mu+1}"'] \arrow[ur, dashed, "h_{\mu+1}"] & Y \end{tikzcd} ,$$

which commutes since, by induction hypothesis, $rh_\mu = g\iota_\mu = g\iota_{\mu+1}l_\mu$. Since l_μ is in $\mathcal{L}$, there is a lift $h_{\mu+1}\colon A_{\mu+1} \to X$ in the above diagram with $h_{\mu+1}l_\mu = h_\mu$ and $rh_{\mu+1} = g\iota_{\mu+1}$. Now given $\kappa < \lambda$ a limit ordinal, let us denote by $\iota^\kappa_\mu\colon A_\mu \to A_\kappa \cong \operatorname{colim}_{\mu<\kappa} A_\mu$ the leg of the colimit cone, for $\mu < \kappa$. We set $h_\kappa\colon A_\kappa \to X$ to be the unique morphism such that $h_\kappa \iota^\kappa_\mu = h_\mu$, for all $\mu < \kappa$. In particular, since $rh_\kappa \iota^\kappa_\mu = rh_\mu = g\iota_\mu = g\iota_\kappa \iota^\kappa_\mu$ for every $\mu < \kappa$, we have $rh_\kappa = g\iota_\kappa$ by uniqueness of such a morphism. Finally, we set $h\colon \operatorname{colim}_{\mu<\lambda} A_\mu \to X$ to be the unique morphism such that $h\iota_\mu = h_\mu$, for all $\mu < \lambda$. It is indeed a lift in the first diagram as $h\iota_0 = h_0 = f$ and, since $rh\iota_\mu = rh_\mu = g\iota_\mu$, for all $\mu < \lambda$, then $rh = g$ by uniqueness of such a morphism. Hence $\iota_0 \in \mathcal{L}$. ☐

We now prove the following "retract argument", which is very useful when trying to prove that a certain morphism is contained in the left or right class of a weak factorization system, since these are closed under retracts by the above result.

Proposition 4.1.6 (Retract argument)**.** *Let $f\colon A \to B$ be a morphism in a category $\mathcal{M}$ which factors as*

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, "i"'] & & B \\ & C \arrow[ur, "p"'] & \end{tikzcd} .$$

(i) If f has the left lifting property with respect to p, then f is a retract of i of the form

$$\begin{tikzcd} A \arrow[r, equal] \arrow[d, "f"'] & A \arrow[r, equal] \arrow[d, "i"] & A \arrow[d, "f"] \\ B \arrow[r, "k"'] \arrow[rr, bend right, equal] & C \arrow[r, "p"'] & B \end{tikzcd} .$$

(ii) If f has the right lifting property with respect to i, then f is a retract of p of the form

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[rr, bend left, equal] \arrow[d, "f"'] & C \arrow[r, "r"] \arrow[d, "p"] & A \arrow[d, "f"] \\ B \arrow[r, equal] & B \arrow[r, equal] & B \end{tikzcd} .$$

Proof. We prove the first statement. Since f has the left lifting property with respect to p, there is a lift $k\colon B \to C$ in the following commutative square.

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d, "f"'] & C \arrow[d, "p"] \\ B \arrow[r, equal] \arrow[ur, dashed, "k"] & B \end{tikzcd}$$

This gives the desired retract since $pk = \mathrm{id}_B$, $pi = f$, and $kf = i$.

The second statement can be proven dually. □

We are now ready to give the definition of a model category in terms of its weak factorization systems.

Definition 4.1.7. A **model category** is a complete and cocomplete category $\mathcal{M}$ together with three classes of morphisms in $\mathcal{M}$,

(i) a class $\mathcal{C}$ of **cofibrations** ($\hookrightarrow$),
(ii) a class $\mathcal{F}$ of **fibrations** ($\twoheadrightarrow$), and
(iii) a class $\mathcal{W}$ of **weak equivalences** ($\xrightarrow{\sim}$),

satisfying the following conditions.

(mc1) The class $\mathcal{W}$ satisfies the 2-out-of-3 property, i.e., given two composable morphisms f and g in $\mathcal{M}$, if two out of the three morphisms f, g, and gf are in $\mathcal{W}$, then so is the third.
(mc2) The pairs $(\mathcal{C}, \mathcal{F} \cap \mathcal{W})$ and $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$ are weak factorization systems on $\mathcal{M}$.

We say that $(\mathcal{C}, \mathcal{F}, \mathcal{W})$ is a **model structure** on $\mathcal{M}$. We denote by $(\mathcal{M}, \mathcal{C}, \mathcal{F}, \mathcal{W})$ or simply by $\mathcal{M}$ the data of a model category.

Definition 4.1.8. Let $(\mathcal{M}, \mathcal{C}, \mathcal{F}, \mathcal{W})$ be a model category. A morphism in $\mathcal{C} \cap \mathcal{W}$ is called a **trivial cofibration** ($\overset{\sim}{\hookrightarrow}$), and a morphism in $\mathcal{F} \cap \mathcal{W}$ is called a **trivial fibration** ($\overset{\sim}{\twoheadrightarrow}$).

In particular, the class of weak equivalences of a model category is closed under compositions by one instance of the 2-out-of-3 property, and it contains all isomorphisms since $\mathcal{C} \cap \mathcal{W} \subseteq \mathcal{W}$ and $\mathcal{C} \cap \mathcal{W}$ contains all isomorphisms by Proposition 4.1.5. We can further show that it is closed under retracts. The proof is taken from [Joy08, Proposition E.1.3].

Proposition 4.1.9. *Let $(\mathcal{M}, \mathcal{C}, \mathcal{F}, \mathcal{W})$ be a model category. Then the class $\mathcal{W}$ of weak equivalences is closed under retracts.*

Proof. Let $f\colon X \xrightarrow{\sim} Y$ be a weak equivalence in $\mathcal{M}$ and suppose that $g\colon A \to B$ is a retract of f through the following commutative diagram.

$$\begin{tikzcd}
A \arrow[r, "k"] \arrow[d, "g"'] \arrow[rr, bend left, equal] & X \arrow[r, "r"] \arrow[d, "f", "\wr"'] & A \arrow[d, "g"] \\
B \arrow[r, "l"'] \arrow[rr, bend right, equal] & Y \arrow[r, "s"'] & B
\end{tikzcd}$$

Since $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$ is a weak factorization system, we can factor the morphism g as

$$\begin{tikzcd}
A \arrow[rr, "g"] \arrow[dr, hook, "\sim", "j"'] & & B \\
& C \arrow[ur, two heads, "p"'] &
\end{tikzcd}$$

with j a trivial cofibration and p a fibration. We then consider the pushout P of j along k and we obtain a commutative diagram

$$\begin{tikzcd}
A \arrow[r, "k"] \arrow[d, hook, "j"', "\wr"] \arrow[rr, bend left, equal] & X \arrow[r, "r"] \arrow[d, hook, "j'"', "\wr"] \arrow[dd, bend left, "f", "\wr"'] & A \arrow[d, hook, "j", "\wr"'] \\
C \arrow[r, "k'"] \arrow[d, two heads, "p"'] \arrow[rr, bend right, equal] & P \arrow[ul, phantom, "\ulcorner", very near start] \arrow[r, dashed, "h"] \arrow[d, dashed, "f'"'] & C \arrow[d, two heads, "p"] \\
B \arrow[r, "l"'] \arrow[rr, bend right, equal] & Y \arrow[r, "s"'] & B,
\end{tikzcd}$$

where $h\colon P \to C$ and $f'\colon P \to Y$ are the unique morphisms given by the universal property of the pushout such that $hk' = \mathrm{id}_C$, $hj' = jr$, $f'k' = lp$, and $f'j' = f$. Note that j' is a trivial cofibration, since it is the pushout of a trivial cofibration and this class is closed under pushouts by Proposition 4.1.5. Moreover, note that the right-below square of the above diagram commutes, since $phj' = pjr = gr = sf = sf'j'$ and $phk' = p = slp = sf'k'$ and hence $ph = sh'$ by uniqueness of such a morphism. By the 2-out-of-3 property, since f and j' are weak equivalences, then so is f'. Hence the pasting of the two bottom squares tells us that the fibration p is a retract of the weak equivalence f'. As before, we can factor f' as

$$\begin{tikzcd}
P \arrow[rr, "f'", "\sim"'] \arrow[dr, hook, "\sim", "i"'] & & Y \\
& Z \arrow[ur, two heads, "q"'] &
\end{tikzcd}$$

with i a trivial cofibration and q a fibration. By the 2-out-of-3 property, we also have that q is a weak equivalence, and hence a trivial fibration. Since p is a fibration and i is a trivial cofibration, there is a lift $t\colon Z \to C$ in the following commutative diagram.

$$\begin{tikzcd}
P \arrow[rr, "h"] \arrow[d, hook, "i"', "\wr"] & & C \arrow[d, two heads, "p"] \\
Z \arrow[r, two heads, "\sim", "q"'] \arrow[rru, dashed, "t"] & Y \arrow[r, "s"'] & B
\end{tikzcd}$$

Since $qik' = f'k' = lp$ and $tik' = hk' = \mathrm{id}_C$, the following diagram also commutes.

$$\begin{tikzcd}
C \arrow[r, "ik'"] \arrow[d, two heads, "p"'] \arrow[rr, bend left, equal] & Z \arrow[r, "t"] \arrow[d, two heads, "q", "\wr"'] & C \arrow[d, two heads, "p"] \\
B \arrow[r, "l"'] \arrow[rr, bend right, equal] & Y \arrow[r, "s"'] & B
\end{tikzcd}$$

This shows that p is a retract of q and, since the class $\mathcal{F} \cap \mathcal{W}$ of trivial fibrations is closed under retracts by Proposition 4.1.5, it follows that p is a trivial fibration. Finally, since $g = pj$ is a composite of weak equivalences, it is also a weak equivalence by the 2-out-of-3 property. $\square$

Since a model category is cocomplete and complete, it admits both an initial object and a terminal object. There are then two classes of objects of interest in a model category, given by requiring that the unique map from the initial object or to the terminal object be a cofibration or a fibration, respectively.

Definition 4.1.10. Let $\mathcal{M}$ be a model category. An object $X \in \mathcal{M}$ is said to be **cofibrant** if the unique morphism $\emptyset \to X$ from the initial object $\emptyset$ of $\mathcal{M}$ is a cofibration. An object $X \in \mathcal{M}$ is said to be **fibrant** if the unique morphism $X \to *$ to the terminal object $*$ of $\mathcal{M}$ is a fibration.

4.2. **Generating sets.** Model categories are often *cofibrantly generated*, in the sense that they admit sets of generating cofibrations and generating trivial cofibrations which generate the whole classes of such under transfinite compositions, pushouts, and retracts. In particular, if the ambient category is locally presentable, cofibrantly generated model categories have nice factorizations into a cofibration (resp. trivial cofibration) followed by a trivial fibration (resp. fibration). The construction of such factorizations is given by the *small object argument.* Some of the results in this section are taken from [Hir03, §10].

In a more general framework, a weak factorization system is said to be cofibrantly generated when its left class is generated by a set of morphisms.

Definition 4.2.1. Let $\mathcal{M}$ be a category. A weak factorization system $(\mathcal{L}, \mathcal{R})$ on $\mathcal{M}$ is **cofibrantly generated** if there is a set $\mathcal{I}$ of morphisms in $\mathcal{M}$ such that $\mathcal{R} = \mathcal{I}^{\boxslash}$. We call $\mathcal{I}$ a **generating set** of morphisms in $\mathcal{L}$. In this case, note that $\mathcal{L} = {}^{\boxslash}(\mathcal{I}^{\boxslash})$.

Remark 4.2.2. If $(\mathcal{L}, \mathcal{R})$ is a cofibrantly generated weak factorization system with generating set $\mathcal{I}$, then $\mathcal{L} = {}^{\boxslash}(\mathcal{I}^{\boxslash})$ and therefore $\mathcal{I} \subseteq \mathcal{L}$.

Notation 4.2.3. Given a set $\mathcal{I}$ of morphisms in a category $\mathcal{M}$, we write $\mathcal{I}{-}\mathrm{inj} := \mathcal{I}^{\boxslash}$ and $\mathcal{I}{-}\mathrm{cof} := {}^{\boxslash}(\mathcal{I}{-}\mathrm{inj}) = {}^{\boxslash}(\mathcal{I}^{\boxslash})$. We call a morphism in $\mathcal{I}{-}\mathrm{inj}$ an **$\mathcal{I}$-injective** and a morphism in $\mathcal{I}{-}\mathrm{cof}$ an **$\mathcal{I}$-cofibration**.

By requiring both weak factorization systems of a model category to be cofibrantly generated, we get the notion of a cofibrantly generated model category.

Definition 4.2.4. A model category $(\mathcal{M}, \mathcal{C}, \mathcal{F}, \mathcal{W})$ is **cofibrantly generated** if the weak factorization systems $(\mathcal{C}, \mathcal{F}\cap\mathcal{W})$ and $(\mathcal{C}\cap\mathcal{W}, \mathcal{F})$ are cofibrantly generated, with generating sets $\mathcal{I}$ and $\mathcal{J}$, respectively. A morphism in $\mathcal{I} \subseteq \mathcal{C}$ is called a **generating cofibration** and a morphism in $\mathcal{J} \subseteq \mathcal{C} \cap \mathcal{W}$ is called a **generating trivial cofibration**.

In particular, a *combinatorial* model category is defined as a locally presentable category endowed with a model structure whose weak factorization systems are cofibrantly generated.

Definition 4.2.5. A model category $\mathcal{M}$ is **combinatorial** if it is locally presentable and cofibrantly generated.

Under these hypotheses, the weak factorization systems of a combinatorial model structure always come with a functorial factorization of its morphisms into a cofibration followed by a trivial fibration, and into a trivial cofibration followed by a fibration. Let us first make precise what we mean by a functorial factorization.

Definition 4.2.6. Let $\mathcal{M}$ be a category. A weak factorization system $(\mathcal{L}, \mathcal{R})$ on $\mathcal{M}$ is **functorial** if there is a functor $\mathcal{M}^{[1]} \to \mathcal{M}^{[2]}$ which sends a morphism f in $\mathcal{M}$ to a factorization $f = rl$ of f into morphisms $l \in \mathcal{L}$ and $r \in \mathcal{R}$.

By the small object argument, given a set $\mathcal{I}$ of morphisms in a locally presentable category $\mathcal{M}$, the weak factorization system generated by this set always exists and is functorial. In particular, every morphism in the category splits as a relative $\mathcal{I}$-cell complex, as defined below, followed by a morphism in $\mathcal{I}-$inj.

Definition 4.2.7. Let $\mathcal{I}$ be a set of morphisms in a category $\mathcal{M}$. A morphism in $\mathcal{M}$ is a **relative $\mathcal{I}$-cell complex** if it is a transfinite composition of pushouts of morphisms in $\mathcal{I}$. We denote by $\mathcal{I}-$cell the class of relative $\mathcal{I}$-cell complexes in $\mathcal{M}$.

Remark 4.2.8. Since the class $\mathcal{I}-$cof is closed under pushouts and transfinite compositions by Proposition 4.1.5, and it contains $\mathcal{I}$, there is an inclusion $\mathcal{I}-\text{cell} \subseteq \mathcal{I}-\text{cof}$.

Before proving the small object argument for locally presentable categories, we give this lemma which tells us that coproducts of morphisms in $\mathcal{I}$ can be seen as transfinite compositions of pushouts of morphisms in $\mathcal{I}$. This appears as [Hir03, Proposition 10.2.7].

Lemma 4.2.9. *Let $\mathcal{I}$ be a set of morphisms in a locally presentable category $\mathcal{M}$ and let $\{i_s \colon A_s \to B_s \mid s \in S\}$ be a family of morphisms in $\mathcal{I}$. Then the coproduct*

$$\bigsqcup_{s\in S} i_s \colon \bigsqcup_{s\in S} A_s \to \bigsqcup_{s\in S} B_s$$

is the composite of a transfinite composition of pushouts of the morphisms i_s.

Proof. Choose a well-ordering of the set S. Then there is a unique ordinal λ with an order-preserving isomorphism $S \cong \lambda$. We write $i_\mu \colon A_\mu \to B_\mu$, for $\mu < \lambda + 1$, for the morphism corresponding to $i_s \colon A_s \to B_s$, for $s \in S$, under the isomorphism $S \cong \lambda$. We define a transfinite composition of pushouts of the morphisms i_μ as follows.

$$X_0 \xrightarrow{l_0} X_1 \xrightarrow{l_1} X_2 \longrightarrow \cdots \longrightarrow X_\mu \xrightarrow{l_\mu} X_{\mu+1} \longrightarrow \cdots \longrightarrow X_\kappa \longrightarrow \cdots$$

We set $X_\mu := \left(\bigsqcup_{\mu\leq\alpha<\lambda+1} A_\alpha\right) \sqcup \left(\bigsqcup_{\alpha<\mu} B_\alpha\right)$ and $l_\mu \colon X_\mu \to X_{\mu+1}$ to be the morphism induced by $i_\mu \colon A_\mu \to B_\mu$, for every $\mu < \lambda+1$. Note that the morphism $l_\mu \colon X_\mu \to X_{\mu+1}$ is the pushout of $i_\mu \colon A_\mu \to B_\mu$ along the inclusion $A_\mu \to X_\mu := \left(\bigsqcup_{\mu\leq\alpha<\lambda+1} A_\alpha\right) \sqcup \left(\bigsqcup_{\alpha<\mu} B_\alpha\right)$, for every $\mu < \lambda + 1$. Moreover, we have that the composite of the transfinite composition $X_0 \to \mathrm{colim}_{\mu<\lambda+1} X_\mu$ is the coproduct $\bigsqcup_{\mu<\lambda+1} i_\mu \colon \bigsqcup_{\mu<\lambda+1} A_\mu \to \bigsqcup_{\mu<\lambda+1} B_\mu$. This shows the result. □

We are finally ready to state and prove the small object argument for locally presentable categories. This appears as [Hir03, Proposition 10.5.16]

Proposition 4.2.10 (Small object argument)**.** *Let $\mathcal{I}$ be a set of morphisms in a locally presentable category $\mathcal{M}$. There is a functorial weak factorization system* ($\mathcal{I}-\mathrm{cof}, \mathcal{I}-\mathrm{inj}$)*. In particular, every morphism in $\mathcal{M}$ factors as a morphism in* $\mathcal{I}-$cell *followed by a morphism in* $\mathcal{I}-$inj*.*

Proof. Since $\mathcal{M}$ is locally presentable, we can choose an ordinal λ such that all the domains of morphisms in $\mathcal{I}$ are λ-small. Let $f \colon X \to Y$ be a morphism in $\mathcal{M}$. We define by transfinite induction a factorization of f as

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "j_\mu"'] & & Y , \\ & C_\mu \arrow[ur, "p_\mu"'] & \end{tikzcd}$$

where j_μ is in $\mathcal{I}-$cell, for every $\mu < \lambda$. For $\mu = 0$, we set $C_0 = X$, $j_0 = \mathrm{id}_X$ and $p_0 = f$. Let $0 < \mu + 1 < \lambda$ be a successor ordinal and let $\mathcal{A}_\mu$ be the collection of all commutative squares in $\mathcal{M}$ of the form

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[d, "i"'] & C_\mu \arrow[d, "p_\mu"] \\ B \arrow[r, "h"'] & Y \end{tikzcd}$$

with $i \colon A \to B$ a morphism in $\mathcal{I}$. We define $C_{\mu+1}$ to be the following pushout.

$$\begin{tikzcd} \bigsqcup_{\mathcal{A}_\mu} A \arrow[r, "\bigsqcup_{\mathcal{A}_\mu} g"] \arrow[d, "\bigsqcup_{\mathcal{A}_\mu} i"'] & C_\mu \arrow[d, "i_\mu"] \arrow[ddr, bend left, "p_\mu"] & \\ \bigsqcup_{\mathcal{A}_\mu} B \arrow[r, "q_{\mu+1}"'] \arrow[drr, bend right, "\bigsqcup_{\mathcal{A}_\mu} h"'] & C_{\mu+1} \arrow[ul, phantom, "\ulcorner", very near start] \arrow[dr, dashed, "p_{\mu+1}"'] & \\ & & Y \end{tikzcd}$$

By definition of $\mathcal{A}_\mu$, the outside square commutes, and hence there is a unique morphism $p_{\mu+1} \colon C_{\mu+1} \to Y$ such that $p_{\mu+1} i_\mu = p_\mu$ and $p_{\mu+1} q_{\mu+1} = \bigsqcup_{\mathcal{A}_\mu} h$. We set $j_{\mu+1} \colon X \to C_{\mu+1}$ to be the composite $j_{\mu+1} = i_\mu j_\mu$, which is in $\mathcal{I}-$cell since j_μ is in $\mathcal{I}-$cell by induction hypothesis and i_μ is in $\mathcal{I}-$cell by Lemma 4.2.9. Then $p_{\mu+1} j_{\mu+1} = p_{\mu+1} i_\mu j_\mu = p_\mu j_\mu = f$ and this gives a factorization of f as desired. Now let $\kappa < \lambda$ be a successor ordinal. We set $C_\kappa = \mathrm{colim}_{\mu<\kappa} C_\mu$ and we denote by $\iota^\kappa_\mu \colon C_\mu \to C_\kappa$ the leg of the colimit cone for $\mu < \kappa$. We set $j_\kappa \coloneqq \iota^\kappa_0 \colon X \to C_\kappa$ where we recall that $X = C_0$. By the universal property of the colimit, there is a unique morphism $p_\kappa \colon C_\kappa \to Y$ such that $p_\kappa \iota^\kappa_\mu = p_\mu$ for all $\mu < \kappa$. Then, we have $p_\kappa j_\kappa = p_\kappa \iota^\kappa_0 = p_0 = f$ and this gives a factorization of f as desired, since $j_\kappa \in \mathcal{I}-$cell as it is a transfinite composition of the morphisms $i_\mu \in \mathcal{I}-$cell for $\mu < \kappa$. Now let $C \coloneqq \mathrm{colim}_{\mu<\lambda} C_\mu$ and let us denote by $i_\mu \colon C_\mu \to C$ the leg of the colimit cone at $\mu < \lambda$. We get a factorization of f as

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "j"'] & & Y \\ & C \arrow[ur, "p"'] & \end{tikzcd},$$

where $j := \iota_0 \colon X = C_0 \to C$ and $p \colon C \to Y$ is the unique morphism given by the universal property of the colimit such that $p\iota_\mu = p_\mu$. By construction, it is clear that the morphism j is a relative $\mathcal{I}$-cell complex. We show that p is an $\mathcal{I}$-injective, i.e., that it has the right lifting property with respect to every morphism in $\mathcal{I}$. Consider a commutative square

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[d, "i"'] & C \arrow[d, "p"] \\ B \arrow[r, "h"'] \arrow[ur, dashed, "l"] & Y \end{tikzcd}$$

with $i \colon A \to B$ a morphism in $\mathcal{I}$. We want to find a lift $l \colon B \to C$ in this diagram. Since the domain A of i is λ-small and C is a λ-filtered colimit of $\{C_\mu\}$, the morphism $g \colon A \to C$ factors through C_μ, for some $\mu < \lambda$, i.e., we have a commutative triangle as below left. This gives a commutative square as below right since $p\iota_\mu = p_\mu$.

$$\begin{tikzcd} A \arrow[rr, "g"] \arrow[dr, "\hat{g}"'] & & C \\ & C_\mu \arrow[ur, "\iota_\mu"'] & \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "\hat{g}"] \arrow[d, "i"'] & C_\mu \arrow[d, "p_\mu"] \\ B \arrow[r, "h"'] & Y \end{tikzcd}$$

Note that this commutative square is in $\mathcal{A}_\mu$, and therefore we can consider the inclusions $k_A \colon A \to \bigsqcup_{\mathcal{A}_\mu} A$ and $k_B \colon B \to \bigsqcup_{\mathcal{A}_\mu} B$ at its components. We define $l \colon B \to C$ to be the composite

$$B \xrightarrow{k_B} \bigsqcup_{\mathcal{A}_\mu} B \xrightarrow{q_{\mu+1}} C_{\mu+1} \xrightarrow{\iota_{\mu+1}} C\,.$$

Then l gives the desired lift since $pl = p\iota_{\mu+1}q_{\mu+1}k_B = p_{\mu+1}q_{\mu+1}k_B = (\bigsqcup_{\mathcal{A}_\mu} h)k_B = h$ and $li = \iota_{\mu+1}q_{\mu+1}k_B i = \iota_{\mu+1}q_{\mu+1}(\bigsqcup_{\mathcal{A}_\mu} i)k_A = \iota_{\mu+1}i_\mu(\bigsqcup_{\mathcal{A}_\mu} g)k_A = \iota_{\mu+1}i_\mu\hat{g} = \iota_\mu\hat{g} = g$. This shows that p is in $\mathcal{I}{-}\mathrm{inj}$. Moreover, it is clear by construction that this factorization is functorial. □

As a direct consequence of this result, we get functorial factorizations in any combinatorial model structure.

Corollary 4.2.11. *Let $(\mathcal{M}, \mathcal{C}, \mathcal{F}, \mathcal{W})$ be a combinatorial model category. Then the weak factorization systems $(\mathcal{C}, \mathcal{F} \cap \mathcal{W})$ and $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$ are functorial.*

Proof. Since the model structure on $\mathcal{M}$ is cofibrantly generated, there are sets $\mathcal{I}$ and $\mathcal{J}$ of morphisms in $\mathcal{M}$ such that

$$(\mathcal{C}, \mathcal{F} \cap \mathcal{W}) = (\mathcal{I}{-}\mathrm{cof}, \mathcal{I}{-}\mathrm{inj}) \ \text{ and } \ (\mathcal{C} \cap \mathcal{W}, \mathcal{F}) = (\mathcal{J}{-}\mathrm{cof}, \mathcal{J}{-}\mathrm{inj}).$$

Since $\mathcal{M}$ is locally presentable, the result then follows from Proposition 4.2.10. □

Finally, it follows from the factorization given by the small object argument and from the retract argument, that every $\mathcal{I}$-cofibration is a retract of a relative $\mathcal{I}$-cell complex.

Proposition 4.2.12. *Let $\mathcal{I}$ be a set of morphisms in a locally presentable category $\mathcal{M}$. Then every morphism $i\colon A \to B$ in $\mathcal{I}-\mathrm{cof}$ is a retract of a morphism $j\colon A \to C$ in $\mathcal{I}-\mathrm{cell}$ of the form*

$$\begin{tikzcd} A \arrow[r, equal] \arrow[d, "i"'] & A \arrow[r, equal] \arrow[d, "j"] & A \arrow[d, "i"] \\ B \arrow[r, "k"'] \arrow[rr, equal, bend right] & C \arrow[r, "p"'] & B . \end{tikzcd}$$

Proof. Let $i\colon A \to B$ be a morphism in $\mathcal{I}-\mathrm{cof}$. By the small object argument (see Proposition 4.2.10), we can factor i as

$$\begin{tikzcd} A \arrow[rr, "i"] \arrow[dr, "j"'] & & B \\ & C \arrow[ur, "p"'] & \end{tikzcd}$$

with $j \in \mathcal{I}-\mathrm{cell}$ and $p \in \mathcal{I}-\mathrm{inj}$. Since i has the left lifting property with respect to p, it is a retract of j of the desired form by the retract argument (see Proposition 4.1.6). $\square$

4.3. **Whitehead Theorem and homotopy category.** Weak equivalences in a model category are interpreted as the morphisms which are "homotopically invertible". Indeed, a Whitehead Theorem for model categories characterizes the weak equivalences between objects which are both fibrant and cofibrant as the morphisms which admit an inverse up to homotopy. Such a notion of weak equivalence therefore gives a "weaker" version of an isomorphism, which is often better to study the relations between objects in a category. For example, an *equivalence of categories* is a more appropriate notion of invertible morphism in the category Cat, than that of isomorphisms. Given a model category, we can localize it at its class of weak equivalences and then study the objects in this new category, where two objects are now isomorphic if and only if they are weakly equivalent in the category we started with. Such a construction is called the *homotopy category* of a model category. The aim of this section is to introduce the notions of (right) homotopies in a model category and we construct its homotopy category based on results in [DS95, §§4-6].

Let us fix a model category $(\mathcal{M}, \mathcal{C}, \mathcal{F}, \mathcal{W})$. We first introduce path objects in $\mathcal{M}$, which allow us to define right homotopies between morphisms of $\mathcal{M}$.

Definition 4.3.1. Let $X \in \mathcal{M}$ be an object. A **path object** for X is an object $\mathrm{Path}(X)$ in $\mathcal{M}$ together with a factorization of the diagonal morphism $\Delta\colon X \to X \times X$

$$X \xrightarrow[\sim]{w} \mathrm{Path}(X) \overset{p}{\twoheadrightarrow} X \times X$$

into a weak equivalence w and a fibration p in $\mathcal{M}$.

When an object $X \in \mathcal{M}$ is fibrant, the projection from its path object to X are actually trivial fibrations, as we see now.

Lemma 4.3.2. *Let X be a fibrant object in $\mathcal{M}$, and $\mathrm{Path}(X)$ be a path object for X. Then the morphisms $p_i\colon \mathrm{Path}(X) \to X$ obtained by composing $p\colon \mathrm{Path}(X) \twoheadrightarrow X \times X$ with the projections $\pi_i\colon X \times X \to X$ are trivial fibrations, for $i = 0, 1$.*

Proof. First note that, since X is fibrant, the projections

$$\pi_0\colon X \times X \twoheadrightarrow X \times * \cong X \quad \text{and} \quad \pi_1\colon X \times X \twoheadrightarrow * \times X \cong X$$

are fibrations since they are obtained as products of the fibrations id_X and $X \twoheadrightarrow *$. Therefore, the morphism p_i is a fibration as a composite of the fibrations p and π_i, for $i = 0, 1$. Moreover, since the following diagram commutes

$$\begin{tikzcd} X \arrow[r, "w", "\sim"'] \arrow[rr, bend left, "\Delta"] \arrow[dr, equal] & \mathrm{Path}(X) \arrow[r, two heads, "p"] \arrow[d, two heads, "p_i"'] & X \times X \arrow[dl, two heads, "\pi_i"] \\ & X & \end{tikzcd},$$

the morphism p_i is a weak equivalence by 2-out-of-3, since id_X and w are weak equivalences, for $i = 0, 1$. □

Remark 4.3.3. In particular, if X is a fibrant object in $\mathcal{M}$, then $\mathrm{Path}(X)$ is also fibrant, since $p_i \colon \mathrm{Path}(X) \overset{\sim}{\twoheadrightarrow} X$ is a fibration, for $i = 0, 1$.

A right homotopy between two morphisms is then defined as a morphism to the path object which retrieves each of the two morphisms when composing with one of the two projections mentioned above.

Definition 4.3.4. Two morphisms $f, g \colon A \to X$ in $\mathcal{M}$ are said to be **right homotopic** if there is a path object

$$X \xrightarrow[\sim]{w} \mathrm{Path}(X) \overset{p}{\twoheadrightarrow} X \times X$$

for X in $\mathcal{M}$ together with a morphism $h \colon A \to \mathrm{Path}(X)$ in $\mathcal{M}$ such that the following diagram commutes.

$$\begin{tikzcd} A \arrow[rr, "{(f,g)}"] \arrow[dr, "h"'] & & X \times X \\ & \mathrm{Path}(X) \arrow[ur, two heads, "p"'] & \end{tikzcd}$$

We call h a **right homotopy** from f to g, and we write $f \sim_r g$.

Remark 4.3.5. Let A and X be objects in $\mathcal{M}$. If X is fibrant, the right homotopy relation is an equivalence relation on the set $\mathcal{M}(A, X)$ of morphisms. We denote the set of right homotopy classes by $\pi_r(A, X) = \mathcal{M}(A, X)/\sim_r$. See [DS95, Lemma 4.14].

Pre-composition with a trivial cofibration induces an isomorphism between sets of right homotopy classes.

Lemma 4.3.6. *Let X be a fibrant object in $\mathcal{M}$, and let $j \colon A \overset{\sim}{\hookrightarrow} B$ be a trivial cofibration in $\mathcal{M}$. Then j induces an isomorphism between sets of right homotopy classes*

$$\pi_r(B, X) \xrightarrow{\cong} \pi_r(A, X), \quad [f]_r \mapsto [fj]_r.$$

Proof. The map $j^* \colon \pi_r(B, X) \to \pi_r(A, X)$ is well-defined. Indeed, if two morphisms $f, g \colon B \to X$ are such that $f \sim_r g$, there is a right homotopy $h \colon B \to \mathrm{Path}(X)$ from f to g, and then $hj \colon A \to \mathrm{Path}(X)$ gives a right homotopy from fj to gj, so that $fj \sim_r gj$.

Given $[g]_r \in \pi_r(A, X)$, consider the following commutative square.

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[d, hook, "j"', "\wr"] & X \arrow[d, two heads] \\ B \arrow[r] \arrow[ur, dashed, "f"] & * \end{tikzcd}$$

Since j is a trivial cofibration and X is fibrant, there is a lift $f\colon B \to X$ in the above diagram. We get that $j^*[f]_r = [fj]_r = [g]_r$, which shows that j^* is surjective. Now given $[f]_r, [f']_r \in \pi_r(B, X)$ such that $[fj]_r = [f'j]_r$, there is a path object $\mathrm{Path}(X)$ for X and a right homotopy $h\colon B \to \mathrm{Path}(X)$ in $\mathcal{M}$ from fj to $f'j$. Consider the following commutative square.

$$\begin{tikzcd} A \arrow[r, "h"] \arrow[d, hook, "j"', "\wr"] & \mathrm{Path}(X) \arrow[d, two heads, "p"] \\ B \arrow[r, "{(f,f')}"'] \arrow[ur, dashed, "h'"] & X \times X \end{tikzcd}$$

Since j is a trivial cofibration and p is a fibration (by definition of path objects), there is a lift $h'\colon B \to \mathrm{Path}(X)$ in the above diagram, which gives a right homotopy from f to f', i.e., $[f]_r = [f']_r$. This shows that j^* is injective and finishes the proof. □

Remark 4.3.7. There is a dual notion of left homotopy defined through a cylinder object; see [DS95, §4]. However, when A is a cofibrant object and X is a fibrant object in $\mathcal{M}$, two morphisms $A \to X$ are right homotopic if and only if they are left homotopic by [DS95, Lemma 4.21]. When the left and right homotopy relations coincide, we then say that the morphisms considered are **homotopic**. This gives an equivalence relation on the set $\mathcal{M}(A, X)$, that we denote by $\sim$, and we write $\pi(A, X) = \mathcal{M}(A, X)/\sim$ for the set of homotopy classes.

A dual statement of Lemma 4.3.6 says that post-composing with a trivial fibration induces an isomorphism between sets of left homotopy classes (see [DS95, Lemma 4.9]). Therefore, as a consequence of the above remark, we get the following result.

Lemma 4.3.8. *Let A be a cofibrant object and Y be a fibrant object in $\mathcal{M}$.*

(i) If $j\colon A \xhookrightarrow{\sim} B$ is a trivial cofibration in $\mathcal{M}$, then j induces an isomorphism between sets of homotopy classes

$$j^*\colon \pi(B, Y) \to \pi(A, Y), \quad [f] \mapsto [fj].$$

(ii) If $q\colon X \overset{\sim}{\twoheadrightarrow} Y$ is a trivial fibration in $\mathcal{M}$, then q induces an isomorphism between sets of homotopy classes

$$q_*\colon \pi(A, X) \to \pi(A, Y), \quad [f] \mapsto [qf].$$

Proof. The first statement follows directly from Lemma 4.3.6 and Remark 4.3.7; note that B is cofibrant as A is cofibrant and j is a cofibration. The second statement can be proven dually. □

The homotopy relation respects weak equivalences in the sense that the morphisms in the same homotopy class as a weak equivalence are also weak equivalences.

Lemma 4.3.9. *Let A be a cofibrant object and X be a fibrant object in $\mathcal{M}$. Suppose that $f\colon A \xrightarrow{\sim} X$ is a weak equivalence in $\mathcal{M}$, and that $g\colon A \to X$ is a morphism in $\mathcal{M}$ such that $f \sim g$. Then g is also a weak equivalence.*

Proof. Let $\mathrm{Path}(X)$ be a path object for X and $h\colon A \to \mathrm{Path}(X)$ be a homotopy from f to g. Recall from Lemma 4.3.2 that, since X is fibrant, the morphism $p_i\colon \mathrm{Path}(X) \overset{\sim}{\twoheadrightarrow} X$ obtained as the composite of $p\colon \mathrm{Path}(X) \twoheadrightarrow X \times X$ with the projection $\pi_i\colon X \times X \twoheadrightarrow X$ is a trivial fibration for $i = 0, 1$. Since $f = p_0 h$, by 2-out-of-3, we get that h is also a weak equivalence. Therefore, by 2-out-of-3 applied to $g = p_1 h$, we conclude that g is also a weak equivalence. □

We now introduce *homotopy equivalences* which are given by the morphisms that have an inverse up to homotopy.

Definition 4.3.10. Let A and X be cofibrant-fibrant objects in $\mathcal{M}$. A morphism $f\colon A \to X$ in $\mathcal{M}$ is a **homotopy equivalence** if there is a morphism $g\colon X \to A$ in $\mathcal{M}$ such that $\mathrm{id}_A \sim gf$ and $fg \sim \mathrm{id}_X$. We say that g is a **homotopy inverse** of f.

The following result, called Whitehead Theorem, characterizes the weak equivalences between cofibrant-fibrant objects as the homotopy equivalences between these objects. This motivates the fact that weak equivalences can be interpreted as a weaker notion of invertibility between objects in a category.

Theorem 4.3.11 (Whitehead Theorem for model categories)**.** *Let $\mathcal{M}$ be a model category, and let A and X be cofibrant-fibrant objects in $\mathcal{M}$. Then a morphism $f\colon A \to X$ in $\mathcal{M}$ is a weak equivalence if and only if it is a homotopy equivalence.*

Proof. Let $f\colon A \xrightarrow{\sim} X$ be a weak equivalence in $\mathcal{M}$. Since $(\mathcal{C}, \mathcal{F}\cap\mathcal{W})$ is a weak factorization system, there is a factorization of f as

$$\begin{tikzcd} A \arrow[rr, "f", "\sim"'] \arrow[dr, hook, "i"'] & & X \\ & C \arrow[ur, two heads, "q"', "\sim"] & \end{tikzcd}$$

with i a cofibration and q a trivial fibration. Note that C is cofibrant since A is cofibrant and i is a cofibration, and C is fibrant since X is fibrant and q is a fibration. Furthermore, by 2-out-of-3, since f and q are weak equivalences, so is i. Since i is a trivial cofibration and A is fibrant, there is a morphism $s\colon C \to A$ in $\mathcal{M}$ such that $si = \mathrm{id}_A$, given by the existence of a lift in the following commutative square.

$$\begin{tikzcd} A \arrow[r, equal] \arrow[d, hook, "i"', "\wr"] & A \arrow[d, two heads] \\ C \arrow[r] \arrow[ur, dashed, "s"] & * \end{tikzcd}$$

Moreover, by Lemma 4.3.8, we have that $i^*\colon \pi(C,C) \to \pi(A,C)$ is an isomorphism. Therefore, since $i^*[is] = [isi] = [i] = i^*[\mathrm{id}_C]$, we get that $[is] = [\mathrm{id}_C]$, i.e., $is \sim \mathrm{id}_C$. This shows that s is a homotopy inverse of i. Dually, we can show that there is a morphism $k\colon X \to C$ such that $qk = \mathrm{id}_X$ and $kq \sim \mathrm{id}_C$. Therefore, the composite $sk\colon X \to A$ is a homotopy inverse for f as $skf = skqi \sim si = \mathrm{id}_A$ and $fsk = qisk \sim qk = \mathrm{id}_X$. This shows that f is a homotopy equivalence.

Now suppose that $f\colon A \to X$ is a homotopy equivalence, and let $g\colon X \to A$ be a homotopy inverse of f. As above, there is factorization of f as

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, hook, "i"'] & & X \\ & C \arrow[ur, two heads, "q"', "\sim"] & \end{tikzcd}$$

with i a cofibration and q a trivial fibration. Again, we have that C is cofibrant-fibrant. By 2-out-of-3, it is enough to show that i is a weak equivalence. Let $\mathrm{Path}(A)$ be a path object for A and $h\colon A \to \mathrm{Path}(A)$ be a homotopy from gf to id_A. Consider the following commutative square.

$$\begin{tikzcd} A \arrow[r, "h"] \arrow[d, hook, "i"'] & \mathrm{Path}(A) \arrow[d, two heads, "p_0", "\wr"'] \\ C \arrow[r, "gq"'] \arrow[ur, dashed, "h'"] & A \end{tikzcd}$$

Since i is a cofibration and p_0 is a trivial fibration by Lemma 4.3.2 as A is fibrant, there is a lift $h' \colon C \to \mathrm{Path}(A)$ in the above diagram. Write $s = p_1 h' \colon C \to A$. Note that $gq \sim s$. Furthermore, we have that $si = p_1 h' i = p_1 h = \mathrm{id}_A$. As $q \colon C \xrightarrow{\sim} X$ is a weak equivalence, we have that q is a homotopy equivalence by the above arguments. Let $k \colon X \to C$ be a homotopy of inverse of q. Since $f = qi$, we get that $kf = kqi \sim i$. Hence $is \sim igq \sim kfgq \sim kq \sim \mathrm{id}_C$. By Lemma 4.3.9, as id_C is a weak equivalence, so is is. Since i is a retract of is of the form

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[rr, bend left, equal] \arrow[d, "i"'] & C \arrow[r, "s"] \arrow[d, "is", "\wr"'] & A \arrow[d, "i"] \\ C \arrow[r, equal] & C \arrow[r, equal] & C \end{tikzcd},$$

and since the class of weak equivalences is closed under retracts by Proposition 4.1.9, we get that i is also a weak equivalence. This shows that $f = qi$ is a weak equivalence, and concludes the proof. □

The Whitehead Theorem suggests that the cofibrant-fibrant objects in a model category are better behaved. In particular, starting from any object of $\mathcal{M}$, we want to replace it with another weakly equivalent object which is both fibrant and cofibrant. This can be done using the following constructions.

Definition 4.3.12. Let $X \in \mathcal{M}$ be an object. Since $(\mathcal{C}, \mathcal{F} \cap \mathcal{W})$ is a weak factorization system on $\mathcal{M}$, the unique morphism $\emptyset \to X$ from the initial object can be factored as

$$\emptyset \hookrightarrow X^c \overset{q_X}{\underset{\sim}{\twoheadrightarrow}} X\,,$$

where X^c is a cofibrant object and q_X is a trivial fibration in $\mathcal{M}$. The data (X^c, q_X) is called a **cofibrant replacement** of X. Dually, since $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$ is a weak factorization system on $\mathcal{M}$, the unique morphism $X \to *$ to the terminal object can be factored as

$$X \overset{j_X}{\underset{\sim}{\hookrightarrow}} X^f \twoheadrightarrow *\,,$$

where X^f is a fibrant object and j_X is a trivial cofibration in $\mathcal{M}$. The data (X^f, j_X) is called a **fibrant replacement** of X.

Remark 4.3.13. When X is cofibrant, a canonical choice of cofibrant replacement for X is given by the identity $\mathrm{id}_X \colon X \to X$. Dually, when X is fibrant, a canonical choice of fibrant replacement for X is given by the identity $\mathrm{id}_X \colon X \to X$.

In particular, a morphism between two objects induces a morphism between their fibrant (resp. cofibrant) replacements, which happens to be unique up to right (resp. left) homotopy.

Lemma 4.3.14. *Let $\mathcal{M}$ be a model category, and let $f \colon X \to Y$ be a morphism in $\mathcal{M}$.*

(i) Given fibrant replacements (X^f, j_X) and (Y^f, j_Y) of X and Y, respectively, there is a morphism $f^f \colon X^f \to Y^f$ in $\mathcal{M}$ making the following square commute.

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, hook, "j_X"', "\wr"] & Y \arrow[d, hook, "j_Y", "\wr"'] \\ X^f \arrow[r, "f^f"'] & Y^f \end{tikzcd}$$

Moreover, such a morphism f^f is unique up to right homotopy, and f^f is a weak equivalence if and only if f is so.

(ii) *Dually, given cofibrant replacements (X^c, q_X) and (Y^c, q_Y) of X and Y, respectively, there is a morphism $f^c\colon X^c \to Y^c$ in $\mathcal{M}$ making the following square commute.*

$$\begin{tikzcd} X^c \arrow[r, "f^c"] \arrow[d, two heads, "q_X"', "\wr"] & Y^c \arrow[d, two heads, "q_Y", "\wr"'] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

Moreover, such a morphism f^c is unique up to left homotopy, and f^c is a weak equivalence if and only if f is so.

Proof. We prove the first statement. Since $j_X\colon X \overset{\sim}{\hookrightarrow} X^f$ is a trivial cofibration and Y^f is fibrant in $\mathcal{M}$, there is a lift in the following diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, hook, "j_X"', "\wr"] & Y \arrow[r, "j_Y"] & Y^f \arrow[d, two heads] \\ X^f \arrow[rr] \arrow[urr, dashed, "f^f"] & & * \end{tikzcd}$$

and we can choose f^f to be this lift. Then, by Lemma 4.3.6, the trivial cofibration j_X induces an isomorphism $(j_X)^*\colon \pi_r(X^f, Y^f) \xrightarrow{\cong} \pi_r(X, Y^f)$ since Y^f is fibrant. Therefore, if $g\colon X^f \to Y^f$ is another morphism such that $gj_X = j_Y f$, then $g \sim_r f^f$ by injectivity of $(j_X)^*$. By 2-out-of-3, it is clear that f^f is a weak equivalence if and only f is so.

The second statement can be proven dually. □

The following remark constructs a cofibrant-fibrant replacement for any object of $\mathcal{M}$.

Remark 4.3.15. Given an object X in $\mathcal{M}$, first take a fibrant replacement $j_X\colon X \overset{\sim}{\hookrightarrow} X^f$ of X and then take a cofibrant replacement $q_{X^f}\colon (X^f)^c \overset{\sim}{\twoheadrightarrow} X^f$ of X^f. Note that the object $(X^f)^c$ is cofibrant-fibrant; it is cofibrant by definition of a cofibrant replacement and it is fibrant since X^f is fibrant and q_{X^f} is a fibration. By applying Lemma 4.3.14 to a morphism $f\colon X \to Y$ in $\mathcal{M}$, we get a morphism $(f^f)^c\colon (X^f)^c \to (Y^f)^c$ which is compatible with the fibrant and cofibrant replacements considered. This morphism $(f^f)^c$ can be seen to be unique up to homotopy with this property, and, furthermore, it is a weak equivalence if and only if f is so.

Using these cofibrant-fibrant replacements, we can define the homotopy category of $\mathcal{M}$ as the category with the same objects as $\mathcal{M}$ and whose morphisms between two objects are given by homotopy classes between their cofibrant-fibrant replacements.

Definition 4.3.16. Let $\mathcal{M}$ be a model category and let us fix for each object $X \in \mathcal{M}$ a cofibrant-fibrant replacement $((X^f)^c, j_X, q_{X^f})$ of X. The **homotopy category** of $\mathcal{M}$ is the category $\mathrm{ho}(\mathcal{M})$ whose objects are the objects of $\mathcal{M}$ and whose hom-sets are given by

$$\mathrm{ho}(\mathcal{M})(X,Y) = \pi((X^f)^c, (Y^f)^c),$$

for all objects X and Y of $\mathcal{M}$.

There is a well-defined functor from $\mathcal{M}$ to its homotopy category, which is the identity on objects and sends a morphism to the homotopy class of its cofibrant-fibrant replacement. This functor exhibits $\mathrm{ho}(\mathcal{M})$ as a localization of $\mathcal{M}$ at its class of weak equivalences.

Lemma 4.3.17. *There is a functor $\gamma\colon \mathcal{M} \to \mathrm{ho}(\mathcal{M})$ which is the identity on objects.*

Proof. Given a morphism $f\colon X \to Y$ in $\mathcal{M}$, we define $\gamma(f) = [(f^f)^c] \in \pi((X^f)^c, (Y^f)^c)$. This is a well-defined functor since the morphisms $(f^f)^c$ are unique up to homotopy, when cofibrant-fibrant replacements are fixed, by Remark 4.3.15. □

Theorem 4.3.18. *The pair $(\mathrm{ho}(\mathcal{M}), \gamma)$ is a localization of $\mathcal{M}$ at the class of weak equivalences $\mathcal{W}$. In other words, for every functor $F\colon \mathcal{M} \to \mathcal{E}$ sending weak equivalences in $\mathcal{M}$ to isomorphisms in $\mathcal{E}$, there is a unique functor $G\colon \mathrm{ho}(\mathcal{M}) \to \mathcal{E}$ such that $G\gamma = F$, i.e., such that the following diagram commutes.*

$$\begin{tikzcd} \mathcal{M} \arrow[r, "F"] \arrow[d, "\gamma"'] & \mathcal{E} \\ \mathrm{ho}(\mathcal{M}) \arrow[ur, dashed, "\exists! G"'] & \end{tikzcd}$$

Proof. Let $F\colon \mathcal{M} \to \mathcal{E}$ be a functor sending weak equivalences in $\mathcal{M}$ to isomorphisms in $\mathcal{E}$. We construct a functor $G\colon \mathrm{ho}(\mathcal{M}) \to \mathcal{E}$ such that $G\gamma = F$. For an object $X \in \mathrm{ho}(\mathcal{M})$, i.e., an object $X \in \mathcal{M}$, we set $GX \coloneqq FX$. Given objects A and X in $\mathcal{M}$, and a morphism $[f] \in \mathrm{ho}(\mathcal{M})(A, X) = \pi((A^f)^c, (X^f)^c)$, we set $G[f]$ to be the following composite in $\mathcal{E}$

$$FA \xrightarrow[\cong]{Fj_A} F(A^f) \xrightarrow[\cong]{(Fq_{A^f})^{-1}} F((A^f)^c) \xrightarrow{Ff} F((X^f)^c) \xrightarrow[\cong]{Fq_{X^f}} F(X^f) \xrightarrow[\cong]{(Fj_X)^{-1}} FX\,,$$

where $((A^f)^c, j_A, q_{A^f})$ and $((X^f)^c, j_X, q_{X^f})$ are cofibrant-fibrant replacements of A and X, respectively. To check that this is well-defined, it is enough to show that if $[f] = [g]$ in $\pi((A^f)^c, (X^f)^c)$, then $Ff = Fg$. Let

$$(X^f)^c \xrightarrow[\sim]{w} \mathrm{Path}((X^f)^c) \xrightarrow{p} (X^f)^c \times (X^f)^c$$

be a path object for $(X^f)^c)$ and $h\colon (A^f)^c \to \mathrm{Path}((X^f)^c)$ be a homotopy from f to g. Since we have that $wp_0 = \mathrm{id}_{(X^f)^c} = wp_1$, where p_0 and p_1 are as in Lemma 4.3.2, then $(Fw)(Fp_0) = F(wp_0) = F(wp_1) = (Fw)(Fp_1)$. Hence, as w is a weak equivalence, its image Fw is an isomorphism in $\mathcal{E}$, and we get that $Fp_0 = Fp_1$. Therefore, we indeed have

$$Ff = F(hp_0) = (Fh)(Fp_0) = (Fh)(Fp_1) = F(hp_1) = Fg.$$

The functoriality of G is straightforward. Furthermore, by construction, we have that $G\gamma = F$ and it is the unique functor with this property. □

The localization $\gamma\colon \mathcal{M} \to \mathrm{ho}(\mathcal{M})$ is *saturated*, which means that the isomorphisms in $\mathrm{ho}(\mathcal{M})$ are precisely the images of the weak equivalences in $\mathcal{M}$ under γ.

Lemma 4.3.19. *A morphism $f\colon X \to Y$ in $\mathcal{M}$ is a weak equivalence if and only if $\gamma(f)$ is an isomorphism in $\mathrm{ho}(\mathcal{M})$.*

Proof. By Lemma 4.3.14, a morphism $f\colon X \to Y$ is a weak equivalence if and only if $(f^f)^c\colon (X^f)^c \to (Y^f)^c$ is a weak equivalence. By Theorem 4.3.11, since $(X^f)^c$ and $(Y^f)^c$ are cofibrant-fibrant, the morphism $(f^f)^c\colon (X^f)^c \to (Y^f)^c$ is a weak equivalence if and only if it is a homotopy equivalence, which holds if and only if $\gamma(f) = [(f^f)^c]$ is an isomorphism in $\mathrm{ho}(\mathcal{M})$. □

Finally, we give two useful lemmas. The first one characterizes the hom-sets with cofibrant source and fibrant source in the homotopy category $\mathrm{ho}(\mathcal{M})$, and the second one says that the weak equivalences in a model category satisfy the 2-out-of-6 property, which is a stronger condition than that of 2-out-of-3.

Lemma 4.3.20. *Let A be a cofibrant object and X be a fibrant object in $\mathcal{M}$. Then there is an isomorphism* $\mathrm{ho}(\mathcal{M})(A, X) \cong \pi(A, X)$.

Proof. Let $j_A \colon A \overset{\sim}{\hookrightarrow} A^f$ be a fibrant replacement of A and $q_X \colon X^c \overset{\sim}{\twoheadrightarrow} X$ be a cofibrant replacement of X. By Lemma 4.3.8, the trivial cofibration j_A and the trivial fibration q_X induce isomorphisms

$$\mathrm{ho}(\mathcal{M})(A, X) = \pi(A^f, X^c) \xrightarrow[(j_A)^*]{\cong} \pi(A, X^c) \xrightarrow[(q_X)_*]{\cong} \pi(A, X)\,,$$

which gives the desired result. □

Lemma 4.3.21 (2-out-of-6 property)**.** *Let $\mathcal{M}$ be a model category and suppose that we have a commutative diagram in $\mathcal{M}$*

$$\begin{tikzcd} X \arrow[r, "gf", "\sim"'] \arrow[d, "f"'] & Z \arrow[d, "h"] \\ Y \arrow[r, "\sim", "hg"'] \arrow[ur, "g"] & T \end{tikzcd}$$

such that gf and hg are weak equivalences. Then the morphisms f, g, and h are also weak equivalences in $\mathcal{M}$.

Proof. Note that isomorphisms satisfy the 2-out-of-6 property. The result then follows from the fact that a morphism f is a weak equivalence in $\mathcal{M}$ if and only if $\gamma(f)$ is an isomorphism in $\mathrm{ho}(\mathcal{M})$ by Lemma 4.3.19. □

4.4. **Quillen pairs, reflections, co-reflections, and equivalences.** Having introduced model categories and their homotopy categories, we now turn our attention to comparisons between two homotopy theories. As for categories, there are homotopical versions of adjunctions and adjoint equivalences between model categories, which induce adjunctions and adjoint equivalences at the level of homotopy categories. Most results of this section are based on [DS95, §9].

Let us fix two model categories $\mathcal{M}$ and $\mathcal{N}$ and an adjunction

$$\begin{tikzcd} \mathcal{M} \arrow[r, bend right, "R"'] \arrow[r, phantom, "\perp"] & \mathcal{N} \arrow[l, bend right, "L"'] \end{tikzcd}$$

between them. We first give the homotopical version of an adjunction between model categories, which is defined as an adjunction satisfying one of the following equivalent conditions.

Proposition 4.4.1. *The following conditions are equivalent for the adjunction $L \dashv R$.*

(i) The left adjoint L preserves cofibrations and trivial cofibrations.
(ii) The right adjoint R preserves fibrations and trivial fibrations.

Proof. We show that the left adjoint L preserves cofibrations if and only if the right adjoint R preserves trivial fibrations. The proof that L preserves trivial cofibrations if and only if R preserves fibrations works similarly.

First note that, by the universal property of the adjunction $L \dashv R$, the below left commutative diagram in $\mathcal{M}$ has a lift if and only if the below right diagram in $\mathcal{N}$ has a lift, for every cofibration $i\colon A \hookrightarrow B$ in $\mathcal{N}$ and every trivial fibration $p\colon X \overset{\sim}{\twoheadrightarrow} Y$ in $\mathcal{M}$.

$$\begin{tikzcd} LA \arrow[r] \arrow[d, "Li"'] & X \arrow[d, two heads, "p", "\wr"'] \\ LB \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd} \qquad\qquad \begin{tikzcd} A \arrow[r] \arrow[d, hook, "i"'] & RX \arrow[d, "Rp"] \\ B \arrow[r] \arrow[ur, dashed] & RY \end{tikzcd}$$

If $L\colon \mathcal{N} \to \mathcal{M}$ preserves cofibrations, for every cofibration $i\colon A \hookrightarrow B$ in $\mathcal{N}$ and every trivial fibration $p\colon X \overset{\sim}{\twoheadrightarrow} Y$ in $\mathcal{M}$, there is a lift in every commutative diagram in $\mathcal{M}$ as above left since Li is a cofibration in $\mathcal{M}$. Hence, there is also a lift in every diagram as above right, which shows that Rp is a trivial fibration in $\mathcal{N}$, for every trivial fibration $p\colon X \overset{\sim}{\twoheadrightarrow} Y$ in $\mathcal{M}$. Conversely, if $R\colon \mathcal{M} \to \mathcal{N}$ preserves trivial fibrations, for every cofibration $i\colon A \hookrightarrow B$ in $\mathcal{N}$ and every trivial fibration $p\colon X \overset{\sim}{\twoheadrightarrow} Y$ in $\mathcal{M}$, there is a lift in every commutative diagram in $\mathcal{N}$ as above right since Rp is a trivial fibration in $\mathcal{N}$. Hence, there is also a lift in every diagram as above left, which shows that Li is a cofibration in $\mathcal{M}$, for every cofibration $i\colon A \hookrightarrow B$ in $\mathcal{N}$. □

Definition 4.4.2. The adjunction $L \dashv R$ is a **Quillen pair** if it satisfies one of the equivalent conditions of Proposition 4.4.1. We call L a **left Quillen** functor, and R a **right Quillen** functor.

Remark 4.4.3. Suppose that $L \dashv R$ is a Quillen pair. Since the left adjoint L preserves initial objects and cofibrations, it sends cofibrant objects in $\mathcal{N}$ to cofibrant objects in $\mathcal{M}$. Dually, since the right adjoint R preserves terminal objects and fibrations, it sends fibrant objects in $\mathcal{M}$ to fibrant objects in $\mathcal{N}$.

Remark 4.4.4. Suppose that the model structure on $\mathcal{N}$ is cofibrantly generated, with generating sets $\mathcal{I}$ and $\mathcal{J}$ of cofibrations and trivial cofibrations, respectively. To show that the left adjoint $L\colon \mathcal{N} \to \mathcal{M}$ is left Quillen, it is then enough to check that L sends the morphisms in $\mathcal{I}$ to cofibrations in $\mathcal{M}$ and the morphisms in $\mathcal{J}$ to trivial cofibrations in $\mathcal{M}$. Indeed, recall that every cofibration (resp. trivial cofibration) in the cofibrantly generated model category $\mathcal{N}$ is a retract of a relative $\mathcal{I}$-cell complex (resp. relative $\mathcal{J}$-cell complex) by Proposition 4.2.12, where a relative $\mathcal{I}$-cell complex (resp. relative $\mathcal{J}$-cell complex) is a transfinite composition of pushouts of morphisms in $\mathcal{I}$ (resp. $\mathcal{J}$). Hence, since L preserves colimits and retracts, and the classes of cofibrations and trivial cofibrations of $\mathcal{M}$ are closed under retracts, pushouts, and transfinite compositions by Proposition 4.1.5, it follows that L preserves all cofibrations and all trivial cofibrations.

The following lemma is a classical result in homotopy theory, and gives conditions on a functor for it to send weak equivalences between fibrant or cofibrant objects to weak equivalences. In particular, these results apply to right and left Quillen functors, respectively.

Lemma 4.4.5 (Ken Brown's Lemma)**.** *Let $F\colon \mathcal{M} \to \mathcal{N}$ be a functor.*

(i) If F sends trivial fibrations between fibrant objects in $\mathcal{M}$ to weak equivalences in $\mathcal{N}$, then F sends all weak equivalences between fibrant objects in $\mathcal{M}$ to weak equivalences in $\mathcal{N}$.

(ii) Dually, if F sends trivial cofibrations between cofibrant objects in $\mathcal{M}$ to weak equivalences in $\mathcal{N}$, then F sends all weak equivalences between cofibrant objects in $\mathcal{M}$ to weak equivalences in $\mathcal{N}$.

Proof. We prove the first statement. Let X and Y be fibrant objects in $\mathcal{M}$, and let $f\colon X \xrightarrow{\sim} Y$ be a weak equivalence in $\mathcal{M}$. Consider the morphism $(\mathrm{id}_X, f)\colon X \to X \times Y$ and denote by $\pi_X\colon X \times Y \to X$ and $\pi_Y\colon X \times Y \to Y$ the projections. Note that, since the objects X and Y are fibrant, the projections π_X and π_Y are fibrations in $\mathcal{M}$. We factor the morphism (id_X, f) as follows

$$\begin{tikzcd} X \arrow[rr, "(\mathrm{id}_X, f)"] \arrow[dr, hook, "\sim", "j"'] & & X \times Y \\ & C \arrow[ur, two heads, "q"'] & \end{tikzcd},$$

where $j\colon X \overset{\sim}{\hookrightarrow} C$ is a trivial cofibration and $q\colon C \twoheadrightarrow X \times Y$ is a fibration in $\mathcal{M}$. Then the composites $\pi_X q\colon C \twoheadrightarrow X$ and $\pi_Y q\colon C \twoheadrightarrow Y$ are fibrations in $\mathcal{M}$, as they are composites of two fibrations. Since $\pi_X qj = \pi_X(\mathrm{id}_X, f) = \mathrm{id}_X$ and $\pi_Y qj = \pi_Y(\mathrm{id}_X, f) = f$, by 2-out-of-3, the composites $\pi_X q$ and $\pi_Y q$ are also weak equivalences in $\mathcal{M}$, and hence trivial fibrations in $\mathcal{M}$. Since C is fibrant, as $X \times Y$ is so, and F sends trivial fibrations between fibrant objects in $\mathcal{M}$ to weak equivalences in $\mathcal{N}$, we have that $F(\pi_X q)\colon FC \xrightarrow{\sim} FX$ and $F(\pi_Y q)\colon FC \xrightarrow{\sim} FY$ are weak equivalence in $\mathcal{N}$. Now, the morphism $Fj\colon FX \xrightarrow{\sim} FC$ is a weak equivalence in $\mathcal{N}$, by 2-out-of-3, as $F(\pi_X q)F(j) = F(\pi_X qj) = F(\mathrm{id}_X) = \mathrm{id}_{FX}$. Then $Ff = F(\pi_Y qj) = F(\pi_Y q)F(j)$ is a composite of weak equivalences, and therefore $Ff\colon FX \xrightarrow{\sim} FY$ is a weak equivalence in $\mathcal{N}$.

The second statement can be proven dually. □

Corollary 4.4.6. *Suppose that the adjunction $L \dashv R$ is a Quillen pair.*

(i) The right Quillen functor R preserves weak equivalences between fibrant objects.
(ii) The left Quillen functor L preserves weak equivalences between cofibrant objects.

Proof. We prove the first statement. Since R is right Quillen, it preserves all trivial fibrations and sends fibrant objects in $\mathcal{M}$ to fibrant objects in $\mathcal{N}$ by Remark 4.4.3. Therefore, by Ken Brown's Lemma (see Lemma 4.4.5), the functor R preserves weak equivalences between fibrant objects.

The second statement can be proven dually. □

Corollary 4.4.7. *Suppose that the adjunction $L \dashv R$ is a Quillen pair.*

(i) If all objects in $\mathcal{M}$ are fibrant, then the right Quillen functor R preserves all weak equivalences.
(ii) If all objects in $\mathcal{N}$ are cofibrant, then the left Quillen functor L preserves all weak equivalences.

Proof. It is a direct consequence of Corollary 4.4.6. □

The counit and unit of an adjunction might be natural isomorphisms, in which case we talk of a *reflection* when the counit is invertible, a *co-reflection* when the unit is invertible, and an *equivalence* when both are invertible. In particular, the right adjoint of a reflection is fully faithful, and dually the left adjoint of a co-reflection is fully faithful. We now present homotopical versions of these notions for model categories.

Definition 4.4.8. Let $\eta\colon \mathrm{id}_{\mathcal{N}} \Rightarrow RL$ and $\epsilon\colon LR \Rightarrow \mathrm{id}_{\mathcal{M}}$ denote the unit and counit, respectively, of the adjunction $L \dashv R$. The adjunction $L \dashv R$ is

(i) a **Quillen reflection** if it is a Quillen pair and, for every fibrant object $X \in \mathcal{M}$, the component of the derived counit

$$L(RX)^c \xrightarrow{L(q^{\mathcal{N}}_{RX})} LRX \xrightarrow{\epsilon_X} X$$

is a weak equivalence in $\mathcal{M}$, where $((RX)^c, q^{\mathcal{N}}_{RX})$ denotes a cofibrant replacement of RX in $\mathcal{N}$,

(ii) a **Quillen co-reflection** if it is a Quillen pair and, for every cofibrant object $A \in \mathcal{N}$, the component of the derived unit

$$A \xrightarrow{\eta_A} RLA \xrightarrow{R(j^{\mathcal{M}}_{LA})} R(LA)^f$$

is a weak equivalence in $\mathcal{N}$, where $((LA)^f, j^{\mathcal{M}}_{LA})$ denotes a fibrant replacement of LA in $\mathcal{M}$,

(iii) a **Quillen equivalence** if it is both a Quillen reflection and a Quillen co-reflection.

In particular, when a Quillen pair is a Quillen reflection, the right adjoint R not only preserves weak equivalences between fibrant objects but also reflects them. And similarly for a Quillen co-reflection. This is summarized in the following proposition.

Proposition 4.4.9. *Suppose that the adjunction $L \dashv R$ is a Quillen pair.*

(i) If $L \dashv R$ is a Quillen reflection, then a morphism f between fibrant objects in $\mathcal{M}$ is a weak equivalence in $\mathcal{M}$ if and only if Rf is a weak equivalence in $\mathcal{N}$,

(ii) If $L \dashv R$ is a Quillen co-reflection, then a morphism g between cofibrant objects in $\mathcal{N}$ is a weak equivalence in $\mathcal{N}$ if and only if Lg is a weak equivalence in $\mathcal{M}$,

(iii) If $L \dashv R$ is a Quillen equivalence, then R creates weak equivalences between fibrant objects and L creates weak equivalences between cofibrant objects.

Proof. We prove the first statement. Suppose that $L \dashv R$ is a Quillen reflection. Let X and Y be fibrant objects in $\mathcal{M}$ and $f\colon X \to Y$ be a morphism in $\mathcal{M}$. By Corollary 4.4.6, since R is right Quillen, it preserves weak equivalences between fibrant objects. Therefore, if f is a weak equivalence, then so is Rf. Conversely, suppose that $Rf\colon RX \xrightarrow{\sim} RY$ is a weak equivalence in $\mathcal{N}$. Then its cofibrant replacement $(Rf)^c\colon (RX)^c \xrightarrow{\sim} (RY)^c$ is also a weak equivalence in $\mathcal{N}$ by Lemma 4.3.14. By Corollary 4.4.6, since L is left Quillen, it preserves weak equivalences between cofibrant objects. Therefore, the morphism $L(Rf)^c$ is a weak equivalence making the following diagram commute in $\mathcal{M}$.

$$\begin{tikzcd}
L(RX)^c \arrow[r, "L(q^{\mathcal{N}}_{RX})"] \arrow[d, "L(Rf)^c"', "\wr"] & LRX \arrow[r, "\epsilon_X"] \arrow[d, "LRf"] & X \arrow[d, "f"] \\
L(RY)^c \arrow[r, "L(q^{\mathcal{N}}_{RY})"'] & LRY \arrow[r, "\epsilon_Y"'] & Y
\end{tikzcd}$$

Since the top and bottom composites are weak equivalences by assumption, it follows by 2-out-of-3 that f is also a weak equivalence.

The second statement can be proven dually, and the last one is a direct consequence of the two other statements. ☐

Given a Quillen pair between model categories, we want to define induced functors between their homotopy categories. The next result together with Theorem 4.3.18 tells us how to construct such functors.

Lemma 4.4.10. *Suppose that the adjunction $L \dashv R$ is a Quillen pair.*

(i) For each object $X \in \mathcal{M}$, fix a fibrant replacement X^f of X. Then the assignment

$$\gamma_{\mathcal{N}} \circ R((-)^f)\colon \mathcal{M} \to \mathrm{ho}(\mathcal{N}), \quad X \mapsto R(X^f), \quad f \mapsto \gamma_{\mathcal{N}}(R(f^f))$$

defines a functor which sends weak equivalences in $\mathcal{M}$ to isomorphisms in $\mathrm{ho}(\mathcal{N})$.

(ii) For each object $A \in \mathcal{N}$, fix a cofibrant replacement A^c of A. Then the assignment

$$\gamma_{\mathcal{M}} \circ L((-)^c)\colon \mathcal{N} \to \mathrm{ho}(\mathcal{M}), \quad A \mapsto L(A^c), \quad g \mapsto \gamma_{\mathcal{M}}(L(g^c))$$

defines a functor which sends weak equivalences in $\mathcal{N}$ to isomorphisms in $\mathrm{ho}(\mathcal{M})$.

Proof. We prove the first statement. Let $f\colon X \to Y$ and $g\colon Y \to Z$ be morphisms in $\mathcal{M}$. By Lemma 4.3.14, the morphisms $(g \circ f)^f$ and $g^f \circ f^f$ both fit in the commutative diagram for $g \circ f$. Therefore, there is a path object $\mathrm{Path}(Z^f)$ and a homotopy $h\colon X^f \to \mathrm{Path}(Z^f)$ from $(g \circ f)^f$ to $g^f \circ f^f$. Since Z^f is fibrant, then $\mathrm{Path}(Z^f)$ is also fibrant by Remark 4.3.3. Since R preserves weak equivalences between fibrant objects by Corollary 4.4.6, then $R(\mathrm{Path}(Z^f))$ is a path object for $R(Z^f)$ in $\mathcal{N}$ and $Rh\colon R(X^f) \to R(\mathrm{Path}(Z^f))$ is a homotopy from $R((g \circ f)^f)$ to $R(g^f) \circ R(f^f)$. This shows that $\gamma_{\mathcal{N}}(R((g \circ f)^f)) = \gamma_{\mathcal{N}}(R(g^f)) \circ \gamma_{\mathcal{N}}(R(f^f))$. Clearly, $\gamma_{\mathcal{N}} \circ R((-)^f$ also preserves identities, and therefore it is a functor.

Now let $f\colon X \xrightarrow{\sim} Y$ be a weak equivalence in $\mathcal{M}$. By Lemma 4.3.14, the morphism $f^f\colon X^f \xrightarrow{\sim} Y^f$ is also a weak equivalence in $\mathcal{M}$. Since R preserves weak equivalences between fibrant objects by Corollary 4.4.6, then $R(f^f)\colon R(X^f) \xrightarrow{\sim} R(Y^f)$ is a weak equivalence in $\mathcal{N}$. By Lemma 4.3.19, it follows that $\gamma_{\mathcal{N}}(R(f^f))$ is an isomorphism in $\mathrm{ho}(\mathcal{N})$.

The second statement can be proven dually. ☐

Definition 4.4.11. Let $L \dashv R$ be a Quillen pair.

(i) For each object $X \in \mathcal{M}$, fix a fibrant replacement X^f of X. The **right derived functor** of R is defined to be the unique functor $\mathbb{R}\colon \mathrm{ho}(\mathcal{M}) \to \mathrm{ho}(\mathcal{N})$, given by Theorem 4.3.18 and Lemma 4.4.10, making the following diagram commute.

$$\begin{tikzcd} \mathcal{M} \arrow[r, "\gamma_{\mathcal{N}}(R(-)^f)"] \arrow[d, "\gamma_{\mathcal{M}}"'] & \mathrm{ho}(\mathcal{N}) \\ \mathrm{ho}(\mathcal{M}) \arrow[ur, "\mathbb{R}"'] & \end{tikzcd}$$

(ii) For each object $A \in \mathcal{N}$, fix a cofibrant replacement A^c of A. The **left derived functor** of L is defined to be the unique functor $\mathbb{L}\colon \mathrm{ho}(\mathcal{N}) \to \mathrm{ho}(\mathcal{M})$, given by Theorem 4.3.18 and Lemma 4.4.10, making the following diagram commute.

$$\begin{tikzcd} \mathcal{N} \arrow[r, "\gamma_{\mathcal{M}}(L(-)^c)"] \arrow[d, "\gamma_{\mathcal{N}}"'] & \mathrm{ho}(\mathcal{M}) \\ \mathrm{ho}(\mathcal{N}) \arrow[ur, "\mathbb{L}"'] & \end{tikzcd}$$

We now show that the derived functors induced from a Quillen pair between model categories form an adjunction between their homotopy categories. Quillen pairs can therefore be interpreted as the correct homotopical version of adjunctions.

Theorem 4.4.12. *Suppose that the adjunction $L \dashv R$ is a Quillen pair. Then their derived functors $\mathbb{L}$ and $\mathbb{R}$ form an adjunction*

$$\begin{tikzcd} \mathrm{ho}(\mathcal{M}) \arrow[r, bend right, "\mathbb{R}"'] \arrow[r, phantom, "\perp"] & \mathrm{ho}(\mathcal{N}) \arrow[l, bend right, "\mathbb{L}"'] \end{tikzcd}$$

between the homotopy categories of $\mathcal{M}$ and $\mathcal{N}$.

Proof. Let $\eta\colon \mathrm{id}_{\mathcal{N}} \Rightarrow RL$ and $\epsilon\colon LR \Rightarrow \mathrm{id}_{\mathcal{M}}$ denote the unit and counit, respectively, of the adjunction $L \dashv R$. Let A be a cofibrant object in $\mathcal{N}$ and X be a fibrant object in $\mathcal{M}$. We first show that there is an isomorphism $\pi(LA, X) \cong \pi(A, RX)$. Note that LA is cofibrant in $\mathcal{M}$ and RX is fibrant in $\mathcal{N}$ by Remark 4.4.3. We define a map

$$\Phi\colon \pi(LA, X) \longrightarrow \pi(A, RX), \quad [f] \mapsto [(Rf)\eta_A],$$

and check that it is well-defined. Given $[f] = [g]$ in $\pi(LA, X)$, let $\mathrm{Path}(X)$ be a path object for X in $\mathcal{M}$ and $h\colon A \to \mathrm{Path}(X)$ be a homotopy from f to g. Recall from Remark 4.3.3 that $\mathrm{Path}(X)$ is fibrant since X is so. As the right Quillen functor R preserves fibrations and weak equivalences between fibrant objects by Corollary 4.4.6, then $R\mathrm{Path}(X)$ is a path object for RX in $\mathcal{N}$. Then $(Rh)\eta_A\colon A \to R\mathrm{Path}(X)$ gives a homotopy from $(Rf)\eta_A$ to $(Rg)\eta_A$. This shows that $[(Rf)\eta_A] = [(Rg)\eta_A]$ and that Φ is well-defined. Dually, one can show that the map

$$\Psi\colon \pi(A, RX) \longrightarrow \pi(LA, X), \quad [g] \mapsto [\epsilon_X(Lg)],$$

is well-defined. By the triangle identities for (η, ϵ), we directly get that $\Phi\Psi = \mathrm{id}_{\pi(A,RX)}$ and $\Psi\Phi = \mathrm{id}_{\pi(LA,X)}$. This gives the desired isomorphism. By Lemma 4.3.20, it follows that there is an induced isomorphism $\overline{\Phi}\colon \mathrm{ho}(\mathcal{M})(LA, X) \xrightarrow{\cong} \mathrm{ho}(\mathcal{N})(A, RX)$ for every cofibrant object $A \in \mathcal{N}$ and every fibrant object $X \in \mathcal{M}$.

Now, let A be any object in $\mathcal{N}$ and X be any object in $\mathcal{M}$. Given a fibrant replacement $(X^f, j_X^{\mathcal{M}})$ of X in $\mathcal{M}$ and a cofibrant replacement $(A^c, q_A^{\mathcal{N}})$ of A in $\mathcal{N}$, the morphisms $\gamma_{\mathcal{M}}(j_X^{\mathcal{M}})$ and $\gamma_{\mathcal{N}}(q_A^{\mathcal{N}})$ are isomorphisms in $\mathrm{ho}(\mathcal{M})$ and $\mathrm{ho}(\mathcal{N})$ respectively, by Lemma 4.3.19. Therefore, we have the following isomorphisms

$$\begin{tikzcd}
\mathrm{ho}(\mathcal{M})(\mathbb{L}A, X) = \mathrm{ho}(\mathcal{M})(L(A^c), X) \arrow[r, "\cong", "\gamma_{\mathcal{M}}(j_X^{\mathcal{M}})_*"'] \arrow[d, dashed, "\cong"'] & \mathrm{ho}(\mathcal{M})(L(A^c), X^f) \arrow[d, "\overline{\Phi}", "\cong"'] \\
\mathrm{ho}(\mathcal{N})(A, \mathbb{R}X) = \mathrm{ho}(\mathcal{N})(A, R(X^f)) \arrow[r, "\cong", "\gamma_{\mathcal{N}}(q_A^{\mathcal{N}})^*"'] & \mathrm{ho}(\mathcal{N})(A^c, R(X^f))\,,
\end{tikzcd}$$

which shows that $\mathbb{L} \dashv \mathbb{R}$ is an adjunction. □

In particular, when a Quillen pair is a Quillen reflection, co-reflection, or equivalence, the induced derived functors give a reflection, co-reflection, or equivalence between the corresponding homotopy categories.

Theorem 4.4.13. *Suppose that the adjunction $L \dashv R$ is a Quillen pair.*

(i) If $L \dashv R$ is a Quillen reflection, the derived adjunction $\mathbb{L} \dashv \mathbb{R}$ is a reflection, i.e., the counit of the adjunction $\mathbb{L} \dashv \mathbb{R}$ is a natural isomorphism.
(ii) If $L \dashv R$ is a Quillen co-reflection, the derived adjunction $\mathbb{L} \dashv \mathbb{R}$ is a co-reflection, i.e., the unit of the adjunction $\mathbb{L} \dashv \mathbb{R}$ is a natural isomorphism.
(iii) If $L \dashv R$ is a Quillen equivalence, the derived adjunction $\mathbb{L} \dashv \mathbb{R}$ is an equivalence of categories between the homotopy categories $\mathrm{ho}(\mathcal{M})$ *and* $\mathrm{ho}(\mathcal{N})$.

Proof. Suppose that $L \dashv R$ is a Quillen pair, and let $\eta\colon \mathrm{id}_{\mathcal{M}} \Rightarrow LR$ and $\epsilon\colon RL \Rightarrow \mathrm{id}_{\mathcal{N}}$ denote the unit and counit, respectively, of the adjunction $L \dashv R$.

We want to describe the unit $\overline{\eta}\colon \mathrm{id}_{\mathrm{ho}(\mathcal{N})} \Rightarrow \mathbb{RL}$ and the counit $\overline{\epsilon}\colon \mathbb{LR} \Rightarrow \mathrm{id}_{\mathrm{ho}(\mathcal{M})}$ of the derived adjunction $\mathbb{L} \dashv \mathbb{R}$. Let $A \in \mathcal{N}$ be an object. Consider the following commutative diagram obtained by taking $X = \mathbb{L}A$ in the diagram of the proof of Theorem 4.4.12.

$$\begin{tikzcd}
\mathrm{ho}(\mathcal{M})(\mathbb{L}A, \mathbb{L}A) = \mathrm{ho}(\mathcal{M})(L(A^c), L(A^c)) \arrow[r, "\cong", "\gamma_{\mathcal{M}}(j_{L(A^c)}^{\mathcal{M}})_*"'] \arrow[d, dashed, "\Theta", "\cong"'] & \mathrm{ho}(\mathcal{M})(L(A^c), (L(A^c))^f) \arrow[d, "\overline{\Phi}", "\cong"'] \\
\mathrm{ho}(\mathcal{N})(A, \mathbb{RL}A) = \mathrm{ho}(\mathcal{N})(A, R((L(A^c))^f)) \arrow[r, "\cong", "\gamma_{\mathcal{N}}(q_A^{\mathcal{N}})^*"'] & \mathrm{ho}(\mathcal{N})(A^c, R((L(A^c))^f))
\end{tikzcd}$$

Then $\overline{\eta}_A = [\Theta(\mathrm{id}_{\mathbb{L}A})] \in \mathrm{ho}(\mathcal{N})(A, \mathbb{R}\mathbb{L}A)$. Therefore, by evaluating the diagram at $\mathrm{id}_{\mathbb{L}A}$, we get that $[R(j^{\mathcal{M}}_{L(A^c)})\eta_{A^c}] = \overline{\eta}_A[q^{\mathcal{N}}_A]$. Similarly, for every object $X \in \mathcal{M}$, we get that $[\epsilon_{X^f} L(q^{\mathcal{N}}_{R(X^f)})] = [j^{\mathcal{M}}_X]\overline{\epsilon}_X$.

We prove the first statement. If $L \dashv R$ is a Quillen reflection, we need to show that $\overline{\epsilon}\colon \mathbb{L}\mathbb{R} \Rightarrow \mathrm{id}_{\mathrm{ho}(\mathcal{M})}$ is a natural isomorphism. Given an object $X \in \mathcal{M}$, by assumption, the derived counit $\epsilon_{X^f} L(q^{\mathcal{N}}_{R(X^f)})$ at the fibrant object X^f of $\mathcal{M}$ is a weak equivalence. Therefore $[\epsilon_{X^f} L(q^{\mathcal{N}}_{R(X^f)})]$ is an isomorphism in $\mathrm{ho}(\mathcal{M})$. By the above, we have that $[\epsilon_{X^f} L(q^{\mathcal{N}}_{R(X^f)})] = [j^{\mathcal{M}}_X]\overline{\epsilon}_X$, and, since $j^{\mathcal{M}}_X$ is a weak equivalence and hence $[j^{\mathcal{M}}_X]$ is an isomorphism in $\mathrm{ho}(\mathcal{M})$, it follows that $\overline{\epsilon}_X$ is an isomorphism in $\mathrm{ho}(\mathcal{M})$. This shows that $\overline{\epsilon}$ is a natural isomorphism, and hence that $\mathbb{R}$ is a reflection.

The second statement can be proven similarly. As a direct consequence, we get that, if $L \dashv R$ is a Quillen equivalence, then both the counit $\overline{\epsilon}$ and the unit $\overline{\eta}$ of the adjunction $\mathbb{L} \dashv \mathbb{R}$ are natural isomorphisms. This says that $\mathbb{L} \dashv \mathbb{R}$ is an equivalence of categories, which proves the last statement. □

Remark 4.4.14. Since the right adjoint of a reflection is fully faithful, if an adjunction $L \dashv R$ is a Quillen reflection, then the right derived functor $\mathbb{R}\colon \mathrm{ho}(\mathcal{M}) \to \mathrm{ho}(\mathcal{N})$ is fully faithful. We therefore say that the right Quillen functor R is *homotopically fully faithful*. Dually, if an adjunction $L \dashv R$ is a Quillen co-reflection, then the left Quillen functor L is homotopically fully faithful.

4.5. **Monoidal model categories and enriched model categories.** Given a model category $\mathcal{M}$, if the category $\mathcal{M}$ is enriched, tensored, and cotensored over a closed monoidal category $\mathcal{T}$ which itself admits a model structure, the model structure on $\mathcal{M}$ is said to be *enriched* if it satisfies an additional axiom, sometimes called the *pushout-product axiom*. In particular, if the model structure on the closed monoidal category $\mathcal{T}$ also satisfies this additional axiom, we call $\mathcal{T}$ a *monoidal model category*. In this section, we introduce enriched model categories and give several ways of expressing the pushout-product axiom.

For this, let us fix a closed monoidal category $(\mathcal{T}, \otimes, I)$ whose underlying category admits a model structure. We use here the definitions and notations for enriched categories introduced in Section 1.1. Before introducing enrichment of a model structure, we recall a construction which builds from two morphisms in a $\mathcal{T}$-enriched category a morphism in $\mathcal{T}$ called the *pullback corner morphism*.

Definition 4.5.1. Let $\mathcal{M}$ be a $\mathcal{T}$-enriched category. Let $i\colon A \to B$ and $p\colon X \to Y$ be two morphisms in its underlying category $\mathcal{M}_0$. We define the **pullback corner morphism** of i and p to be the unique morphism $(i^*, p_*)\colon \mathcal{M}(B,X) \to \mathcal{M}(A,X) \times_{\mathcal{M}(A,Y)} \mathcal{M}(B,Y)$ in $\mathcal{T}$ given by the universal property of the pullback as in the following diagram.

$$\begin{tikzcd}
\mathcal{M}(B,X) \arrow[dr, dashed, "{(i^*,p_*)}"] \arrow[drr, bend left, "i^*"] \arrow[ddr, bend right, "p_*"'] & & \\
& \mathcal{M}(A,X) \times_{\mathcal{M}(A,Y)} \mathcal{M}(B,Y) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{M}(A,X) \arrow[d, "p_*"] \\
& \mathcal{M}(B,Y) \arrow[r, "i^*"'] & \mathcal{M}(A,Y)
\end{tikzcd}$$

We are now ready to introduce the notion of enriched model categories.

Definition 4.5.2. A $\mathcal{T}$-enriched category $\mathcal{M}$ is a $\mathcal{T}$**-enriched model category** if

(emc1) its underlying category $\mathcal{M}_0$ admits a model structure $(\mathcal{C}, \mathcal{F}, \mathcal{W})$, (emc2) the $\mathcal{T}$-enriched category $\mathcal{M}$ is tensored and cotensored over $\mathcal{T}$ (see Definition 1.1.11),

(emc3) for every cofibration $i\colon A \hookrightarrow B$ in $\mathcal{M}_0$ and every fibration $p\colon X \twoheadrightarrow Y$ in $\mathcal{M}_0$, the pullback corner morphism

$$(i^*, p_*)\colon \mathcal{M}(B, X) \to \mathcal{M}(A, X) \times_{\mathcal{M}(A,Y)} \mathcal{M}(B, Y)$$

is a fibration in $\mathcal{T}$, which is trivial if either i or p is trivial.

Remark 4.5.3. Note that, if $\emptyset$ is an initial object in $\mathcal{M}$ and $*$ is a terminal object in $\mathcal{M}$, then, for every object $A \in \mathcal{M}$, the hom objects $\mathcal{M}(\emptyset, A)$ and $\mathcal{M}(A, *)$ are isomorphic to the terminal object in $\mathcal{T}$. Hence, we can deduce from (emc3) of Definition 4.5.2 that:

(i) if A is a cofibrant object in $\mathcal{M}_0$ and $p\colon X \twoheadrightarrow Y$ is a fibration in $\mathcal{M}_0$, then the induced morphism

$$p_*\colon \mathcal{M}(A, X) \twoheadrightarrow \mathcal{M}(A, Y)$$

is a fibration in $\mathcal{T}$, which is trivial if p is so, and

(ii) if $i\colon A \hookrightarrow B$ is a cofibration in $\mathcal{M}_0$ and X is a fibrant object in $\mathcal{M}_0$, then the morphism

$$i^*\colon \mathcal{M}(B, X) \twoheadrightarrow \mathcal{M}(A, X)$$

is a fibration in $\mathcal{T}$, which is trivial if i is so.

In particular, if A is a cofibrant object in $\mathcal{M}_0$ and X is a fibrant object in $\mathcal{M}_0$, then the hom object $\mathcal{M}(A, X)$ is fibrant in $\mathcal{T}$.

Since a $\mathcal{T}$-enriched model category is assumed to be both tensored and cotensored over $\mathcal{T}$, we can rephrase (emc3) by using tensors and cotensors instead of internal homs. The condition using tensors is often convenient to check that a model structure is enriched. We first introduce the construction given by the *pushout-product morphism* and *pullback corner morphism*, defined using tensors and cotensors.

Definition 4.5.4. Let $\mathcal{M}$ be a tensored $\mathcal{T}$-enriched category. Let $i\colon A \to B$ be a morphism in $\mathcal{M}_0$ and $k\colon S \to T$ be a morphism in $\mathcal{T}$. The **pushout-product** of i and k is the unique morphism $i \,\square_{\mathcal{M}}\, k\colon A \otimes_{\mathcal{M}} T \coprod_{A \otimes_{\mathcal{M}} S} B \otimes_{\mathcal{M}} S \to B \otimes_{\mathcal{M}} T$ in $\mathcal{M}_0$ given by the universal property of the pushout as in the following diagram.

$$\begin{tikzcd}
A \otimes_{\mathcal{M}} S \arrow[r, "i \otimes_{\mathcal{M}} S"] \arrow[d, "A \otimes_{\mathcal{M}} k"'] & B \otimes_{\mathcal{M}} S \arrow[d] \arrow[ddr, bend left, "B \otimes_{\mathcal{M}} k"] & \\
A \otimes_{\mathcal{M}} T \arrow[r] \arrow[drr, bend right, "i \otimes_{\mathcal{M}} T"'] & A \otimes_{\mathcal{M}} T \coprod_{A \otimes_{\mathcal{M}} S} B \otimes_{\mathcal{M}} S \arrow[ul, phantom, "\ulcorner", very near start] \arrow[dr, dashed, "i \,\square_{\mathcal{M}}\, k"'] & \\
& & B \otimes_{\mathcal{M}} T
\end{tikzcd}$$

Definition 4.5.5. Let $\mathcal{M}$ be a cotensored $\mathcal{T}$-enriched category. Let $p\colon X \to Y$ be a morphism in $\mathcal{M}_0$ and $k\colon S \to T$ be a morphism in $\mathcal{T}$. The **pullback corner morphism** of k and p is the unique morphism $p^k\colon X^T \to X^S \times_{Y^S} Y^T$ in $\mathcal{M}_0$ given by the universal property of the pullback as in the following diagram.

$$\begin{tikzcd}
X^T \arrow[dr, dashed, "p^k"] \arrow[drr, bend left, "X^k"] \arrow[ddr, bend right, "p^T"'] & & \\
& X^S \times_{Y^S} Y^T \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X^S \arrow[d, "p^S"] \\
& Y^T \arrow[r, "Y^k"'] & Y^S
\end{tikzcd}$$

Using the constructions defined above, we can reformulate (emc3) as follows.

Proposition 4.5.6. *Let $\mathcal{M}$ be a $\mathcal{T}$-enriched category which satisfies* (emc1-2) *of Definition 4.5.2. Then it satisfies* (emc3) *if and only if it satisfies one of the following equivalent conditions.*

(emc3') *For every cofibration $i\colon A \hookrightarrow B$ in $\mathcal{M}_0$ and every cofibration $k\colon S \hookrightarrow T$ in $\mathcal{T}$, the pushout-product morphism*
$$i \,\square_{\mathcal{M}}\, k\colon A \otimes_{\mathcal{M}} T \coprod_{A \otimes_{\mathcal{M}} S} B \otimes_{\mathcal{M}} S \to B \otimes_{\mathcal{M}} T$$
is a cofibration in $\mathcal{M}_0$, which is trivial if either i or k is trivial.

(emc3") *For every fibration $p\colon X \twoheadrightarrow Y$ in $\mathcal{M}_0$ and every cofibration $k\colon S \hookrightarrow T$ in $\mathcal{T}$, the pullback corner morphism*
$$p^k\colon X^T \to X^S \times_{Y^S} Y^T$$
is a fibration in $\mathcal{M}_0$, which is trivial if either p or k is trivial.

Proof. Since $\mathcal{M}$ is tensored and cotensored over $\mathcal{T}$ by (emc2) of Definition 4.5.2, for every pair of objects $A, B \in \mathcal{M}$ and every object $S \in \mathcal{T}$, we have isomorphisms
$$\mathcal{M}_0(A \otimes_{\mathcal{M}} S, B) \cong \mathcal{T}(S, \mathcal{M}(A,B)) \cong \mathcal{M}_0(A, B^S)$$
natural in A, B, and S, by Remark 1.1.12. Let $i\colon A \to B$ and $p\colon X \to Y$ be morphisms in $\mathcal{M}_0$ and $k\colon S \to T$ be a morphism in $\mathcal{T}$. Then there is a lift in the diagram in $\mathcal{T}$ as depicted below
$$\begin{tikzcd} S \arrow[r] \arrow[d, "k"'] & \mathcal{M}(B,X) \arrow[d, "(i^*, p_*)"] \\ T \arrow[r] \arrow[ur, dashed] & \mathcal{M}(A,X) \times_{\mathcal{M}(A,Y)} \mathcal{M}(B,Y) \end{tikzcd}$$
if and only if there is a lift in one of the following diagrams in $\mathcal{M}_0$, respectively.
$$\begin{tikzcd} A \otimes_{\mathcal{M}} T \coprod_{A \otimes_{\mathcal{M}} S} B \otimes_{\mathcal{M}} S \arrow[r] \arrow[d, "i \,\square_{\mathcal{M}}\, k"'] & X \arrow[d, "p"] \\ B \otimes_{\mathcal{M}} T \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & X^T \arrow[d, "p^k"] \\ B \arrow[r] \arrow[ur, dashed] & X^S \times_{Y^S} Y^T \end{tikzcd}$$
By taking i to be a cofibration in $\mathcal{M}_0$, p to be a fibration in $\mathcal{M}_0$, and k to be a cofibration in $\mathcal{T}$, with one of them being trivial for each case, we can see that (ecm3) is equivalent to (ecm3') and (ecm3"), respectively. ☐

Remark 4.5.7. Suppose that the model structure on $\mathcal{T}$ and $\mathcal{M}_0$ are cofibrantly generated. Then, in order to prove (ecm3'), it is enough to check that this condition holds for generating cofibrations and generating trivial cofibrations. Indeed, recall that all cofibrations and trivial cofibrations in a cofibrantly generated model category are retracts of a transfinite composition of pushouts of morphisms in the generating sets. Hence, since the tensoring functor $\otimes_{\mathcal{M}}$ preserves colimits and retracts in both variables by Remark 1.1.14, then the pushout-product construction also preserves colimits in each variable as it is constructed as a pushout of tensors. Hence, since the classes of cofibrations and trivial cofibrations of $\mathcal{M}_0$ are closed under retracts, pushouts, and transfinite compositions by Proposition 4.1.5, it follows that (emc3') holds for all cofibrations and trivial cofibrations.

When the model structure on the closed monoidal category $\mathcal{T}$ is enriched over itself, we say that it is *monoidal*. Note that the three constructions in (emc3), (emc3'), and (emc3") in this case all take morphisms in $\mathcal{T}$ and produce a morphism in $\mathcal{T}$.

Definition 4.5.8. A closed monoidal category $\mathcal{T}$ whose underlying category admits a model structure is a **monoidal model category** if it is an enriched model category over itself.

Finally, we show that a weak equivalence in a $\mathcal{T}$-enriched model category induces weak equivalences between hom objects in $\mathcal{T}$ under correct cofibrancy and fibrancy conditions. In order to prove this result, we first need the following factorization lemma, which holds in any model category.

Lemma 4.5.9. *Let $\mathcal{M}$ be a model category.*

(i) Let $f\colon X \to Y$ be a morphism in $\mathcal{M}$ between fibrant objects X and Y in $\mathcal{M}$. Then there is a factorization of f as

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "\sim", "j"'] & & Y \\ & Z \arrow[ur, two heads, "r"'] & \end{tikzcd},$$

where $r\colon Z \twoheadrightarrow Y$ is a fibration in $\mathcal{M}$ and $j\colon X \xrightarrow{\sim} Z$ is a weak equivalence which is a section of a trivial fibration $q\colon Z \overset{\sim}{\twoheadrightarrow} X$ in $\mathcal{M}$, i.e., we have $qj = \mathrm{id}_X$.

(ii) Let $f\colon A \to B$ be a morphism in $\mathcal{M}$ between cofibrant objects A and B in $\mathcal{M}$. Then there is a factorization of f as

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, hook, "i"'] & & B \\ & C \arrow[ur, "\sim", "q"'] & \end{tikzcd},$$

where $i\colon A \hookrightarrow B$ is a cofibration in $\mathcal{M}$ and $q\colon C \xrightarrow{\sim} B$ is a weak equivalence which is a retract of a trivial cofibration $j\colon B \overset{\sim}{\hookrightarrow} C$ in $\mathcal{M}$, i.e., we have $qj = \mathrm{id}_B$.

Proof. We prove the first statement. Let $f\colon X \to Y$ be a morphism in $\mathcal{M}$ between fibrant objects X and Y in $\mathcal{M}$. Let us fix a path object for Y

$$Y \xrightarrow[\sim]{w} \mathrm{Path}(Y) \overset{p}{\twoheadrightarrow} Y \times Y .$$

We recall that $p_i\colon \mathrm{Path}(Y) \overset{\sim}{\twoheadrightarrow} Y$ denotes the composite of $p\colon \mathrm{Path}(Y) \to Y \times Y$ with the projection $\pi_i\colon Y \times Y \to Y$, for $i = 0,1$, and that p_i is a trivial fibration by Lemma 4.3.2 since Y is fibrant, for $i = 0,1$. In particular, we have $p_i w = \mathrm{id}_Y$, for $i = 0,1$. We define $q\colon Z \overset{\sim}{\twoheadrightarrow} X$ to be the pullback of p_0 along f, as depicted in the below diagram.

$$\begin{tikzcd} X \arrow[d, "f"'] \arrow[dr, dashed, "j"] \arrow[drr, equal, bend left] & & \\ Y \arrow[dr, "w"'] & Z \arrow[r, two heads, "q", "\sim"'] \arrow[d, "f'"] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "f"] \\ & \mathrm{Path}(Y) \arrow[r, two heads, "\sim", "p_0"'] & Y \end{tikzcd}$$

Note that q is a trivial fibration since p_0 is a trivial fibration and the class of trivial fibrations is closed under pullbacks by Proposition 4.1.5. Then, since $p_0 wf = f$, there is a unique morphism $j\colon X \to Z$, as depicted above, such that $qj = \mathrm{id}_X$ and $f'j = wf$. Since $q\colon Z \overset{\sim}{\twoheadrightarrow} X$ is a weak equivalence, by 2-out-of-3, we get that $j\colon X \xrightarrow{\sim} Z$ is also a weak equivalence. Finally, we set $r \coloneqq p_1 f'\colon Z \to Y$. With this definition of r, we have that $rj = p_1 f'j = p_1 wf = f$. Hence the morphisms r and j give a factorization of f. It remains to show that r is a fibration. Consider the following commutative diagram.

$$
\begin{tikzcd}
Z \arrow[r, dashed, two heads, "q'"] \arrow[d, "f'"'] \arrow[rr, bend left, "q"] \arrow[dd, bend right=60, "r"'] & X \times Y \arrow[r, "\pi_0"] \arrow[d, "f \times \mathrm{id}_Y"] \arrow[dr, phantom, "\lrcorner", very near start] \arrow[ddl, dashed, two heads, "\pi_1"'] & X \arrow[d, "f"] \\
\mathrm{Path}(Y) \arrow[r, two heads, "p"] \arrow[d, "p_1"'] & Y \times Y \arrow[r, "\pi_0"'] \arrow[dl, two heads, "\pi_1"] & Y \\
Y & &
\end{tikzcd}
$$

Since the right-hand square is a pullback, there is a unique morphism $q' \colon Z \to X \times Y$ such that $\pi_0 q' = q$ and $(f \times \mathrm{id}_Y)q' = pf'$. Since the outer rectangle is also a pullback, the left-hand square is a pullback. Henc, since fibrations are closed under pullbacks and p is a fibration, the morphism $q' \colon Z \twoheadrightarrow X \times Y$ is a fibration. Note that the projections $\pi_1 \colon X \times Y \twoheadrightarrow Y$ and $\pi_1 \colon Y \times Y \twoheadrightarrow Y$ are fibrations, as X and Y are fibrant objects. Finally, we have $r = p_1 f' = \pi_1 p f' = \pi_1 (f \times \mathrm{id}_Y) q' = \pi_1 q'$ is a composite of two fibrations, and therefore is a fibration. This shows the desired result.

The second statement can be proven dually □

Proposition 4.5.10. *Let $\mathcal{M}$ be a $\mathcal{T}$-enriched model category.*

(i) Let $f \colon X \xrightarrow{\sim} Y$ be a weak equivalence between fibrant objects X and Y in $\mathcal{M}_0$. Then, for every cofibrant object $A \in \mathcal{M}_0$, the induced morphism

$$f_* \colon \mathcal{M}(A, X) \to \mathcal{M}(A, Y)$$

is a weak equivalence in $\mathcal{T}$.

(ii) Let $f \colon A \xrightarrow{\sim} B$ be a weak equivalence between cofibrant objects A and B in $\mathcal{M}_0$. Then, for every fibrant object $X \in \mathcal{M}_0$, the induced morphism

$$f^* \colon \mathcal{M}(B, X) \to \mathcal{M}(A, X)$$

is a weak equivalence in $\mathcal{T}$.

Proof. We prove the first statement. Let $f \colon X \xrightarrow{\sim} Y$ be a weak equivalence between fibrant objects X and Y in $\mathcal{M}_0$. By Lemma 4.5.9, since X and Y are fibrant, there is a factorization of f as

$$
\begin{tikzcd}
X \arrow[rr, "f", "\sim"'] \arrow[dr, "\sim", "j"'] & & Y , \\
& Z \arrow[ur, two heads, "r"'] &
\end{tikzcd}
$$

where $r \colon Z \twoheadrightarrow Y$ is a fibration and $j \colon X \xrightarrow{\sim} Z$ is a weak equivalence which is a section of a trivial fibration $q \colon Z \overset{\sim}{\twoheadrightarrow} X$. By 2-out-of-3, since f and j are weak equivalences, we get that $r \colon Z \overset{\sim}{\twoheadrightarrow} Y$ is a trivial fibration. Let A be a cofibrant object in $\mathcal{M}_0$. By Remark 4.5.3, since q and r are trivial fibrations, the induced morphisms

$$q_* \colon \mathcal{M}(A, Z) \overset{\sim}{\twoheadrightarrow} \mathcal{M}(A, X) \quad \text{and} \quad r_* \colon \mathcal{M}(A, Z) \overset{\sim}{\twoheadrightarrow} \mathcal{M}(A, Y)$$

are trivial fibrations in $\mathcal{T}$. Since j is a section of q, i.e., we have that $qj = \mathrm{id}_X$, then $q_* j_* = (qj)_* = \mathrm{id}_{\mathcal{M}(A,X)}$ and $j_* \colon \mathcal{M}(A, X) \xrightarrow{\sim} \mathcal{M}(A, Z)$ is a weak equivalence in $\mathcal{T}$, by 2-out-of-3. Finally, since $f = rj$, then $f_* = r_* j_*$ and $f_* \colon \mathcal{M}(A, X) \xrightarrow{\sim} \mathcal{M}(A, Y)$ is a weak equivalence in $\mathcal{T}$, as a composite of two weak equivalences.

The second statement can be proven dually. □

5. Constructions of model structures

In this section, we introduce two different constructions, which build, from an existing model category, another model category. These constructions are very useful in practice as we will see in Section 7 and Part IV..

The first construction, presented in Section 5.1, is given by inducing a model structure along an adjunction. More precisely, given a right adjoint $U\colon \mathcal{K} \to \mathcal{M}$, where $\mathcal{M}$ is a model category, we can define, if it exists, a model structure on the category $\mathcal{K}$, whose fibrations and weak equivalences are precisely the morphisms in $\mathcal{K}$ whose image under U is a fibration or a weak equivalence in $\mathcal{M}$, respectively. Such a model structure is called *right-induced*. Dually, given a left adjoint $L\colon \mathcal{N} \to \mathcal{M}$, we can construct the *left-induced* model structure on $\mathcal{N}$, if it exists, by defining its cofibrations and weak equivalences to be the morphisms in $\mathcal{N}$ whose image under L is a cofibration or a weak equivalence in $\mathcal{M}$, respectively. We state here a theorem by Hess, Kędziorek, Riehl, and Shipley, which gives a criterion for the left- and right-induced model structure to exist, called the *acyclicity condition*, when the categories considered are locally presentable. In particular, we use these results in Section 7 to right-induce a model structure on DblCat along the functors $(\mathbf{H}, \mathcal{V})\colon \mathrm{DblCat} \to 2\mathrm{Cat} \times 2\mathrm{Cat}$ introduced in Section 3.4. The results of this section are based on [HKRS17, GKR20].

In Section 5.2, we give another construction of model categories, that of *left Bousfield localizations*. In this context, the ambient category does not change, and we only change the model structure. The idea is that of a localization, where we add to the class of weak equivalences new morphisms which we would like to be invertible. In particular, this construction is very useful when one wants to restrict the class of fibrant objects to a certain class of objects of interest. For example, in Part IV., we build models of $(\infty, 1)$-categories, $(\infty, 2)$-categories, and double $(\infty, 1)$-categories by localizing categories of simplicial spaces and bisimplicial spaces, so that the fibrant objects are precisely the objects of interest. To construct the left Bousfield localization of a model category, we restrict ourselves to *simplicial categories* whose objects are all cofibrant, since this is the setting in which we will apply the theorems. The theory holds however in a more general setting and we refer the reader to [Hir03, Wer16] for the general theory of left Bousfield localizations. As mentioned in the introduction, a simplicial model category is a model category enriched over the model structure on simplicial sets for Kan complexes, and we also recall here the main features of this model structure. The results of this section are based on [Hir03, Wer16].

5.1. **Left- and right-induced model structures.** We now turn our attention towards model structures induced along adjunctions. This is a useful tool to define new model structures on a category, when it is related by an interesting adjunction to another category that already admits a model structure. The definitions and results here are based on [HKRS17, §§2-3] and [GKR20, §2].

Let us fix a model category $(\mathcal{M}, \mathcal{C}, \mathcal{F}, \mathcal{W})$ and two adjunctions

$$\begin{tikzcd} \mathcal{K} \arrow[r, bend right, "U"'] \arrow[r, phantom, "\perp"] & \mathcal{M} \arrow[l, bend right, "F"'] \arrow[r, bend right, "R"'] \arrow[r, phantom, "\perp"] & \mathcal{N} \arrow[l, bend right, "L"'] \end{tikzcd}.$$

We define what it means to induce the model structure of $\mathcal{M}$ on the categories $\mathcal{K}$ and $\mathcal{N}$ along the right adjoint U and the left adjoint L, respectively.

Definition 5.1.1. Consider the adjunctions $F \dashv U$ and $L \dashv R$.

(i) The **right-induced model structure** on $\mathcal{K}$, if it exists, is given by

$$({}^{\boxslash}(U^{-1}(\mathcal{F} \cap \mathcal{W})), U^{-1}\mathcal{F}, U^{-1}\mathcal{W}).$$

(ii) The **left-induced model structure** on $\mathcal{N}$, if it exists, is given by

$$(L^{-1}\mathcal{C}, (L^{-1}(\mathcal{C} \cap \mathcal{W}))^{\boxslash}, L^{-1}\mathcal{W}).$$

Remark 5.1.2. Note that, since $\mathcal{W}$ satisfies the 2-out-of-3 property, then the classes $U^{-1}\mathcal{W}$ and $L^{-1}\mathcal{W}$ of weak equivalences in the right- and left-induced model structures, respectively, also satisfy the 2-out-of-3 property.

In particular, when we consider the model structure on a category obtained as the right- or left-induced model structure along an adjunction, this adjunction becomes a Quillen pair.

Proposition 5.1.3. *Consider the adjunctions $F \dashv U$ and $L \dashv R$.*

(i) If the right-induced model structure on $\mathcal{K}$ exists, then $F \dashv U$ is a Quillen pair.
(ii) If the left-induced model structure on $\mathcal{N}$ exists, then $L \dashv R$ is a Quillen pair.

Proof. We prove the first statement. By definition, if f is a weak equivalence (resp. fibration) in $\mathcal{K}$, then Uf is a weak equivalence (resp. fibration) in $\mathcal{M}$. Therefore the right adjoint U preserves fibrations and trivial fibrations. This shows that $F \dashv U$ is a Quillen pair.

The second statement can be proven dually. □

The following is a useful criterion for the existence of the right- or left-induced model structure along an adjunction. Under smallness conditions on the categories involved, it is enough to check the *acyclicity conditions*. This is a result by Hess, Kędziorek, Riehl, and Shipley, whose proof goes beyond the scope of this introduction to model categories and is therefore not displayed here.

Proposition 5.1.4. *Suppose that the model category $(\mathcal{M}, \mathcal{C}, \mathcal{F}, \mathcal{W})$ is combinatorial and that the categories $\mathcal{K}$ and $\mathcal{N}$ are locally presentable.*

(i) The right-induced model structure on $\mathcal{K}$ exists if and only if ${}^{\boxslash}(U^{-1}\mathcal{F}) \subseteq U^{-1}\mathcal{W}$,
(ii) The left-induced model structure on $\mathcal{N}$ exists if and only if $(L^{-1}\mathcal{C})^{\boxslash} \subseteq L^{-1}\mathcal{W}$.

Proof. This result is obtained by applying [HKRS17, Corollary 3.3.4] or [GKR20, Corollary 2.7]. Note that, by [HKRS17, Corollary 3.1.7], every combinatorial model structure is accessible. □

The next result gives a useful criterion for proving the acyclicity condition corresponding to a right-induced model structure. A dual statement for the acyclicity condition corresponding to a left-induced model structure can be proven similarly, and is a straightforward consequence of [HKRS17, Theorem 2.2.1].

Corollary 5.1.5. *Suppose that the model category $(\mathcal{M}, \mathcal{C}, \mathcal{F}, \mathcal{W})$ is combinatorial and that the category $\mathcal{K}$ is locally presentable. Suppose that all objects in $\mathcal{M}$ are fibrant and that, for each object $X \in \mathcal{K}$, there is a factorization of the diagonal morphism in $\mathcal{K}$*

$$X \xrightarrow{w} \mathrm{Path}(X) \xrightarrow{p} X \times X$$

such that Uw is a weak equivalence in $\mathcal{M}$ and Up is a fibration in $\mathcal{M}$. Then the right-induced model structure on $\mathcal{K}$ exists.

Proof. By Proposition 5.1.4, it suffices to show that ${}^{\boxslash}(U^{-1}\mathcal{F}) \subseteq U^{-1}\mathcal{W}$. Let $j \colon A \to B$ be a morphism in ${}^{\boxslash}(U^{-1}\mathcal{F})$. Since every object in $\mathcal{M}$ is fibrant, the morphism $A \to *$ is in $U^{-1}\mathcal{F}$. Therefore, there is a lift $r \colon B \to A$ in the following commutative square.

$$\begin{tikzcd} A \arrow[r, equal] \arrow[d, "j"'] & A \arrow[d] \\ B \arrow[r] \arrow[ur, dashed, "r"] & * \end{tikzcd}$$

As $p\colon \mathrm{Path}(B) \to B \times B$ is in $U^{-1}\mathcal{F}$, there is also a lift $h\colon B \to \mathrm{Path}(B)$ in the following commutative diagram.

$$\begin{tikzcd} A \arrow[r, "j"] \arrow[d, "j"'] & B \arrow[r, "w"] & \mathrm{Path}(B) \arrow[d, "p"] \\ B \arrow[rr, "(jr, \mathrm{id}_B)"'] \arrow[urr, dashed, "h"] & & B \times B \end{tikzcd}$$

Note that the composites

$$\mathrm{Path}(B) \xrightarrow{p} B \times B \xrightarrow{\pi_0} B \quad \text{and} \quad \mathrm{Path}(B) \xrightarrow{p} B \times B \xrightarrow{\pi_1} B$$

are in $U^{-1}\mathcal{W}$, by 2-out-of-3, since $\mathrm{id}_B = \pi_0 pw$ and $\mathrm{id}_B = \pi_1 pw$. Therefore, as $\mathrm{id}_B = \pi_1 ph$, the morphism h is also in $U^{-1}\mathcal{W}$, by 2-out-of-3. We deduce that jr is in $U^{-1}\mathcal{W}$, by 2-out-of-3, since $jr = \pi_0 ph$. By applying 2-out-of-6 (see Lemma 4.3.21) to the following diagram

$$\begin{tikzcd} A \arrow[r, equal] \arrow[d, "j"'] & A \arrow[d, "j"] \\ B \arrow[r, "jr"'] \arrow[ur, "r"] & B \end{tikzcd},$$

we get that j is in $U^{-1}\mathcal{W}$. $\square$

Finally, the following proposition gives us generating sets of cofibrations and trivial cofibrations for a model structure which is right-induced from a cofibrantly generated model structure.

Proposition 5.1.6. *Suppose that the model category $(\mathcal{M}, \mathcal{C}, \mathcal{F}, \mathcal{W})$ is combinatorial with generating sets $\mathcal{I}$ of cofibrations and $\mathcal{J}$ of trivial cofibrations, and that the category $\mathcal{K}$ is locally presentable. Then the right-induced model structure on $\mathcal{K}$ along the adjunction $F \dashv U$ is also cofibrantly generated with generating sets $F\mathcal{I}$ of cofibrations and $F\mathcal{J}$ of trivial cofibrations.*

Proof. To show that $F\mathcal{I}$ and $F\mathcal{J}$ are generating sets of cofibrations and trivial cofibrations, respectively, for the right-induced model structure on $\mathcal{K}$, it is enough to check that $U^{-1}\mathcal{F} = (F\mathcal{J})^{\boxslash}$ and that $U^{-1}(\mathcal{F} \cap \mathcal{W}) = (F\mathcal{I})^{\boxslash}$. First note that, since the functor F is left Quillen by Proposition 5.1.3, then F preserves trivial cofibrations and hence $F\mathcal{J} \subseteq F(\mathcal{C} \cap \mathcal{W}) \subseteq {}^{\boxslash}(U^{-1}\mathcal{F})$. It follows that $U^{-1}\mathcal{F} = ({}^{\boxslash}(U^{-1}\mathcal{F}))^{\boxslash} \subseteq (F\mathcal{J})^{\boxslash}$. Now, let $p\colon X \twoheadrightarrow Y$ in $\mathcal{K}$ be a morphism in $(F\mathcal{J})^{\boxslash}$. Then, for every morphism $j\colon A \overset{\sim}{\hookrightarrow} B$ in $\mathcal{J}$, there is a lift in the below left commutative diagram.

$$\begin{tikzcd} FA \arrow[r] \arrow[d, "Fj"'] & X \arrow[d, two heads, "p"] \\ FB \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r] \arrow[d, hook, "j"', "\wr"] & UX \arrow[d, "Up"] \\ B \arrow[r] \arrow[ur, dashed] & UY \end{tikzcd}$$

By the universal property of the adjunction $F \dashv U$, the above left commutative square corresponds uniquely to a commutative square as above right, for every morphism $j\colon A \overset{\sim}{\hookrightarrow} B$ in $\mathcal{J}$. Since $\mathcal{J}$ is a generating set of trivial cofibrations in $\mathcal{M}$, it follows that Up is a fibration in $\mathcal{M}$. This shows that $(F\mathcal{J})^{\boxslash} \subseteq U^{-1}\mathcal{F}$. Hence we indeed have that $U^{-1}\mathcal{F} = (F\mathcal{J})^{\boxslash}$. Similarly, one can show that the other equality holds. $\square$

5.2. **Left Bousfield localizations.** In this section, we introduce *left Bousfield localizations* of model categories. These are given by localizing a model category at some set of morphisms. In other words, we are requiring some specific morphisms to become invertible in the homotopy category. This construction is very useful when one wants to study a specific class of objects in a model category. Indeed, this allows one to restrict the class of fibrant objects of a model category to be precisely the class of objects of study. To simplify the approach, we only consider here left Bousfield localizations of *simplicial* model categories at a set of cofibrations. We refer the reader to [Hir03, Wer16] for the theory in a more general setting.

Simplicial model categories are defined to be enriched model categories over the Kan-Quillen model structure on the category of simplicial sets. We start by recalling the definition of a simplicial set and the main features of the model structure for simplicial sets. We first introduce the simplex category Δ.

Definition 5.2.1. We define the **simplex category** Δ to be the category whose objects are the ordered sets $[n] = \{0 < 1 < \ldots < n\}$, for $n \geq 0$, and whose morphisms are the order-preserving maps. In particular, we denote by $d^i \colon [n-1] \to [n]$ the **ith face map** which skips i, for $0 \leq i \leq n$ and $n \geq 1$, and by $s^j \colon [n+1] \to [n]$ the **jth degeneracy map** which doubles j, for $0 \leq j \leq n$ and $n \geq 0$. These maps generate all the morphisms in Δ under compositions.

A simplicial set is then defined to be a functor $\Delta^{\mathrm{op}} \to \mathrm{Set}$.

Definition 5.2.2. We define $\mathrm{sSet} \coloneqq \mathrm{Set}^{\Delta^{\mathrm{op}}}$ to be the category of simplicial sets and simplicial maps.

Notation 5.2.3. Given a simplicial set $X \colon \Delta^{\mathrm{op}} \to \mathrm{Set}$, we denote by $X_n \coloneqq X([n])$ its set of **n-simplices**, for all $n \geq 0$. We denote by $d_i \coloneqq X(d^i) \colon X_n \to X_{n-1}$ and $s_j \coloneqq X(s^j) \colon X_n \to X_{n+1}$ the images under X of the face and degeneracy maps, for all $0 \leq i \leq n$ and $0 \leq j \leq n$.

We now recall the basic constructions of boundary and horn inclusions of standard n-simplices.

Definition 5.2.4. We define the **standard n-simplex** $\Delta[n]$ to be the representable functor $\Delta(-,[n]) \colon \Delta^{\mathrm{op}} \to \mathrm{Set}$, for $n \geq 0$. By the Yoneda Lemma, the face maps d^i induce simplicial maps $d^i \colon \Delta[n-1] \to \Delta[n]$, for all $0 \leq i \leq n$ and $n \geq 1$. We define the **boundary** of $\Delta[n]$ to be

$$\delta\Delta[n] \coloneqq \bigcup_{0 \leq i \leq n} \mathrm{Im}\left(d^i \colon \Delta[n-1] \to \Delta[n]\right),$$

for $n \geq 1$, and we set $\delta\Delta[0] \coloneqq \emptyset$. For $n \geq 1$ and $0 \leq t \leq n$, we define the **(n,t)-horn** of $\Delta[n]$ to be

$$\Lambda^t[n] \coloneqq \bigcup_{0 \leq i \leq n,\, i \neq t} \mathrm{Im}\left(d^i \colon \Delta[n-1] \to \Delta[n]\right).$$

These simplicial sets come with inclusion maps $\iota_n \colon \delta\Delta[n] \to \Delta[n]$, for all $n \geq 0$, and $\ell_{n,t} \colon \Lambda^t[n] \to \Delta[n]$ for all $n \geq 1$ and $0 \leq t \leq n$.

Remark 5.2.5. Boundary inclusions generate all monomorphisms in sSet under transfinite compositions of pushouts and retracts.

These boundary and horn inclusions form generating sets of cofibrations and trivial cofibrations, respectively, for the model structure on sSet described below in Theorem 5.2.7. The fibrations and fibrant objects are then defined by their lifting properties with respect to horn inclusions.

Definition 5.2.6. A simplicial map $p\colon X \to Y$ is a **Kan fibration** if it has the right lifting property with respect to the horn inclusion $\ell_{n,t}\colon \Lambda^t[n] \to \Delta[n]$, for all $n \geq 1$ and $0 \leq t \leq n$. A simplicial set X is a **Kan complex** if the unique map $X \to \Delta[0]$ is a Kan fibration.

We now state the existence of the Kan-Quillen model structure on sSet and describe its relevant classes of maps. This model structure is enriched over itself through the cartesian monoidal structure on sSet. We refer the reader to [JT99] for a proof.

Theorem 5.2.7. *There is a combinatorial, simplicial model structure on* sSet *whose*

(i) cofibrations are the monomorphisms; in particular, every object is cofibrant,
(ii) fibrations are the Kan fibrations; in particular, the fibrant objects are the Kan complexes,
(iii) weak equivalences are the simplicial maps $f\colon X \to Y$ such that the induced maps $f^*\colon \pi(Y,K) \to \pi(X,K)$ between homotopy classes is an isomorphism, for every Kan complex K.

Generating sets of cofibrations and trivial cofibrations are given by the sets of monomorphisms

$$\mathcal{I} = \{\iota_n\colon \delta\Delta[n] \to \Delta[n] \mid n \geq 0\} \quad \textit{and} \quad \mathcal{J} = \{\ell_{n,t}\colon \Lambda^t[n] \to \Delta[n] \mid n \geq 1, 0 \leq t \leq n\}.$$

Proof. The existence of the model structure is [JT99, Theorem 1.7.1]. The fact that it is simplicial follows from [JT99, Theorem 1.5.2]. □

Remark 5.2.8. The category sSet is locally presentable. Moreover, every simplicial set can be obtained as a colimit of the standard n-simplices. Using this description, we can define a realization functor $|-|\colon \mathrm{sSet} \to \mathrm{Top}$ to the category of topological spaces and continuous maps as follows. A simplicial set $X \cong \operatorname{colim}_{\sigma\in X_n,\, n\geq 0} \Delta[n]$ is sent to the topological space $|X| \coloneqq \operatorname{colim}_{\sigma\in X_n,\, n\geq 0} \Delta^n$, where Δ^n is the geometric n-simplex. A weak equivalence in sSet can then be characterized through this geometric realization functor: a simplicial map $f\colon X \to Y$ is a weak equivalence in sSet if and only if its realization $|f|\colon |X| \to |Y|$ is a weak (homotopy) equivalence of topological spaces. This realization functor further induces a Quillen equivalence between the Kan-Quillen model structure on sSet and the Quillen model structure for topological spaces.

With this model structure on sSet, we can introduce simplicial model categories as the simplicial categories which admit a model structure that is enriched over sSet.

Definition 5.2.9. A **simplicial model category** $\mathcal{M}$ is an sSet-enriched model category, where sSet is endowed with the Kan-Quillen model structure of Theorem 5.2.7.

We are now ready to introduce left Bousfield localizations of simplicial model categories. For simplicity and since it is the case for all the model structures in this paper to which we will apply this construction, we suppose that all objects in the simplicial model categories considered are cofibrant. If it is not the case, then one needs to work with cofibrant replacements and left properness in order to make the theory work. We first introduce what will be the fibrant objects and the weak equivalences in the left Bousfield localization.

Definition 5.2.10. Let $\mathcal{M}$ be a simplicial model category such that all objects in $\mathcal{M}$ are cofibrant and let $\mathcal{S}$ be a set of cofibrations in $\mathcal{M}$.

(i) An object $X \in \mathcal{M}$ is $\mathcal{S}$-local if it is fibrant in $\mathcal{M}$ and, for every cofibration $s\colon A \to B$ in $\mathcal{S}$, the induced map
$$s^*\colon \mathcal{M}(B,X) \to \mathcal{M}(A,X)$$
is a trivial fibration in sSet.

(ii) A morphism $f\colon A \to B$ in $\mathcal{M}$ is an $\mathcal{S}$-local equivalence if, for every $\mathcal{S}$-local object X in $\mathcal{M}$, the induced map
$$f^*\colon \mathcal{M}(B,X) \to \mathcal{M}(A,X)$$
is a weak equivalence in sSet.

We now describe the model structure given by localizing at a set of cofibrations.

Definition 5.2.11. Let $\mathcal{M}$ be a simplicial model category such that all objects in $\mathcal{M}$ are cofibrant and let $\mathcal{S}$ be a set of cofibrations in $\mathcal{M}$. The **left Bousfield localization** $L_{\mathcal{S}}\mathcal{M}$ of $\mathcal{M}$ at $\mathcal{S}$, if it exists, is the model structure on $\mathcal{M}$ whose

(i) cofibrations are precisely the cofibrations in $\mathcal{M}$,
(ii) weak equivalences are the $\mathcal{S}$-local equivalences.

The rest of the model structure is determined by lifting properties.

Remark 5.2.12. Note that every cofibration $s\colon A \hookrightarrow B$ in $\mathcal{S}$ is a trivial cofibration in the left Bousfield localization $L_{\mathcal{S}}\mathcal{M}$ since it is both a cofibration and an $\mathcal{S}$-local equivalence. Furthermore, it follows from Remark 4.5.3 that a cofibration $j\colon A \hookrightarrow B$ is a trivial cofibration in $L_{\mathcal{S}}\mathcal{M}$ if and only if the induced map
$$j^*\colon \mathcal{M}(B,X) \twoheadrightarrow \mathcal{M}(A,X)$$
is a trivial fibration in sSet, for every $\mathcal{S}$-local object $X \in \mathcal{M}$.

Remark 5.2.13. By Remark 4.5.3, every weak equivalence $f\colon \mathcal{A} \xrightarrow{\sim} \mathcal{B}$ in $\mathcal{M}$ induces a weak equivalence
$$f^*\colon \mathcal{M}(B,X) \xrightarrow{\sim} \mathcal{M}(A,X)$$
in sSet, for every fibrant object $X \in \mathcal{M}$. Since $\mathcal{S}$-local objects are in particular fibrant in $\mathcal{M}$, we get that every weak equivalence in $\mathcal{M}$ is in particular a weak equivalence in $L_{\mathcal{S}}\mathcal{M}$. As a consequence, every trivial cofibration in $\mathcal{M}$ is a trivial cofibration in $L_{\mathcal{S}}\mathcal{M}$.

The following proposition implies that the homotopy category of the left Bousfield localization $L_{\mathcal{S}}\mathcal{M}$ is embedded in that of $\mathcal{M}$ in a reflective way.

Proposition 5.2.14. *Let $\mathcal{M}$ be a simplicial model category such that all objects in $\mathcal{M}$ are cofibrant. Let $\mathcal{S}$ be a set of cofibrations in $\mathcal{M}$ such that the left Bousfield localization $L_{\mathcal{S}}\mathcal{M}$ exists. Then the identity adjunction*
$$\begin{tikzcd} L_{\mathcal{S}}\mathcal{M} \arrow[r, bend right, "\mathrm{id}_{\mathcal{M}}"'] \arrow[r, phantom, "\perp"] & \mathcal{M} \arrow[l, bend right, "\mathrm{id}_{\mathcal{M}}"'] \end{tikzcd}$$
is a Quillen reflection.

Proof. First, note that the left adjoint $\mathrm{id}_{\mathcal{M}}\colon \mathcal{M} \to L_{\mathcal{S}}\mathcal{M}$ preserves all cofibrations, by definition of $L_{\mathcal{S}}\mathcal{M}$, and all weak equivalences, by Remark 5.2.13. Hence it is left Quillen. Furthermore, the component of the derived counit at a fibrant object $X \in L_{\mathcal{S}}\mathcal{M}$ is given by the cofibrant replacement morphism $q_X^{\mathcal{M}}\colon X^c \to X$ in $\mathcal{M}$. Since this is a weak equivalence in $\mathcal{M}$, it is also a weak equivalence in $L_{\mathcal{S}}\mathcal{M}$ by Remark 5.2.13. This shows that the identity adjunction is a Quillen reflection. $\square$

When the simplicial model category is combinatorial, the left Bousfield localization at any set of cofibrations always exists. We do not prove this result here, as it goes beyond the scope of this introduction to model categories.

Theorem 5.2.15. *Let $\mathcal{M}$ be a combinatorial, simplicial model category such that all objects in $\mathcal{M}$ are cofibrant. Let $\mathcal{S}$ be a set of cofibrations in $\mathcal{M}$. Then the left Bousfield localization $L_{\mathcal{S}}\mathcal{M}$ on $\mathcal{M}$ exists and it is combinatorial.*

Proof. This follows from [Wer16, Theorem 6.1]. Note that the model category $\mathcal{M}$ is left proper since all objects in $\mathcal{M}$ are cofibrant. A proof in the case of cellular model categories can also be found in [Hir03, Theorem 4.1.1]. $\square$

Furthermore, we can show that the left Bousfield localization of a simplicial model category is also simplicial.

Proposition 5.2.16. *Let $\mathcal{M}$ be a simplicial model category such that all objects in $\mathcal{M}$ are cofibrant. Let $\mathcal{S}$ be a set of cofibrations in $\mathcal{M}$ such that the left Bousfield localization $L_{\mathcal{S}}\mathcal{M}$ exists. Then the model structure $L_{\mathcal{S}}\mathcal{M}$ of $\mathcal{M}$ is simplicial.*

More precisely, if $j\colon A \hookrightarrow B$ is a cofibration in $L_{\mathcal{S}}\mathcal{M}$ and $i\colon K \hookrightarrow L$ is a cofibration in sSet*, then the pushout-product morphism $j \,\square\, i\colon A \otimes L \coprod_{A\otimes K} B \otimes K \hookrightarrow B \otimes L$ is a cofibration in $L_{\mathcal{S}}\mathcal{M}$, which is trivial if either j is a weak equivalence in $L_{\mathcal{S}}\mathcal{M}$ or i is a weak equivalence in* sSet*, where $\otimes$*

Proof. Since the model structure on $\mathcal{M}$ is simplicial, the pushout-product $j \,\square\, i$ is a cofibration in $\mathcal{M}$ by (emc3') of Proposition 4.5.6, and hence it is also a cofibration in $L_{\mathcal{S}}\mathcal{M}$. It remains to show that $j \,\square\, i$ is an $\mathcal{S}$-local equivalence, whenever j or i is a weak equivalence. Let X be an $\mathcal{S}$-local object in $\mathcal{M}$. We show that the map

$$(j \,\square\, i)^*\colon \mathcal{M}(B \otimes L, X) \to \mathcal{M}(A \otimes L \coprod_{A\otimes K} B \otimes K, X)$$

is a trivial fibration in sSet, whenever j or i is a weak equivalence. For $n \geq 0$, by the universal property of the tensor $\otimes$ and by Proposition 1.1.15, there is a lift in the below left diagram in sSet if and only if there is a lift in the below right diagram in sSet.

$$\begin{tikzcd} \delta\Delta[n] \arrow[r] \arrow[d, hook, "\iota_n"'] & \mathcal{M}(B\otimes L, X) \arrow[d, two heads, "(j\,\square\, i)^*"] \\ \Delta[n] \arrow[r] \arrow[ur, dashed] & \mathcal{M}(A\otimes L \coprod_{A\otimes K} B\otimes K, X) \end{tikzcd} \qquad \begin{tikzcd} \delta\Delta[n]\times L \coprod_{\delta\Delta[n]\times K} \Delta[n]\times K \arrow[r] \arrow[d, hook, "\iota_n \,\square\, i"'] & \mathcal{M}(B, X) \arrow[d, two heads, "j^*"] \\ \Delta[n]\times L \arrow[r] \arrow[ur, dashed] & \mathcal{M}(A, X) \end{tikzcd}$$

Since the model structure on sSet is simplicial, the pushout-product map $\iota_n \,\square\, i$ is a cofibration in sSet, which is trivial if i is so. Moreover, by Remark 4.5.3, we have that j^* is a fibration. Furthermore, the fibration j^* is trivial if j is an $\mathcal{S}$-local equivalence. Hence, if either j is an $\mathcal{S}$-local equivalence in $\mathcal{M}$ or i is a weak equivalence in sSet, there is a lift in the above right diagram. This shows that there is a lift in the above left diagram, and hence that $(j \,\square\, i)^*$ is a trivial fibration in sSet. We conclude that $j \,\square\, i$ is a trivial cofibration in $L_{\mathcal{S}}\mathcal{M}$, whenever j is a weak equivalence in $L_{\mathcal{S}}\mathcal{M}$ or i is a weak equivalence in sSet. $\square$

We now give a characterization of the $\mathcal{S}$-local objects in terms of lifting properties.

Lemma 5.2.17. *Let $\mathcal{M}$ be a combinatorial, simplicial model category such that all objects in $\mathcal{M}$ are cofibrant. Let $\mathcal{S}$ be a set of cofibrations in $\mathcal{M}$ and let $\mathcal{J}$ denote a set of generating trivial cofibrations for $\mathcal{M}$. We define the set $\mathcal{J}_{\mathcal{S}}$ to be*

$$\mathcal{J}_{\mathcal{S}} = \mathcal{J} \bigcup \{ s \,\square\, \iota_n \colon A \otimes \Delta[n] \coprod_{A\otimes\delta\Delta[n]} B \otimes \delta\Delta[n] \to B \otimes \Delta[n] \mid s\colon A \to B \in \mathcal{S}, n \geq 0 \},$$

where $\otimes$ denotes the tensor of $\mathcal{M}$ over sSet*. Then, an object $X \in \mathcal{M}$ is $\mathcal{S}$-local if and only if the unique morphism $X \to *$ to the terminal object $*$ in $\mathcal{M}$ is in $\mathcal{J}_{\mathcal{S}}-$*inj*, i.e., it has the right lifting property with respect to every morphism in $\mathcal{J}_{\mathcal{S}}$, where $\otimes$ denotes the tensor of $\mathcal{M}$ over* sSet*.*

Proof. Let X be an object in $\mathcal{M}$. First note that X is fibrant in $\mathcal{M}$ if and only if the morphism $X \to *$ is in $\mathcal{J}-$inj. Now, let $s\colon A \hookrightarrow B$ be a cofibration in $\mathcal{S}$ and $n \geq 0$. Then, by the universal property of the tensor $\otimes$, there is a lift in the below left diagram in $\mathcal{M}$ if and only if there is a lift in the below right diagram in sSet.

$$\begin{tikzcd} A \otimes \Delta[n] \coprod_{A\otimes \delta\Delta[n]} B \otimes \delta\Delta[n] \arrow[r] \arrow[d, "s \,\square\, \iota_n"'] & X \arrow[d] \\ B \otimes \Delta[n] \arrow[r] \arrow[ur, dashed] & * \end{tikzcd} \qquad \begin{tikzcd} \delta\Delta[n] \arrow[r] \arrow[d, hook, "\iota_n"'] & \mathcal{M}(B,X) \arrow[d, "s^*"] \\ \Delta[n] \arrow[r] \arrow[ur, dashed] & \mathcal{M}(A,X) \end{tikzcd}$$

This says that the morphism $X \to *$ has the right lifting property with respect to $s \,\square\, \iota_n$, for every $s \in \mathcal{S}$ and $n \geq 0$, if and only if the map $s^*\colon \mathcal{M}(B,X) \to \mathcal{M}(A,X)$ is a trivial fibration in sSet, for every $s \in \mathcal{S}$. Hence, this shows that an object X is $\mathcal{S}$-local if and only if $X \to *$ is in $\mathcal{J}_{\mathcal{S}}-$inj. □

Using this result, we can show that the fibrant objects in the left Bousfield localization at a set $\mathcal{S}$ of cofibrations are $\mathcal{S}$-local objects. The converse, i.e., that $\mathcal{S}$-local objects are fibrant in the left Bousfield localization, requires to find a set of generating trivial cofibrations which is, in general, larger than the set $\mathcal{J}_{\mathcal{S}}$ introduced in Lemma 5.2.17, and to prove that $\mathcal{S}$-local objects also lift against the additional trivial cofibrations. Since constructing this generating set of trivial cofibrations involves a lot of technicalities, we do not prove it here.

Proposition 5.2.18. *Let $\mathcal{M}$ be a combinatorial, simplicial model category such that all objects in $\mathcal{M}$ are cofibrant and let $\mathcal{S}$ be a set of cofibrations in $\mathcal{M}$. An object is fibrant in $L_{\mathcal{S}}\mathcal{M}$ if and only if it is $\mathcal{S}$-local.*

Proof. Let X be a fibrant object in $L_{\mathcal{S}}\mathcal{M}$. By Lemma 5.2.17, it is enough to show that the morphism $X \to *$, which is a fibration in $L_{\mathcal{S}}\mathcal{M}$ by assumption, is in $\mathcal{J}_{\mathcal{S}}-$inj. By Remark 5.2.13 and Proposition 5.2.16, the set $\mathcal{J}_{\mathcal{S}}$ is contained in the class of trivial cofibrations of $L_{\mathcal{S}}\mathcal{M}$. Hence, since $X \to *$ has the right lifting property with respect to all trivial cofibrations, it is in $\mathcal{J}_{\mathcal{S}}-$inj.

For the converse, i.e., that an $\mathcal{S}$-local object is fibrant in $L_{\mathcal{S}}\mathcal{M}$, we refer the reader to [Wer16, Proposition 5.3] or [Hir03, Theorem 4.1.1]. □

We now want to prove that a morphism between $\mathcal{S}$-local objects is a weak equivalence (resp. fibration) in $L_{\mathcal{S}}\mathcal{M}$ if and only if it is a weak equivalence (resp. fibration) in $\mathcal{M}$. For this, we first compute path objects for simplicial model categories.

Lemma 5.2.19. *Let $\mathcal{M}$ be a simplicial model category and let X be a fibrant object in $\mathcal{M}$. Then the following diagram*

$$X \xrightarrow[\sim]{w} X^{\Delta[1]} \overset{p}{\twoheadrightarrow} X \times X$$

gives a path object for X in $\mathcal{M}$, where the morphisms w and p are induced by the maps

$$\Delta[0] \cup \Delta[0] = \delta\Delta[1] \xrightarrow{\iota_1} \Delta[1] \xrightarrow{!} \Delta[0]$$

in sSet*.*

Proof. Since the map $\ell_{1,0}\colon \Delta[0] \overset{\sim}{\hookrightarrow} \Delta[1]$ is a trivial cofibration in sSet, and the map $\iota_1\colon \Delta[0] \cup \Delta[0] = \delta\Delta[1] \hookrightarrow \Delta[1]$ is a cofibration in sSet, if X is fibrant in $\mathcal{M}$, then the induced morphisms

$$X^{\ell_{1,0}}\colon X^{\Delta[1]} \overset{\sim}{\twoheadrightarrow} X^{\Delta[0]} \cong X \quad \text{and} \quad p = X^{\iota_1}\colon X^{\Delta[1]} \twoheadrightarrow X^{\Delta[0]\cup\Delta[0]} \cong X \times X$$

are fibrations in $\mathcal{M}$ such that the first morphism is trivial, by (emc3") of Proposition 4.5.6. Since $!\ell_{1,0} = \mathrm{id}_{\Delta[0]}$, then $X^{\ell_{1,0}}w = X^{\ell_{1,0}}X^{!} = \mathrm{id}_X$ and, by 2-out-of-3, we get that w is a weak equivalence. This shows that $X^{\Delta[1]}$ is a path object for X. $\square$

We are now ready to prove the desired result.

Proposition 5.2.20. *Let $\mathcal{M}$ be a combinatorial, simplicial model category such that all objects in $\mathcal{M}$ are cofibrant and let $\mathcal{S}$ be a set of cofibrations in $\mathcal{M}$.*

(i) A morphism between $\mathcal{S}$-local objects is a weak equivalence in $L_{\mathcal{S}}\mathcal{M}$ if and only if it is a weak equivalence in $\mathcal{M}$.
(ii) A morphism between $\mathcal{S}$-local objects is a fibration in $L_{\mathcal{S}}\mathcal{M}$ if and only if it is a fibration in $\mathcal{M}$.

Proof. We first prove (i). By Remark 5.2.13, we already know that weak equivalences in $\mathcal{M}$ are weak equivalences in $L_{\mathcal{S}}\mathcal{M}$. So let $f\colon X \xrightarrow{\sim} Y$ be a weak equivalence in $L_{\mathcal{S}}\mathcal{M}$ between $\mathcal{S}$-local objects. Since $\mathcal{S}$-local objects are precisely the fibrant objects in $L_{\mathcal{S}}\mathcal{M}$ by Proposition 5.2.18 and all objects in $L_{\mathcal{S}}\mathcal{M}$ are cofibrant, by the Whitehead Theorem (see Theorem 4.3.11), the morphism $f\colon X \xrightarrow{\sim} Y$ is a homotopy equivalence in $L_{\mathcal{S}}\mathcal{M}$. To make sense of the notion of homotopy in the simplicial model category $L_{\mathcal{S}}\mathcal{M}$, we are using the path objects $X^{\Delta[1]}$ and $Y^{\Delta[1]}$ given by Lemma 5.2.19. Then $X^{\Delta[1]}$ and $Y^{\Delta[1]}$ are also path objects in $\mathcal{M}$, as $\mathcal{M}$ is also simplicial and the simplicial enrichment of $L_{\mathcal{S}}\mathcal{M}$ and $\mathcal{M}$ are the same. This implies that f is a homotopy equivalence in $\mathcal{M}$. In particular, it is a weak equivalence in $\mathcal{M}$, by Theorem 4.3.11.

We now prove (ii). Since $\mathrm{id}_{\mathcal{M}}\colon L_{\mathcal{S}}\mathcal{M} \to \mathcal{M}$ is right Quillen by Proposition 5.2.14, a fibration in $L_{\mathcal{S}}\mathcal{M}$ is in particular a fibration in $\mathcal{M}$. So let $p\colon X \to Y$ be a fibration in $\mathcal{M}$ between $\mathcal{S}$-local objects. We factor p as

$$\begin{tikzcd} X \arrow[rr, "p"] \arrow[dr, hook, "\sim", "j"'] & & Y \\ & Z \arrow[ur, two heads, "r"'] & \end{tikzcd}$$

with j a trivial cofibration in $L_{\mathcal{S}}\mathcal{M}$ and r a fibration in $L_{\mathcal{S}}\mathcal{M}$. Since Y is $\mathcal{S}$-local and p is a fibration in $L_{\mathcal{S}}\mathcal{M}$, then Z is also $\mathcal{S}$-local by Proposition 5.2.18. By (i), the trivial cofibration j between $\mathcal{S}$-local objects is a weak equivalence in $\mathcal{M}$. Hence j is a trivial cofibration in $\mathcal{M}$, as $\mathcal{M}$ and $L_{\mathcal{S}}\mathcal{M}$ have the same cofibrations. Since p has the left lifting property with respect to j, it is a retract of the fibration r by the retract argument (see Proposition 4.1.6) and therefore it is also a fibration in $L_{\mathcal{S}}\mathcal{M}$. $\square$

We finally want to prove that a left Quillen functor $L\colon \mathcal{M} \to \mathcal{N}$ which sends all cofibrations in $\mathcal{S}$ to weak equivalences in $\mathcal{N}$ induces a left Quillen functor $L\colon L_{\mathcal{S}}\mathcal{M} \to \mathcal{N}$. For this, we first need the following result. We do not give a full proof of this result since it involves more knowledge about Reedy model categories, which has not been covered in this introduction. We refer the reader to [Hir03] for a complete proof.

Lemma 5.2.21. *Let $\mathcal{M}$ be a simplicial model category such that all objects in $\mathcal{M}$ are cofibrant and let $\mathcal{N}$ be a model category. Suppose that*

$$\begin{tikzcd} \mathcal{N} \arrow[r, bend right, "R"'] \arrow[r, phantom, "\perp"] & \mathcal{M} \arrow[l, bend right, "L"'] \end{tikzcd}$$

is a Quillen pair. Let $f\colon A \to B$ be a morphism in $\mathcal{M}$. Then the morphism $Lf\colon LA \to LB$ is a weak equivalence in $\mathcal{N}$ if and only if the induced map of simplicial sets

$$L(f \otimes \Delta[-])^*\colon \mathcal{N}(L(B \otimes \Delta[-]), X) \to \mathcal{N}(L(A \otimes \Delta[-]), X)$$

is a weak equivalence in sSet*, for every fibrant object* $X \in \mathcal{N}$*.*

Proof. Since $\mathcal{M}$ is simplicial, the cosimplicial objects

$$A \otimes \Delta[-], B \otimes \Delta[-] \colon \Delta \to \mathrm{Set}$$

are cosimplicial resolutions of the cofibrant objects A and B in $\mathcal{M}$ by [Hir03, Proposition 16.1.3]. Then, since L is left Quillen, by [Hir03, Proposition 16.2.1], the cosimplicial objects

$$L(A \otimes \Delta[-]), L(B \otimes \Delta[-]) \colon \Delta \to \mathrm{Set}$$

are cosimplicial resolutions of the cofibrant objects LA and LB in $\mathcal{N}$. Finally, by the equivalence of (1) and (5) in [Hir03, Theorem 17.7.7], we get that Lf is a weak equivalence if and only if the induced map

$$L(f \otimes \Delta[-])^* \colon \mathcal{N}(L(B \otimes \Delta[-]), X) \to \mathcal{N}(L(A \otimes \Delta[-]), X)$$

is a weak equivalence in sSet, for every fibrant object $X \in \mathcal{N}$. □

Before proving the desired result, we first consider the right adjoint of the left Quillen functor $L \colon \mathcal{M} \to \mathcal{N}$ that sends cofibrations in $\mathcal{S}$ to weak equivalences in $\mathcal{N}$, and we prove that it sends fibrant objects in $\mathcal{N}$ to $\mathcal{S}$-local objects in $\mathcal{M}$.

Lemma 5.2.22. *Let $\mathcal{M}$ be a combinatorial, simplicial model category such that all objects in $\mathcal{M}$ are cofibrant and let $\mathcal{N}$ be a model category. Suppose that*

$$\begin{tikzcd} \mathcal{N} \arrow[r, bend right, "R"'] \arrow[r, phantom, "\perp"] & \mathcal{M} \arrow[l, bend right, "L"'] \end{tikzcd}$$

is a Quillen pair. Let $\mathcal{S}$ be a set of cofibrations in $\mathcal{M}$ such that $Ls \colon LA \xrightarrow{\sim} LB$ is a weak equivalence in $\mathcal{N}$, for all $s \in \mathcal{S}$. Then the right adjoint $R \colon \mathcal{N} \to \mathcal{M}$ sends fibrant objects in $\mathcal{N}$ to $\mathcal{S}$-local objects in $\mathcal{M}$.

Proof. Let X be a fibrant object in $\mathcal{N}$. Since $Ls \colon LA \xrightarrow{\sim} LB$ is a weak equivalence in $\mathcal{N}$, for all $s \in \mathcal{S}$, then the induced map

$$L(s \otimes \Delta[-])^* \colon \mathcal{N}(L(B \otimes \Delta[-]), X) \xrightarrow{\sim} \mathcal{N}(L(A \otimes \Delta[-]), X)$$

is a weak equivalence in sSet, by Lemma 5.2.21, for all $s \in \mathcal{S}$. Since $L \dashv R$ is an adjunction, there is an isomorphism

$$\mathcal{N}(L(A \otimes \Delta[n]), X) \cong \mathcal{M}_0(A \otimes \Delta[n], RX) \cong \mathcal{M}(A, RX)_n$$

natural in $A \in \mathcal{M}$, $X \in \mathcal{N}$, and $n \geq 0$. Hence this induces an isomorphism of simplicial sets $\mathcal{N}(L(A \otimes \Delta[-]), X) \cong \mathcal{M}(A, RX)$, natural in $A \in \mathcal{M}$ and $X \in \mathcal{N}$. Hence we have a commutative square in sSet of the form

$$\begin{tikzcd} \mathcal{N}(L(B \otimes \Delta[-]), X) \arrow[r, "L(s \otimes \Delta[-])^*", "\sim"'] \arrow[d, "\cong"'] & \mathcal{N}(L(A \otimes \Delta[-]), X) \arrow[d, "\cong"] \\ \mathcal{M}(B, RX) \arrow[r, "s^*"'] & \mathcal{M}(A, RX)\, , \end{tikzcd}$$

for all $s \in \mathcal{S}$. By 2-out-of-3, we get that the bottom map s^* is a weak equivalence in sSet, for all $s \in \mathcal{S}$. This shows that RX is $\mathcal{S}$-local. □

Theorem 5.2.23. *Let $\mathcal{M}$ be a combinatorial, simplicial model category such that all objects in $\mathcal{M}$ are cofibrant and let $\mathcal{N}$ be a model category. Suppose that*

$$\begin{tikzcd} \mathcal{N} \arrow[r, bend right, "R"'] \arrow[r, phantom, "\perp"] & \mathcal{M} \arrow[l, bend right, "L"'] \end{tikzcd}$$

is a Quillen pair. Let $\mathcal{S}$ be a set of cofibrations in $\mathcal{M}$ such that $Ls\colon LA \xrightarrow{\sim} LB$ is a weak equivalence in $\mathcal{N}$, for all $s \in \mathcal{S}$. Then the adjunction $L \dashv R$ induces a Quillen pair

$$\begin{tikzcd} \mathcal{N} \arrow[r, bend right, "R"'] \arrow[r, phantom, "\perp"] & L_{\mathcal{S}}\mathcal{M} \arrow[l, bend right, "L"'] \end{tikzcd}$$

between $\mathcal{N}$ and the left Bousfield localization $L_{\mathcal{S}}\mathcal{M}$ of $\mathcal{M}$ at $\mathcal{S}$.

Proof. Since $L\colon \mathcal{M} \to \mathcal{N}$ preserves cofibrations and the cofibrations in $L_{\mathcal{S}}\mathcal{M}$ are the cofibrations in $\mathcal{M}$, it follows that $L\colon L_{\mathcal{S}}\mathcal{M} \to \mathcal{N}$ preserves cofibrations. We show that it also preserves trivial cofibrations. Let $j\colon A \overset{\sim}{\hookrightarrow} B$ be a trivial cofibration in $L_{\mathcal{S}}\mathcal{M}$. Since L preserves cofibrations, then $Lj\colon LA \hookrightarrow LB$ is a cofibration in $\mathcal{N}$. It remains to show that Lj is a weak equivalence in $\mathcal{N}$. Since j is an $\mathcal{S}$-local equivalence in $\mathcal{M}$, then the induced map

$$j^*\colon \mathcal{M}(B,Y) \to \mathcal{M}(A,Y)$$

is a weak equivalence in sSet, for every $\mathcal{S}$-local object $Y \in \mathcal{M}$. By Lemma 5.2.22, the image RX of a fibrant object $X \in \mathcal{N}$ is $\mathcal{S}$-local. Hence the induced map

$$j^*\colon \mathcal{M}(B,RX) \xrightarrow{\sim} \mathcal{M}(A,RX)$$

is a weak equivalence in sSet, for every fibrant object $X \in \mathcal{N}$. Since there is an isomorphism $\mathcal{N}(L(A \otimes \Delta[-]),X) \cong \mathcal{M}(A,RX)$ in sSet natural in $A \in \mathcal{M}$ and $X \in \mathcal{N}$ and by 2-out-of-3, we get that the map

$$L(j \otimes \Delta[-])^*\colon \mathcal{N}(L(B \otimes \Delta[-]),X) \to \mathcal{N}(L(A \otimes \Delta[-]),X)$$

is a weak equivalence in sSet, for every fibrant object $X \in \mathcal{N}$. Finally, by Lemma 5.2.21, this says that $Lj\colon LA \overset{\sim}{\hookrightarrow} LB$ is a weak equivalence in $\mathcal{N}$. Hence L preserves trivial cofibrations and this shows that $L\colon L_{\mathcal{S}}\mathcal{M} \to \mathcal{N}$ is left Quillen. $\square$

PART III.

HOMOTOPY THEORY OF 2-DIMENSIONAL CATEGORIES

In category theory as well as in homotopy theory, we strive to find the correct notion of "sameness". When working with categories themselves, we have already discussed the fact that having an isomorphism between categories is much too strong a requirement, and we would instead concur that the right condition to demand is that of an equivalence of categories. At heart, this is due to the fact that the category Cat of categories and functors can be extended to a 2-category with natural transformations as 2-morphisms and, as we have seen in Definition 2.4.1, there is a weaker version of an invertible morphism in that context. In particular, an equivalence of categories is defined to be a functor which admits an inverse up to natural isomorphisms, and can equivalently be characterized as a functor which is surjective on objects up to isomorphism, and fully faithful on morphisms.

Since these equivalences seem to give a good notion of invertibility for functors between categories, we expect them to model a certain homotopy theory of categories. It is indeed the case since equivalences of categories form a class of weak equivalences in a model structure on Cat, called the *canonical model structure*.

Going one dimension up and focusing on 2-categories, as we have seen in Definition 2.3.3, the 2-functors themselves form a 2-category, with morphisms given by the pseudo-natural transformations and 2-morphisms given by the modifications. We can then define a higher version of an equivalence of categories, by saying that a 2-functor is a *biequivalence* if it admits an inverse up to pseudo-natural equivalences (see Definition 2.4.6), which correspond to equivalences in the pseudo-hom 2-category. Note that, in this case, the inverse functor might not be a 2-functor anymore, but rather a *pseudo-functor*. By a Whitehead theorem for 2-categories, and in analogy with the 1-dimensional case, these biequivalences can be characterized as the 2-functors which are surjective on objects up to equivalence, full on morphisms up to 2-isomorphism, and fully faithful on 2-morphisms.

As is the case of equivalences of categories, biequivalences form a class of weak equivalences for a model structure on the category 2Cat of 2-categories and 2-functors, constructed by Lack in [Lac02, Lac04]. In particular, the inclusion of Cat into 2Cat gives a homotopically full embedding from the canonical model structure on Cat into Lack's model structure on 2Cat.

By considering the relations between 2-categories and double categories, namely that every 2-category can be seen as a horizontal double category with only trivial vertical morphisms, we expect to find a homotopy theory of double categories containing that of 2-categories through this horizontal embedding. Constructing such a homotopy theory for double categories is the aim of this part.

The idea of defining a model structure on the category DblCat of double categories and double functors is scarcely a new one. In [FP10, FPP08], Fiore, Paoli, and Pronk construct several model structures on DblCat. However, the horizontal embedding of 2-categories does not induce a Quillen pair between Lack's model structure on 2Cat and any of these model structures on DblCat (see [MSV20a, §8]). Some intuition is provided by the fact that their categorical model structures on DblCat are constructed from the canonical model structure on Cat, and hence the weak equivalences in there induce equivalences of categories between the underlying horizontal (1-)categories, and between the (1-)categories of vertical morphisms and squares. However, a biequivalence of 2-categories does not in general induce an equivalence between underlying categories, which explains partially why the horizontal embedding of 2Cat into DblCat does not preserve weak equivalences with respect to their model structures.

In order to remedy this loss of higher data, we define new model structures on DblCat in joint works with Maru Sarazola and Paula Verdugo [MSV20a, MSV20b], which are compatible with the horizontal embedding of 2-categories into double categories. We first construct in [MSV20a] a model structure on DblCat by upgrading the 1-categorical definitions above to 2-categorical ones. For this, instead of considering the underlying horizontal categories and the categories of vertical morphisms and squares, we consider instead the following 2-categories: the underlying horizontal 2-category $\mathbf{H}\mathbb{A}$ as given in Definition 3.4.3 and the 2-category $\mathcal{V}\mathbb{A}$ as given in Definition 3.4.9 of a double category $\mathbb{A}$. In particular, these 2-categories have the desired underlying categories, and hence extend the definitions above.

We then define a weak equivalence in DblCat to be a double functor F such that the induced 2-functors $\mathbf{H}F$ and $\mathcal{V}F$ are biequivalences; we call them suggestively *double biequivalences*. They can be described analogously to the biequivalences; namely, a double biequivalence is precisely a double functor which is surjective on objects up to horizontal equivalence, full on horizontal morphisms up to vertically invertible square (with trivial vertical boundaries), surjective on vertical morphisms up to weakly horizontally invertible square, and fully faithful on squares. We then construct a model structure on DblCat whose weak equivalences are precisely the double biequivalences. In particular, the horizontal embedding of 2Cat into DblCat both preserves and reflects weak equivalences, and this model structure is actually designed in such a way that it is as homotopically compatible as possible with the horizontal embedding.

Unsurprisingly, this model structure is not well-behaved with respect to the vertical direction, as it is constructed with a pronounced horizontal bias. In particular, it is not compatible with the nerve construction from double categories to double $(\infty,1)$-categories, as we will see in Section 11. It is for instance due to the fact that all objects of this first model structure on DblCat are fibrant, while the double categories whose nerve is fibrant are precisely the weakly horizontally invariant double categories (see Definition 3.6.5 and Theorem 11.4.8). Hence the nerve does not preserve fibrant objects, which should actually be the case. To remedy this issue, we construct in [MSV20b] another model structure on DblCat in which the fibrant objects are precisely the weakly horizontally invariant double categories and the weak equivalences are such that they induce a double biequivalence between fibrant replacements. The existence of such a model structure was also noticed at roughly the same time by Campbell. This new model structure on DblCat is again compatible with the (homotopical) horizontal embedding of 2-categories into double categories, and further provides the right homotopical context to apply the nerve construction.

In Section 6, we first recall the main features of Lack's model structure on 2Cat. In particular, we provide sets of generating (trivial) cofibrations, and show that it is monoidal with respect to the Gray tensor product for 2-categories. Then, in Section 7, we construct a model structure on DblCat by right-inducing it from two copies of Lack's model structure on 2Cat along the right adjoint $(\mathbf{H},\mathcal{V})\colon \mathrm{DblCat}\to 2\mathrm{Cat}\times 2\mathrm{Cat}$. We show that this model structure on DblCat shares some features with Lack's model structure. In particular, it is cofibrantly generated and enriched over 2Cat with respect to the enrichment given in Proposition 3.5.2. Due to the asymmetry between the horizontal and vertical directions of this model structure, it is not monoidal with respect to the Gray tensor product for double categories. Finally, in Section 8, we construct the other model structure on DblCat for weakly horizontally invariant double categories. While we were not able to show that there is a set of generating trivial cofibrations for this model structure, it should also be cofibrantly generated, and this will be studied in forthcoming work. However, unlike the other one, this model structure is monoidal with respect to the Gray tensor product for double categories, and hence also enriched over 2Cat with respect to the enrichment given in Proposition 3.5.2.

6. Lack's model structure for 2-categories

In this section, we recall the main features of the canonical model structure on Cat – the category of categories and functors – and Lack's model structure on 2Cat – the category of 2-categories and 2-functors. As already mentioned in the introduction, the weak equivalences of these model structures are given by the equivalences of categories and the biequivalences of 2-categories, respectively. The fibrations in Cat are the *isofibrations*, and the fibrations in 2Cat are 2-categorical versions of these isofibrations. In particular, all categories and all 2-categories in these model structures are fibrant.

In Section 6.1, we first introduce these model structures by describing their weak equivalences and fibrations, and show that the discrete embedding $D\colon \mathrm{Cat} \to 2\mathrm{Cat}$ is homotopically full. We further show that the canonical model structure on Cat is right-induced from Lack's model structure on 2Cat along D. Then, in Section 6.2, we turn our attention to the cofibrations in Lack's model structure on 2Cat. In particular, we show that the cofibrant objects are precisely the 2-categories whose underlying category is free. We further provide sets of generating (trivial) cofibrations for this model structure, by studying the lifting properties of the (trivial) fibrations described in Section 6.1. Finally, in Section 6.3, we show that Lack's model structure on 2Cat is not monoidal with respect to the cartesian product, but that it is so with respect to the Gray tensor product for 2-categories.

The results here are based on [Lac02, Lac04].

6.1. **Lack's model structure.** Before introducing the model structure for 2-categories constructed by Lack in [Lac02, §3] and [Lac04, §2], we recall the canonical model structure on the category Cat of categories and functors. The weak equivalences in this model structure are given by the equivalences of categories, which correspond to the equivalences in the 2-category of categories, functors, and natural transformations. These can equivalently be defined as follows (see e.g. [Rez00, Proposition 1.1]).

Definition 6.1.1. Let $\mathcal{C}$ and $\mathcal{D}$ be categories. A functor $F\colon \mathcal{C} \to \mathcal{D}$ is an **equivalence of categories** if

- (eq1) for every object $D \in \mathcal{D}$, there is an object $C \in \mathcal{C}$ together with an isomorphism $D \xrightarrow{\cong} FC$ in $\mathcal{D}$,
- (eq2) for every pair of objects $C, E \in \mathcal{C}$ and every morphism $d\colon FC \to FE$ in $\mathcal{D}$, there is a unique morphism $c\colon C \to E$ in $\mathcal{C}$ such that $d = Fc$.

The fibrations in the canonical model structure are given by the *isofibrations*, which are the functors such that every isomorphism to an object in the image can be lifted to an isomorphism in the source category.

Definition 6.1.2. Let $\mathcal{C}$ and $\mathcal{D}$ be categories. A functor $F\colon \mathcal{C} \to \mathcal{D}$ is an **isofibration** if, for every object $E \in \mathcal{C}$ and every isomorphism $d\colon D \xrightarrow{\cong} FE$ in $\mathcal{D}$, there is an isomorphism $c\colon C \xrightarrow{\cong} E$ in $\mathcal{C}$ such that $d = Fc$.

These two classes of functors just defined determine a model structure on Cat, called the *canonical model structure*.

Theorem 6.1.3. *There is a model structure on* Cat *in which the weak equivalences are the equivalences of categories and the fibrations are the isofibrations.*

Proof. A proof can be found in [JT91, Theorem 3.4] or [Rez00, Theorem 3.1]. □

Remark 6.1.4. Note that every category $\mathcal{C}$ is fibrant, since every functor $\mathcal{C} \to [0]$ is trivially an isofibration. Moreover, a trivial fibration in the canonical model structure on Cat is precisely a functor which is surjective on objects and fully faithful on morphisms.

Remark 6.1.5. The functors which have the left lifting property with respect to functors that are surjective on objects and fully faithful on morphisms are precisely the functors which are injective on objects. This gives a characterization of the cofibrations in the canonical model structure on Cat. Furthermore, every category $\mathcal{C}$ is cofibrant, since $\emptyset \to \mathcal{C}$ is trivially injective on objects.

A 2-categorical analogue of an equivalence of categories is given by the notion of *biequivalences*. These are defined as the 2-functors which are surjective on objects up to equivalence – the higher categorical version of invertibility for a morphism – and which induce an equivalence between hom-categories.

Definition 6.1.6. Let $\mathcal{A}$ and $\mathcal{B}$ be 2-categories. A 2-functor $F\colon \mathcal{A} \to \mathcal{B}$ is a **biequivalence** if

(b1) for every object $B \in \mathcal{B}$, there is an object $A \in \mathcal{A}$ together with an equivalence $B \xrightarrow{\simeq} FA$ in $\mathcal{B}$,
(b2) for every pair of objects $A, C \in \mathcal{A}$ and every morphism $b\colon FA \to FC$ in $\mathcal{B}$, there is a morphism $a\colon A \to C$ in $\mathcal{A}$ together with a 2-isomorphism $b \cong Fa$ in $\mathcal{B}$,
(b3) for every pair of morphisms $a, c\colon A \to C$ in $\mathcal{A}$ and every 2-morphism $\beta\colon Fa \Rightarrow Fc$ in $\mathcal{B}$, there is a unique 2-morphism $\alpha\colon a \Rightarrow c$ in $\mathcal{A}$ such that $\beta = F\alpha$.

Similarly, a 2-categorical version of an isofibration is a 2-functor which is such that every equivalence to an object in the image can be lifted to an equivalence in the source 2-category, and which induces an isofibration on hom-categories.

Definition 6.1.7. Let $\mathcal{A}$ and $\mathcal{B}$ be 2-categories. A 2-functor $F\colon \mathcal{A} \to \mathcal{B}$ is a **Lack fibration** if

(f1) for every object $C \in \mathcal{A}$ and every equivalence $b\colon B \xrightarrow{\simeq} FC$ in $\mathcal{B}$, there is an equivalence $a\colon A \xrightarrow{\simeq} C$ in $\mathcal{A}$ such that $b = Fa$,
(f2) for every morphism $c\colon A \to C$ in $\mathcal{A}$ and every 2-isomorphism $\beta\colon b \cong Fc$ in $\mathcal{B}$, there is a 2-isomorphism $\alpha\colon a \cong c$ in $\mathcal{A}$ such that $\beta = F\alpha$.

These classes of 2-functors determine a model structure on 2Cat, constructed by Lack.

Theorem 6.1.8. *There is a model structure on* 2Cat *in which the weak equivalences are the biequivalences and the fibrations are the Lack fibrations.*

Proof. This appears as [Lac02, Theorem 3.3] or [Lac04, Theorem 4]. □

Remark 6.1.9. Note that every 2-category $\mathcal{A}$ is fibrant in 2Cat, as every 2-functor $\mathcal{A} \to [0]$ trivially satisfies (f1-2).

We now show that the trivial fibrations in this model structure are also a 2-categorical analogue of the trivial fibrations in the canonical model structure in Cat. Namely, we show that they are precisely the 2-functors which are surjective on objects and which induce a trivial fibration between hom-categories. For this, we need the following result saying that a 2-functor which induces a trivial fibration between hom-categories reflects equivalences and 2-isomorphisms.

Lemma 6.1.10. *Let $\mathcal{A}$ and $\mathcal{B}$ be 2-categories, and let $F\colon \mathcal{A} \to \mathcal{B}$ be a 2-functor.*

(i) Suppose that F is fully faithful on 2-morphisms. Then, if $\alpha\colon a \Rightarrow c$ is a 2-morphism in $\mathcal{A}$ such that $F\alpha\colon Fa \cong Fc$ is a 2-isomorphism in $\mathcal{B}$, then α is a 2-isomorphism in $\mathcal{A}$.

(ii) Suppose that F is full on morphisms and fully faithful on 2-morphisms. Then, if $a\colon A \to C$ is a morphism in $\mathcal{A}$ such that $Fa\colon FA \xrightarrow{\simeq} FC$ is an equivalence in $\mathcal{B}$, then a is an equivalence in $\mathcal{A}$.

Proof. We first prove (i). Since $F\alpha\colon Fa \cong Fc$ is a 2-isomorphism, it has an inverse $(F\alpha)^{-1}\colon Fc \Rightarrow Fa$. By fully faithfulness of F on 2-morphisms, there is a unique 2-morphism $\gamma\colon c \Rightarrow a$ in $\mathcal{A}$ such that $(F\alpha)^{-1} = F\gamma$. Since we have that

$$F(\gamma\alpha) = (F\gamma)(F\alpha) = (F\alpha)^{-1}F(\alpha) = \mathrm{id}_{Fa} = F(\mathrm{id}_a),$$

by fully faithfulness on 2-morphisms, we get that $\gamma\alpha = \mathrm{id}_a$. Similarly, we can show that $\alpha\gamma = \mathrm{id}_c$, which shows that γ is an inverse of α. Hence α is a 2-isomorphism in $\mathcal{A}$.

We now prove (ii). Since $Fa\colon FA \xrightarrow{\simeq} FC$ is an equivalence in $\mathcal{B}$, there is a morphism $b\colon FC \to FA$ in $\mathcal{B}$ together with 2-isomorphisms $\eta\colon \mathrm{id}_{FA} \cong b(Fa)$ and $\epsilon\colon (Fa)b \cong \mathrm{id}_{FC}$. By fullness of F on morphisms, there is a morphism $c\colon C \to A$ in $\mathcal{A}$ such that $b = Fc$. Then $\eta\colon F(\mathrm{id}_A) = \mathrm{id}_{FA} \Rightarrow b(Fa) = (Fc)(Fa) = F(ca)$ and, by fully faithfulness of F on 2-morphisms, there is a unique 2-morphism $\overline{\eta}\colon \mathrm{id}_A \Rightarrow ca$ in $\mathcal{A}$ such that $\eta = F(\overline{\eta})$. Moreover, $\overline{\eta}$ is a 2-isomorphism by (i), as η is so. Similarly, there is a unique 2-isomorphism $\overline{\epsilon}\colon ac \cong \mathrm{id}_C$ such that $\epsilon = F\overline{\epsilon}$. This shows that a is an equivalence in $\mathcal{A}$. □

Proposition 6.1.11. *Let $\mathcal{A}$ and $\mathcal{B}$ be 2-categories. A 2-functor $F\colon \mathcal{A} \to \mathcal{B}$ is a trivial fibration in the model structure on* 2Cat *of Theorem 6.1.8 if and only if it is surjective on objects, full on morphisms, and fully faithful on 2-morphisms.*

Proof. Suppose that F is surjective on objects, full on morphisms, and fully faithful on 2-morphisms. Then it satisfies trivially (b1-3) of Definition 6.1.6. We show that F satisfies (f1-2) of Definition 6.1.7. Let $b\colon B \xrightarrow{\simeq} FC$ be an equivalence in $\mathcal{B}$. By surjectivity of F on objects, there is an object $A \in \mathcal{A}$ such that $B = FA$. Then, by fullness of F on morphisms, there is a morphism $a\colon A \to C$ such that $b = Fa$. Since F is full on morphisms and fully faithful on 2-morphisms, it follows from Lemma 6.1.10 (ii) that a is an equivalence in $\mathcal{A}$, which proves (f1). Now, let $c\colon A \to C$ be a morphism in $\mathcal{A}$ and $\beta\colon b \cong Fc$ be a 2-isomorphism in $\mathcal{B}$. By fullness of F on morphisms, there is a morphism $a\colon A \to C$ such that $b = Fa$. Then $\beta\colon Fa = b \cong Fc$, and by fully faithfulness of F on 2-morphisms, there is a unique 2-morphism $\alpha\colon a \Rightarrow c$ in $\mathcal{A}$ such that $\beta = F\alpha$. By Lemma 6.1.10 (i), we have that α is a 2-isomorphism, which proves (f2).

Now suppose that F is a trivial fibration, i.e., it is both a biequivalence and a Lack fibration. Let $B \in \mathcal{B}$ be an object. By (b1), there is an object $C \in \mathcal{A}$ together with an equivalence $b\colon B \xrightarrow{\simeq} FC$ in $\mathcal{B}$, and, by (f1), there is an equivalence $a\colon A \xrightarrow{\simeq} C$ in $\mathcal{A}$ such that $b = Fa$. In particular, we have $B = FA$, so that F is surjective on objects. Now, let $b\colon FA \to FC$ be a morphism in $\mathcal{B}$. By (b2), there is a morphism $a\colon A \to C$ in $\mathcal{A}$ together with a 2-isomorphism $\beta\colon b \cong Fc$ in $\mathcal{B}$, and, by (f2), there is a 2-isomorphism $\alpha\colon a \cong c$ in $\mathcal{A}$ such that $\beta = F\alpha$. In particular, we have $b = Fa$, so that F is full on morphisms. Fully faithfulness on 2-morphisms is precisely (b3). □

Finally, we show that there is a Quillen reflection (see Definition 4.4.8) between the canonical model structure on Cat and Lack's model structure on 2Cat. This is given by the adjunction $P \dashv D$ of Proposition 2.1.13, where we recall that $D\colon \mathrm{Cat} \to 2\mathrm{Cat}$ is the functor which sends a category to its associated locally discrete 2-category, and $P\colon 2\mathrm{Cat} \to \mathrm{Cat}$ is obtained by taking the path components of the hom-categories of a 2-category.

Remark 6.1.12. Given a category $\mathcal{C}$, note that an equivalence in the 2-category $D\mathcal{C}$ is precisely an isomorphism in the category $\mathcal{C}$, since $D\mathcal{C}$ has only trivial 2-morphisms.

Furthermore, given a 2-category $\mathcal{A}$, the path component of an equivalence in $\mathcal{A}$ is an isomorphism in $P\mathcal{A}$. Indeed, given an equivalence $a\colon A \xrightarrow{\simeq} C$ in $\mathcal{A}$, there is a morphism $c\colon C \to A$ together with 2-isomorphisms $ca \cong \mathrm{id}_A$ and $\mathrm{id}_B \cong ac$. Therefore, in the category $P\mathcal{A}$, we get that $[c][a] = [ca] = [\mathrm{id}_A] = \mathrm{id}_A$ and similarly that $[a][c] = \mathrm{id}_B$. This shows that $[a]$ is an isomorphism in $P\mathcal{A}$.

Before showing the result, we show that the left adjoint P preserves weak equivalences.

Lemma 6.1.13. *The functor $P\colon 2\mathrm{Cat} \to \mathrm{Cat}$ sends biequivalences to equivalences of categories.*

Proof. Let $F\colon \mathcal{A} \to \mathcal{B}$ be a biequivalence. We show that $PF\colon P\mathcal{A} \to P\mathcal{B}$ satisfies (eq1-2) of Definition 6.1.1. Let $B \in P\mathcal{B}$ be an object. By (b1) of Definition 6.1.6, there is an object $A \in \mathcal{A}$ together with an equivalence $b\colon B \xrightarrow{\simeq} FA$ in $\mathcal{B}$. By Remark 6.1.12, this gives an object $A \in P\mathcal{A}$ and an isomorphism $[b]\colon B \xrightarrow{\cong} FA$ in $P\mathcal{B}$, which shows (eq1). Now, let A and C be objects in $P\mathcal{A}$ and $[b]\colon FA \to FC$ be a morphism in $P\mathcal{B}$. By (b2) of Definition 6.1.6, there is a morphism $a\colon A \to C$ in $\mathcal{A}$ together with a 2-isomorphism $b \cong Fa$ in $\mathcal{B}$. In particular, this says that the morphism $[a]\colon A \to C$ in $P\mathcal{A}$ is such that $[b] = F[a]$, which shows that PF is full on morphisms. Moreover, if $c\colon A \to C$ is another morphism in $\mathcal{A}$ such that $F[c] = [b] = F[a]$, then there is a 2-morphism $\beta\colon Fa \Rightarrow Fc$ in $\mathcal{B}$ by definition of $P\mathcal{B}$. By (b3) of Definition 6.1.6, there is a unique 2-morphism $\alpha\colon a \Rightarrow c$ in $\mathcal{A}$ such that $F\alpha = \beta$. This shows that $[a] = [c]$ in $P\mathcal{A}$ and hence proves (eq2). $\square$

The following result, which appears as [Lac02, Theorem 8.2], implies that the homotopy category of categories is embedded into that of 2-categories in a reflective way.

Theorem 6.1.14. *The adjunction*

$$\begin{tikzcd} \mathrm{Cat} \arrow[r, phantom, "\perp"] \arrow[r, bend right, "D"'] & 2\mathrm{Cat} \arrow[l, bend right, "P"'] \end{tikzcd}$$

is a Quillen reflection, where Cat *is endowed with the model structure of Theorem 6.1.3 and* 2Cat *is endowed with the model structure of Theorem 6.1.8.*

Proof. We show that the right adjoint $D\colon \mathrm{Cat} \to 2\mathrm{Cat}$ preserves fibrations and trivial fibrations. Let $F\colon \mathcal{C} \to \mathcal{D}$ be an isofibration. We show that the 2-functor $DF\colon D\mathcal{C} \to D\mathcal{D}$ satisfies (f1-2) of Definition 6.1.7. Let E be an object in $D\mathcal{C}$ and $d\colon D \xrightarrow{\simeq} FE$ be an equivalence in $D\mathcal{D}$. By Remark 6.1.12, the equivalence d is an isomorphism $d\colon D \xrightarrow{\cong} FE$ in $\mathcal{D}$. Since F is an isofibration, there is an isomorphism $c\colon C \xrightarrow{\cong} E$ in $\mathcal{C}$ such that $d = Fc$, which shows (f1). The 2-functor DF trivially satisfies (f2) since all 2-morphisms in $D\mathcal{D}$ are trivial. Hence DF is a fibration in 2Cat. Now let $F\colon \mathcal{C} \to \mathcal{D}$ be a trivial fibration in Cat, i.e., it is surjective on objects and fully faithful on morphisms. Then $DF\colon D\mathcal{C} \to D\mathcal{D}$ is clearly surjective on objects and full on morphisms. It is further fully faithful on 2-morphisms, since F is fully faithful on morphisms and all 2-morphisms in $D\mathcal{C}$ and $D\mathcal{D}$ are trivial. This shows that DF is a trivial fibration in 2Cat. Therefore, the functor $D\colon \mathrm{Cat} \to 2\mathrm{Cat}$ is right Quillen.

It remains to show that the derived counit is level-wise an equivalence of categories. Let $\mathcal{C}$ be a category. First note that $PD\mathcal{C} = \mathcal{C}$, so that the counit $\epsilon_{\mathcal{C}}$ is given by the identity at $\mathcal{C}$. Now let $q_{D\mathcal{C}}\colon (D\mathcal{C})^c \to D\mathcal{C}$ denote a cofibrant replacement of $D\mathcal{C}$ in 2Cat. Since P preserves weak equivalences by Lemma 6.1.13, we have that $P(q_{D\mathcal{C}})\colon P(D\mathcal{C})^c \to PD\mathcal{C} = \mathcal{C}$ is an equivalence of categories. But this is precisely the component of the derived counit at $\mathcal{C}$. This shows that $P \dashv D$ is a Quillen reflection. $\square$

Remark 6.1.15. The components of the derived counit of the adjunction $P \dashv D$ are not biequivalences in general. To see this, let $\Sigma[1]$ be the 2-category free on a 2-morphism (see Notation 2.1.8). Then $\Sigma[1]$ is cofibrant, since its underlying category is free (see Corollary 6.2.4 below). We have that $DP\Sigma[1] = [1]$ is the 2-category free on a morphism. Then the unit $\eta_{\Sigma[1]}\colon \Sigma[1] \to DP\Sigma[1] = [1]$, which is also the derived counit as all objects in Cat are fibrant, is not a biequivalence, as it identifies the two non trivial morphisms of $\Sigma[1]$ which are not related by a 2-isomorphism and therefore does not satisfy (b3) of Definition 6.1.6.

Theorem 6.1.16. *The model structure on* Cat *of Theorem 6.1.3 is right-induced along the adjunction*

$$\begin{tikzcd} \mathrm{Cat} \arrow[r, bend right, "D"'] \arrow[r, phantom, "\perp"] & 2\mathrm{Cat} \arrow[l, bend right, "P"'] \end{tikzcd}$$

from the model structure on 2Cat *of Theorem 6.1.8.*

Proof. We need to show that a functor $F\colon \mathcal{C} \to \mathcal{D}$ is an equivalence (resp. isofibration) in Cat if and only if $DF\colon D\mathcal{C} \to D\mathcal{D}$ is a biequivalence (resp. Lack fibration) in 2Cat. Since the functor $D\colon \mathrm{Cat} \to 2\mathrm{Cat}$ is right Quillen, it preserves fibrations and, since all objects in Cat are fibrant, it also preserves weak equivalences by Corollary 4.4.7. This shows that, if F is an equivalence (resp. isofibration), then DF is a biequivalence (resp. Lack fibration).

Now suppose that DF is a biequivalence. Since $P\colon 2\mathrm{Cat} \to \mathrm{Cat}$ sends biequivalences to equivalences by Lemma 6.1.13 and $PD = \mathrm{id}_{\mathrm{Cat}}$, then $PDF = F$ is an equivalence.

Finally, suppose that DF is a Lack fibration. We show that F is an isofibration. Let $E \in \mathcal{C}$ be an object and $d\colon D \xrightarrow{\cong} FE$ be an isomorphism in $\mathcal{D}$. Then the isomorphism $d\colon D \xrightarrow{\cong} FE$ is in particular an equivalence in $D\mathcal{D}$. By (f1) of Definition 6.1.7 for DF, there is an equivalence $c\colon C \xrightarrow{\simeq} E$ in $D\mathcal{C}$ such that $d = (DF)c$. By Remark 6.1.12, this corresponds to an isomorphism $c\colon C \xrightarrow{\cong} E$ in $\mathcal{C}$ such that $d = Fc$, which shows that F is an isofibration. $\square$

6.2. **Cofibrations, cofibrant objects, and generating sets.** While all categories are cofibrant in the canonical model structure on Cat, not every 2-category is cofibrant. However, cofibrations and cofibrant objects in Lack's model structure on 2Cat can be characterized in terms of their underlying functor and their underlying category, respectively. The characterizations of this section come from [Lac02, §4], and the generating sets of cofibrations and trivial cofibrations given in Notation 6.2.5 from [Lac02, §3] and [Lac04, §2].

Recall the functor $U\colon 2\mathrm{Cat} \to \mathrm{Cat}$ from Definition 2.1.11 which sends a 2-category $\mathcal{A}$ to its underlying category $U\mathcal{A}$.

Remark 6.2.1. This functor $U\colon 2\mathrm{Cat} \to \mathrm{Cat}$ has a right adjoint $R\colon \mathrm{Cat} \to 2\mathrm{Cat}$, which sends a category $\mathcal{C}$ to the 2-category $R\mathcal{C}$ with the same objects and morphisms as $\mathcal{C}$, and with a unique 2-morphism between each pair of parallel morphisms. A functor $F\colon \mathcal{C} \to \mathcal{D}$ is sent to the 2-functor $RF\colon R\mathcal{C} \to R\mathcal{D}$ which acts as F on objects and morphisms, and sends the unique 2-morphism $!\colon f \Rightarrow g$ in $R\mathcal{C}$ to the unique 2-morphism $!\colon Ff \Rightarrow Fg$ in $R\mathcal{D}$, for all morphisms f, g in $\mathcal{C}$. Note that the functor RF is fully faithful on 2-morphims, as there is a unique 2-morphism between each pair of parallel morphisms in $R\mathcal{C}$ and in $R\mathcal{D}$.

Theorem 6.2.2. *Let $\mathcal{A}$ and $\mathcal{B}$ be 2-categories. A 2-functor $F\colon \mathcal{A} \to \mathcal{B}$ is a cofibration in the model structure on* 2Cat *of Theorem 6.1.8 if and only if its underlying functor*

$UF\colon U\mathcal{A}\to U\mathcal{B}$ has the left lifting property with respect to all surjective on objects and full functors.

Proof. Suppose that $F\colon \mathcal{A}\to\mathcal{B}$ is a cofibration in 2Cat. Let $G\colon \mathcal{C}\to\mathcal{D}$ be a surjective on objects and full functor in Cat. We need to show that there is a lift in every commutative diagram as below left.

$$\begin{tikzcd} U\mathcal{A} \arrow[r] \arrow[d, "UF"'] & \mathcal{C} \arrow[d, "G"] \\ U\mathcal{B} \arrow[r] \arrow[ur, dashed] & \mathcal{D} \end{tikzcd} \qquad \begin{tikzcd} \mathcal{A} \arrow[r] \arrow[d, "F"'] & R\mathcal{C} \arrow[d, "RG"] \\ \mathcal{B} \arrow[r] \arrow[ur, dashed] & R\mathcal{D} \end{tikzcd}$$

By the adjunction $U \dashv R$, such a lift exists if and only if there is a lift in the above right commutative diagram. This is indeed the case since $RG\colon R\mathcal{C}\to R\mathcal{D}$ is a trivial fibration in 2Cat: it is surjective on objects and full on morphisms since G is so, and fully faithful on 2-morphisms by Remark 6.2.1; see Proposition 6.1.11. This shows that UF has the left lifting property with respect to every surjective on objects and full functor.

Now suppose that UF has such a lifting property. Let $P\colon \mathcal{X}\to\mathcal{Y}$ be a trivial fibration in 2Cat. We need to show that there is a lift in every commutative diagram as below left.

$$\begin{tikzcd} \mathcal{A} \arrow[r] \arrow[d, "F"'] & \mathcal{X} \arrow[d, "P"] \\ \mathcal{B} \arrow[r, "Q"'] \arrow[ur, dashed, "L"] & \mathcal{Y} \end{tikzcd} \qquad \begin{tikzcd} U\mathcal{A} \arrow[r] \arrow[d, "UF"'] & U\mathcal{X} \arrow[d, "UP"] \\ U\mathcal{B} \arrow[r, "UQ"'] \arrow[ur, dashed, "UL"] & U\mathcal{Y} \end{tikzcd}$$

By applying U to this diagram, we get a commutative diagram as above right in which there is a lift $UL\colon U\mathcal{B}\to U\mathcal{X}$, since $UP\colon U\mathcal{X}\to U\mathcal{Y}$ is surjective on objects and full on morphisms as P is so by Proposition 6.1.11. This defines the 2-functor $L\colon \mathcal{B}\to\mathcal{X}$ on objects and morphisms. Given a 2-morphism $\beta\colon b\Rightarrow d$ in $\mathcal{B}$, by fully faithfulness on 2-morphisms of P (see Proposition 6.1.11), there is a unique 2-morphism $\chi\colon Lb\Rightarrow Ld$ in $\mathcal{X}$ such that $Q\beta = P\chi\colon Qb = PLb \Rightarrow Qd = PLd$. We set $L\beta := \chi$ and this gives a well-defined 2-functor making the diagram above left commute. Hence F is a cofibration in 2Cat. □

Corollary 6.2.3. *Let $\mathcal{A}$ and $\mathcal{B}$ be 2-categories. A 2-functor $F\colon\mathcal{A}\to\mathcal{B}$ is a cofibration in the model structure on* 2Cat *of Theorem 6.1.8 if and only if*

(i) it is injective on objects and faithful on morphisms,
(ii) the underlying category $U\mathcal{B}$ is a retract of a category obtained from the image of $U\mathcal{A}$ under UF by freely adjoining objects and then morphisms between objects.

Proof. There is a cofibrantly generated weak factorization system $(\mathcal{L},\mathcal{R})$ on Cat, where the right class $\mathcal{R}$ contains all surjective on objects and full functors. A generating set of morphisms is given by the set containing the unique functor $\emptyset\to[0]$ and the inclusion functor $[0]\sqcup[0]\to[1]$. Then the left class $\mathcal{L}$ contains all functors $G\colon\mathcal{C}\to\mathcal{D}$ which are injective on objects and faithful on morphisms, and such that $\mathcal{D}$ is a retract of a category obtained from the image of $\mathcal{C}$ under G by freely adjoining objects and then morphisms between objects. Then the result follows from Theorem 6.2.2 and the above characterization of functors in Cat which have the left lifting property with respect to the surjective on objects and full functors. □

By applying this result to a 2-functor whose source is the initial object $\emptyset$ in 2Cat, we get the following characterization of cofibrant 2-categories.

Corollary 6.2.4. *A* 2*-category $\mathcal{A}$ is cofibrant in the model structure on* 2Cat *of Theorem 6.1.8 if and only if its underlying category $U\mathcal{A}$ is free.*

Proof. By Corollary 6.2.3, the 2-functor $\emptyset \to \mathcal{A}$ is a cofibration if and only if the category $U\mathcal{A}$ is a retract of a free category. However, a retract of a free category is itself free. □

Finally, we give sets of generating cofibrations, and generating trivial cofibrations for Lack's model structure on 2Cat. For this, recall the construction Σ from Notation 2.1.8.

Notation 6.2.5. Let $\mathcal{I}_2$ denote the set containing the following 2-functors:

(i) the unique 2-functor $i_1 \colon \emptyset \to [0]$,
(ii) the inclusion 2-functor $i_2 \colon [0] \sqcup [0] \to [1]$,
(iii) the inclusion 2-functor $i_3 \colon \delta C \to C$, where $C = \Sigma[1]$ is the free 2-category on a 2-morphism, and $\delta C = \Sigma([0] \sqcup [0])$ is its sub-2-category containing the boundary of the 2-morphism, i.e., it is free on two parallel morphisms,
(iv) the 2-functor $i_4 \colon C_2 \to C$ sending the two non trivial 2-morphisms of C_2 to the non trivial 2-morphism of C, where $C_2 = \Sigma(\{0 \rightrightarrows 1\})$ is the free 2-category on two parallel 2-morphisms.

Let $\mathcal{J}_2$ denote the set containing the following 2-functors:

(i) the inclusion 2-functor $j_1 \colon [0] \to E_{\mathrm{adj}}$, where the 2-category E_{adj} is the "free-living adjoint equivalence",
(ii) the inclusion 2-functor $j_2 \colon [1] \to C_{\mathrm{inv}}$, where the 2-category $C_{\mathrm{inv}} = \Sigma I$ is the "free-living 2-isomorphism", where the category $I = \{0 \cong 1\}$ denotes the "free-living isomorphism".

Theorem 6.2.6. *The model structure on* 2Cat *of Theorem 6.1.8 is cofibrantly generated, and sets of generating cofibrations and generating trivial cofibrations can be given by the sets $\mathcal{I}_2$ and $\mathcal{J}_2$, respectively.*

Proof. Using the characterization of cofibrations of Corollary 6.2.3, it is straightforward to see that the 2-functors in $\mathcal{I}_2$ are cofibrations in 2Cat. A direct computation shows that a 2-functor has the right lifting property with respect to each 2-functor in $\mathcal{I}_2$ precisely when it is surjective on objects, full on morphisms, and fully faithful on 2-morphisms. By Proposition 6.1.11, such a 2-functor is precisely a trivial fibration in 2Cat.

Similarly, one can check that the 2-functors in $\mathcal{J}_2$ are trivial cofibrations in 2Cat and that a 2-functor has the right lifting property with respect to each 2-functor in $\mathcal{J}_2$ precisely when it satisfies (f1-2) of Definition 6.1.7, i.e., it is a Lack fibration. □

6.3. **Monoidality.** We now study the monoidality of Lack's model structure on 2Cat, following [Lac02, §7]. While it is not monoidal with respect to the cartesian product, it is monoidal with respect to the Gray tensor product.

Remark 6.3.1. The model structure on 2Cat of Theorem 6.1.8 is not monoidal with respect to the cartesian product. Indeed, given the generating cofibration $i_2 \colon [0] \sqcup [0] \to [1]$, then the pushout-product

$$i_2 \,\square_\times\, i_2 \colon [1] \sqcup [1] \coprod_{[0]\sqcup[0]\sqcup[0]\sqcup[0]} [1] \sqcup [1] \to [1] \times [1]$$

is not a cofibration. To see this, the domain of $i_2 \,\square_\times\, i_2$ can be described as the 2-category freely generated by four morphisms $(f,0) \colon (0,0) \to (1,0)$, $(f,1) \colon (0,1) \to (1,1)$, $(0,f) \colon (0,0) \to (0,1)$, and $(1,f) \colon (1,0) \to (1,1)$, while the codomain of $i_2 \,\square_\times\, i_2$ is the 2-category containing the same four morphisms subject to the following relation: $(1,f)(f,0) = (f,1)(0,f)$. Therefore, the pushout-product $i_2 \,\square_\times\, i_2$ is not faithful on morphisms, as it sends the two distinct composites $(1,f)(f,0)$ and $(f,1)(0,f)$ to the same morphism in $[1] \times [1]$. This shows that $i_2 \,\square_\times\, i_2$ is not a cofibration in 2Cat by Corollary 6.2.3.

To show that Lack's model structure on 2Cat is monoidal with respect to the Gray tensor product $\otimes_2$, as defined in Proposition 2.3.4, we first describe the data of the Gray tensor product of two 2-categories, in order to compare it with the cartesian product.

Description 6.3.2. Let $\mathcal{A}$ and $\mathcal{X}$ be 2-categories. Their Gray tensor product $\mathcal{A} \otimes_2 \mathcal{X}$ is the 2-category described by the following data:

(i) an object in $\mathcal{A} \otimes_2 \mathcal{X}$ is a pair (A, X) of an object $A \in \mathcal{A}$ and an object $X \in \mathcal{X}$,

(ii) morphisms in $\mathcal{A} \otimes_2 \mathcal{X}$ are generated by the following morphisms:
- a morphism $(a, X)\colon (A, X) \to (C, X)$, for each pair (a, X) of a morphism $a\colon A \to C$ in $\mathcal{A}$ and an object $X \in \mathcal{X}$, and
- a morphism $(A, x)\colon (A, X) \to (A, Z)$, for each pair (A, x) of an object $A \in \mathcal{A}$ and a morphism $x\colon X \to Z$ in $\mathcal{X}$,

(iii) 2-morphisms in $\mathcal{A} \otimes_2 \mathcal{X}$ are generated by the following 2-morphisms:
- a 2-morphism $(\alpha, X)\colon (a, X) \Rightarrow (c, X)$, for each pair (α, X) of a 2-morphism $\alpha\colon a \Rightarrow c$ in $\mathcal{A}$ and an object $X \in \mathcal{X}$,
- a 2-morphism $(A, \chi)\colon (A, x) \Rightarrow (A, z)$, for each pair (A, χ) of an object $A \in \mathcal{A}$ and a 2-morphism $\chi\colon x \Rightarrow z$ in $\mathcal{X}$, and
- a 2-isomorphism $(a, x)\colon (C, x)(a, X) \cong (a, Z)(A, x)$, for each pair (a, x) of a morphism $a\colon A \to C$ in $\mathcal{A}$ and a morphism $x\colon X \to Z$ in $\mathcal{X}$,

subject to conditions which are equivalent to requiring that the below 2-functor $\pi_{\mathcal{A},\mathcal{X}}\colon \mathcal{A} \otimes_2 \mathcal{X} \to \mathcal{A} \times \mathcal{X}$ is fully faithful on 2-morphisms.

There is a 2-functor $\pi_{\mathcal{A},\mathcal{X}}\colon \mathcal{A} \otimes_2 \mathcal{X} \to \mathcal{A} \times \mathcal{X}$, which is the identity on objects, sends the generating morphisms (a, X) and (A, x) of $\mathcal{A} \otimes_2 \mathcal{X}$ to the morphisms (a, id_X) and (id_A, x) in $\mathcal{A} \times \mathcal{X}$, respectively, and sends the generating 2-morphisms (α, X), (A, χ), and (a, x) of $\mathcal{A} \otimes_2 \mathcal{X}$ to the 2-morphisms $(\alpha, \mathrm{id}_{\mathrm{id}_X})$, $(\mathrm{id}_{\mathrm{id}_A}, \chi)$, and $\mathrm{id}_{(a,x)}$ in $\mathcal{A} \times \mathcal{X}$, respectively.

Lemma 6.3.3. *Let $\mathcal{A}$ and $\mathcal{X}$ be 2-categories. Then the 2-functor $\pi_{\mathcal{A},\mathcal{X}}\colon \mathcal{A} \otimes_2 \mathcal{X} \to \mathcal{A} \times \mathcal{X}$ is a trivial fibration in the model structure on* 2Cat *of Theorem 6.1.8.*

Proof. To show that $\pi_{\mathcal{A},\mathcal{X}}$ is a trivial fibration, we use the characterization in Proposition 6.1.11. Since $\pi_{\mathcal{A},\mathcal{X}}$ is the identity on objects, it is clearly surjective on objects. Given a morphism $(a, x)\colon (A, X) \to (C, Z)$ in $\mathcal{A} \times \mathcal{X}$, the composite

$$(A, X) \xrightarrow{(a,X)} (C, X) \xrightarrow{(C,x)} (C, Z)$$

in $\mathcal{A} \otimes_2 \mathcal{X}$ is sent by $\pi_{\mathcal{A},\mathcal{X}}$ to (a, x), which shows that $\pi_{\mathcal{A},\mathcal{X}}$ is full on morphisms. Fully faithfulness on 2-morphisms holds by Description 6.3.2 (iii). □

Using this result, we can prove the following lemma.

Lemma 6.3.4. *Let $\mathcal{A}$ and $\mathcal{B}$ be 2-categories, and $F\colon \mathcal{A} \to \mathcal{B}$ be a biequivalence. Then, for every 2-category $\mathcal{X}$, the induced 2-functors*

$$F \times \mathrm{id}_{\mathcal{X}}\colon \mathcal{A} \times \mathcal{X} \to \mathcal{B} \times \mathcal{X} \quad \text{and} \quad F \otimes_2 \mathrm{id}_{\mathcal{X}}\colon \mathcal{A} \otimes_2 \mathcal{X} \to \mathcal{B} \otimes_2 \mathcal{X}$$

are also biequivalences.

Proof. It is straightforward to see that (b1-3) of Definition 6.1.6 hold for the 2-functor $F \times \mathrm{id}_{\mathcal{X}}\colon \mathcal{A} \times \mathcal{X} \to \mathcal{B} \times \mathcal{X}$ since they hold for F. Therefore $F \times \mathrm{id}_{\mathcal{X}}$ is a biequivalence.

By Lemma 6.3.3, the 2-functors $\pi_{\mathcal{A},\mathcal{X}}\colon \mathcal{A} \otimes_2 \mathcal{X} \to \mathcal{A} \times \mathcal{X}$ and $\pi_{\mathcal{B},\mathcal{X}}\colon \mathcal{B} \otimes_2 \mathcal{X} \to \mathcal{B} \times \mathcal{X}$ are trivial fibrations, and hence they are in particular biequivalences. Since the following diagram commutes

$$\begin{tikzcd}
\mathcal{A}\otimes_2\mathcal{X} \arrow[r, "F\otimes_2 \mathrm{id}_{\mathcal{X}}"] \arrow[d, "\pi_{\mathcal{A},\mathcal{X}}"'] & \mathcal{B}\otimes_2\mathcal{X} \arrow[d, "\pi_{\mathcal{B},\mathcal{X}}"] \\
\mathcal{A}\times\mathcal{X} \arrow[r, "F\times \mathrm{id}_{\mathcal{X}}"'] & \mathcal{B}\times\mathcal{X}
\end{tikzcd},$$

and $F\times\mathrm{id}_{\mathcal{X}}$ is a biequivalence, it follows by 2-out-of-3 that $F\otimes_2\mathrm{id}_{\mathcal{X}}$ is also a biequivalence. □

We are now ready to prove that Lack's model structure on 2Cat is monoidal with respect to the Gray tensor product.

Theorem 6.3.5. *The model structure on* 2Cat *of Theorem 6.1.8 is monoidal with respect to the Gray tensor product* $\otimes_2$*.*

Proof. We first show that the pushout-product $i\,\square_{\otimes_2} j$ is a cofibration in 2Cat, whenever i and j are cofibrations in 2Cat. By Remark 4.5.7, it is enough to show the result when i and j are in the set of generating cofibrations $\mathcal{I}_2=\{i_1,i_2,i_3,i_4\}$ described in Notation 6.2.5. Furthermore, note that $i\,\square_{\otimes_2} j\cong j\,\square_{\otimes_2} i$ since the Gray tensor product is symmetric, and therefore it is enough to show the result for one of the two pushout-products.

Suppose that $i=i_1\colon\emptyset\to[0]$. Since $\emptyset\otimes_2\mathcal{A}\cong\emptyset$ and $[0]\otimes_2\mathcal{A}\cong\mathcal{A}$ for every 2-category $\mathcal{A}$, then $i_1\,\square_{\otimes_2} j\cong j$ and it is a cofibration, for every $j\in\mathcal{I}_2$. Now suppose that i is one of the generating cofibrations $i_3\colon\delta C\to C$ or $i_4\colon C_2\to C$. Then i is an isomorphism on underlying categories. Since U preserves pushouts and the underlying category of the Gray tensor product $\mathcal{A}\otimes_2\mathcal{B}$ only depends on the underlying categories of $\mathcal{A}$ and $\mathcal{B}$, for every pair of 2-categories $\mathcal{A}$ and $\mathcal{B}$, it follows that the functor

$$U(i\,\square_{\otimes_2} j)\colon U(D\otimes_2\mathcal{B})\coprod_{U(D\otimes_2\mathcal{A})}U(C\otimes_2\mathcal{A})\to U(C\otimes_2\mathcal{B})$$

is an isomorphism of categories, where D is either δC or C_2, and $j\colon\mathcal{A}\to\mathcal{B}$ is in $\{i_2,i_3,i_4\}$. It follows from Corollary 6.2.3 that $i\,\square_{\otimes_2} j$ is a cofibration. It remains to show that $i_2\,\square_{\otimes_2} i_2$ is a cofibration, for the generating cofibration $i_2\colon[0]\sqcup[0]\to[1]$. The domain of $i_2\,\square_{\otimes_2} i_2$ can be described as the 2-category freely generated on four morphisms $(f,0)\colon(0,0)\to(1,0)$, $(f,1)\colon(0,1)\to(1,1)$, $(0,f)\colon(0,0)\to(0,1)$, and $(1,f)\colon(1,0)\to(1,1)$. Then $i_2\,\square_{\otimes_2} i_2$ is the inclusion 2-functor of this 2-category into the 2-category with the same underlying category, and an additional 2-isomorphism $(1,f)(f,0)\cong(f,1)(0,f)$ (compare with Remark 6.3.1). Hence the pushout-product $i_2\,\square_{\otimes_2} i_2$ is an isomorphism on underlying categories, and therefore it is a cofibration by Corollary 6.2.3. This shows that $i\,\square_{\otimes_2} j$ is a cofibration in 2Cat whenever i and j are cofibrations in 2Cat.

We now show that the pushout-product $i\,\square_{\otimes_2} j$ is a trivial cofibration in 2Cat, whenever i is a cofibration in 2Cat and $j\colon\mathcal{A}\to\mathcal{B}$ is a trivial cofibration in 2Cat. Again, it is enough to show the result for $i\in\mathcal{I}_2$. Note that all domains of the generating cofibrations in $\mathcal{I}_2$ are cofibrant by Corollary 6.2.4, since they have free underlying categories. Therefore, the generating cofibration $i\in\mathcal{I}_2$ is of the form $i\colon D\to E$ with D cofibrant. By Lemma 6.3.4, the 2-functors $\mathrm{id}_D\otimes_2 j$ and $\mathrm{id}_E\otimes_2 j$ are biequivalences, since j is a biequivalence. Furthermore, since D is cofibrant, j is a cofibration, and $\mathrm{id}_D\otimes_2 j=(\emptyset\to D)\,\square_{\otimes_2} j$, it follows by the first part of the proof that $\mathrm{id}_D\otimes_2 j$ is a cofibration in 2Cat. Consider the following diagram.

$$\begin{tikzcd}
D \otimes_2 \mathcal{A} \arrow[r, hook, "\mathrm{id}_D \otimes_2 j", "\sim"'] \arrow[d, "i \otimes_2 \mathrm{id}_{\mathcal{A}}"'] & D \otimes_2 \mathcal{B} \arrow[d] \arrow[ddr, bend left, "i \otimes_2 \mathrm{id}_{\mathcal{B}}"] & \\
E \otimes_2 \mathcal{A} \arrow[r, hook, "k", "\sim"'] \arrow[drr, bend right, "\sim", "\mathrm{id}_E \otimes_2 j"'] & P \arrow[ul, phantom, "\ulcorner", very near start] \arrow[dr, dashed, "i \,\square_{\otimes_2} j"] & \\
& & E \otimes_2 \mathcal{B}
\end{tikzcd}$$

Since trivial cofibrations are closed under pushouts and $\mathrm{id}_D \otimes_2 j$ is a trivial cofibration by the above discussion, then k is also a trivial cofibration. Then $i \,\square_{\otimes_2} j$ is a biequivalence by 2-out-of-3 applied to $\mathrm{id}_E \otimes_2 j = (i \,\square_{\otimes_2} j)k$. This shows that $i \,\square_{\otimes_2} j$ is a trivial cofibration in 2Cat whenever i and j are cofibrations in 2Cat such that one of i and j is trivial. This concludes the proof. □

7. The first model structure for double categories

In this section, we construct a first model structure on DblCat – the category of double categories and double functors –, in such a way that it is compatible with the horizontal embedding functor $\mathbb{H}\colon 2\mathrm{Cat} \to \mathrm{DblCat}$, which sends a 2-category to its associated horizontal double category. For this, we consider the functor $(\mathbf{H}, \mathcal{V})\colon \mathrm{DblCat} \to 2\mathrm{Cat} \times 2\mathrm{Cat}$, which extracts from a double category $\mathbb{A}$ its underlying horizontal 2-category $\mathbf{H}\mathbb{A}$ (see Definition 3.4.3) and the 2-category $\mathcal{V}\mathbb{A} = \mathbf{H}[\mathbb{V}[1], \mathbb{A}]$ (see Definition 3.4.9) whose objects are the vertical morphisms of $\mathbb{A}$ and whose morphisms are the squares of $\mathbb{A}$. We first show in Section 7.1 that the right-induced model structure on DblCat along the functor $(\mathbf{H}, \mathcal{V})\colon \mathrm{DblCat} \to 2\mathrm{Cat} \times 2\mathrm{Cat}$ exists, where both copies of 2Cat are endowed with Lack's model structure.

In particular, the weak equivalences in this model structure can be characterized as the *double biequivalences*, i.e., the double functors which are surjective on objects up to horizontal equivalence, full on horizontal morphisms up to vertically invertible square (with trivial vertical boundaries), surjective on vertical morphisms up to weakly horizontally invertible square, and fully faithful on squares (see Definition 7.2.1). In Section 7.2, we show that this characterization of weak equivalences holds, and we further characterize the fibrations and trivial fibrations in right-induced model structure on DblCat.

Then, in Section 7.3, we turn our attention to the cofibrations in this model structure on DblCat. In particular, we show that they admit a similar characterization to the one of the cofibrations in 2Cat given in Theorem 6.2.2 in the horizontal direction. As a consequence, cofibrant double categories have a free underlying horizontal category. However, their underlying vertical category is not free, but is instead a copy of disjoint unions of the categories [0] and [1]. This gives an instance of the fact that this model structure is not well-behaved with respect to the vertical direction.

In Section 3.4, we show that the horizontal embedding $\mathbb{H}\colon 2\mathrm{Cat} \to \mathrm{DblCat}$ is both a left and a right Quillen functor and that it is homotopically fully faithful. Moreover, we show that Lack's model structure on 2Cat is both left- and right-induced from this model structure on DblCat along $\mathbb{H}$. In particular, this says that the horizontal embedding $\mathbb{H}$ both preserves and reflects the whole homotopical structure, and hence that the homotopy theory of 2-categories is completely determined by that of double categories under $\mathbb{H}$. With these very nice results, we more than achieved our goal of defining a model structure for double categories compatible with the horizontal embedding.

Finally, in Section 7.5, we note that this model structure on DblCat is neither monoidal for the cartesian product nor for the Gray tensor product for double categories. The fact that it is not cartesian closed can be deduced by using a similar argument to the one for

Lack's model structure on 2Cat. But the fact that it is not monoidal for the Gray tensor product for double categories is a consequence of the asymmetry between the horizontal and vertical directions in the characterization of cofibrations. However, by restricting the Gray tensor product in one variable along $\mathbb{H}$, we remove this issue and this provides a 2Cat-enrichment of the model structure on DblCat.

All the results in this section are based on joint work [MSV20a] with Maru Sarazola and Paula Verdugo.

7.1. **The model structure.** Recall the functors $\mathbf{H}\colon \mathrm{DblCat} \to 2\mathrm{Cat}$ introduced in Definition 3.4.3 and $\mathcal{V}\colon \mathrm{DblCat} \to 2\mathrm{Cat}$ introduced in Definition 3.4.9. We aim to induce a model structure on DblCat along the right adjoint $(\mathbf{H}, \mathcal{V})\colon \mathrm{DblCat} \to 2\mathrm{Cat} \times 2\mathrm{Cat}$, where each copy of 2Cat is endowed with Lack's model structure. To prove the existence of such a right-induced model structure, we apply Corollary 5.1.5, which requires the existence of a path object for DblCat. Recall that the 2-category E_{adj} is the "free-living adjoint equivalence", and that $[-,-]_{\mathrm{ps}}$ denote the pseudo-hom double categories introduced in Definition 3.3.4.

Definition 7.1.1. Let $\mathbb{A}$ be a double category. We define a **path object** for $\mathbb{A}$ to be the double category $\mathrm{Path}(\mathbb{A}) = [\mathbb{H}E_{\mathrm{adj}}, \mathbb{A}]_{\mathrm{ps}}$ together with the following factorization of the diagonal double functor $\Delta\colon \mathbb{A} \to \mathbb{A} \times \mathbb{A}$

$$\mathbb{A} \xrightarrow{W} \mathrm{Path}(\mathbb{A}) \xrightarrow{P} \mathbb{A} \times \mathbb{A}\,,$$

where $W\colon \mathbb{A} \cong [[0], \mathbb{A}]_{\mathrm{ps}} \to \mathrm{Path}(\mathbb{A}) = [\mathbb{H}E_{\mathrm{adj}}, \mathbb{A}]_{\mathrm{ps}}$ is induced by the unique double functor $\mathbb{H}E_{\mathrm{adj}} \to [0]$, and $P\colon \mathrm{Path}(\mathbb{A}) = [\mathbb{H}E_{\mathrm{adj}}, \mathbb{A}]_{\mathrm{ps}} \to \mathbb{A} \times \mathbb{A} \cong [[0] \sqcup [0], \mathbb{A}]_{\mathrm{ps}}$ is induced by the inclusion double functor $[0] \sqcup [0] \to \mathbb{H}E_{\mathrm{adj}}$.

We show that the image under $(\mathbf{H}, \mathcal{V})\colon \mathrm{DblCat} \to 2\mathrm{Cat}\times 2\mathrm{Cat}$ of the double functors W and P is a weak equivalence and a fibration, respectively, in $2\mathrm{Cat} \times 2\mathrm{Cat}$.

Proposition 7.1.2. *Let $\mathbb{A}$ be a double category. The path object*

$$\mathbb{A} \xrightarrow{W} \mathrm{Path}(\mathbb{A}) \xrightarrow{P} \mathbb{A} \times \mathbb{A}$$

of Definition 7.1.1 is such that $(\mathbf{H}, \mathcal{V})W$ is a weak equivalence in $2\mathrm{Cat}\times 2\mathrm{Cat}$ and $(\mathbf{H}, \mathcal{V})P$ is a fibration in $2\mathrm{Cat}\times 2\mathrm{Cat}$, where each copy of 2Cat is endowed with the model structure of Theorem 6.1.8.

Proof. We first prove that $\mathbf{H}W$ and $\mathcal{V}W$ are biequivalences. By Lemma 3.5.6 and Corollary 3.5.8, we have commutative squares as depicted below,

$$\begin{tikzcd}
\mathbf{H}[[0], \mathbb{A}]_{\mathrm{ps}} \arrow[r, "\cong"] \arrow[d, "\mathbf{H}W = \mathbf{H}[!, \mathbb{A}]_{\mathrm{ps}}"'] & {[[0], \mathbf{H}\mathbb{A}]_{2,\mathrm{ps}}} \arrow[d, "{[!, \mathbf{H}\mathbb{A}]_{2,\mathrm{ps}}}"] \\
\mathbf{H}[\mathbb{H}E_{\mathrm{adj}}, \mathbb{A}]_{\mathrm{ps}} \arrow[r, "\cong"'] & {[E_{\mathrm{adj}}, \mathbf{H}\mathbb{A}]_{2,\mathrm{ps}}}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{V}[[0], \mathbb{A}]_{\mathrm{ps}} \arrow[r, "\cong"] \arrow[d, "\mathcal{V}W = \mathcal{V}[!, \mathbb{A}]_{\mathrm{ps}}"'] & {[[0], \mathcal{V}\mathbb{A}]_{2,\mathrm{ps}}} \arrow[d, "{[!, \mathcal{V}\mathbb{A}]_{2,\mathrm{ps}}}"] \\
\mathcal{V}[\mathbb{H}E_{\mathrm{adj}}, \mathbb{A}]_{\mathrm{ps}} \arrow[r, "\cong"'] & {[E_{\mathrm{adj}}, \mathcal{V}\mathbb{A}]_{2,\mathrm{ps}}}
\end{tikzcd}$$

where $[-,-]_{2,\mathrm{ps}}$ denote the pseudo-hom 2-categories introduced in Definition 2.3.3. Now recall from Notation 6.2.5 that $j_1\colon [0] \to E_{\mathrm{adj}}$ is a trivial cofibration in 2Cat and also that every 2-category is fibrant. Therefore, by monoidality of the model structure on 2Cat with respect to the Gray tensor product (see Proposition 2.3.4 and Theorem 6.3.5), the 2-functors

$$[j_1, \mathbf{H}\mathbb{A}]_{2,\mathrm{ps}}\colon [E_{\mathrm{adj}}, \mathbf{H}\mathbb{A}]_{2,\mathrm{ps}} \to [[0], \mathbf{H}\mathbb{A}]_{2,\mathrm{ps}}, \quad [j_1, \mathcal{V}\mathbb{A}]_{2,\mathrm{ps}}\colon [E_{\mathrm{adj}}, \mathcal{V}\mathbb{A}]_{2,\mathrm{ps}} \to [[0], \mathcal{V}\mathbb{A}]_{2,\mathrm{ps}}$$

are trivial fibrations in 2Cat. By pre-composing these 2-functors with $[!, \mathbf{H}\mathbb{A}]_{2,\mathrm{ps}}$ and $[!, \mathcal{V}]_{2,\mathrm{ps}}$, respectively, we get the identity at $\mathbf{H}\mathbb{A}$ and $\mathcal{V}\mathbb{A}$. Therefore, by 2-out-of-3, we get that $[!, \mathbf{H}\mathbb{A}]_{2,\mathrm{ps}}$ and $[!, \mathcal{V}]_{2,\mathrm{ps}}$ are biequivalences. It follows from the commutativity of the above diagrams that $\mathbf{H}W$ and $\mathcal{V}W$ are also biequivalences. This shows that $(\mathbf{H},\mathcal{V})W$ is a weak equivalence in $2\mathrm{Cat}\times 2\mathrm{Cat}$.

We now prove that $\mathbf{H}P$ and $\mathcal{V}P$ are Lack fibrations. Again, by Lemma 3.5.6 and Corollary 3.5.8, we have commutative squares as follows,

$$\begin{tikzcd}
\mathbf{H}[\mathbb{H}E_{\mathrm{adj}},\mathbb{A}]_{\mathrm{ps}} \arrow[r, "\cong"] \arrow[d, "\mathbf{H}P=\mathbf{H}[\mathbb{H}i,\mathbb{A}]_{\mathrm{ps}}"'] & {[E_{\mathrm{adj}},\mathbf{H}\mathbb{A}]_{2,\mathrm{ps}}} \arrow[d, "{[i,\mathbf{H}\mathbb{A}]_{2,\mathrm{ps}}}"] \\
\mathbf{H}[[0]\sqcup[0],\mathbb{A}]_{\mathrm{ps}} \arrow[r, "\cong"'] & {[[0]\sqcup[0],\mathbf{H}\mathbb{A}]_{2,\mathrm{ps}}}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{V}[\mathbb{H}E_{\mathrm{adj}},\mathbb{A}]_{\mathrm{ps}} \arrow[r, "\cong"] \arrow[d, "\mathcal{V}P=\mathcal{V}[\mathbb{H}i,\mathbb{A}]_{\mathrm{ps}}"'] & {[E_{\mathrm{adj}},\mathcal{V}\mathbb{A}]_{2,\mathrm{ps}}} \arrow[d, "{[i,\mathcal{V}\mathbb{A}]_{2,\mathrm{ps}}}"] \\
\mathcal{V}[[0]\sqcup[0],\mathbb{A}]_{\mathrm{ps}} \arrow[r, "\cong"'] & {[[0]\sqcup[0],\mathcal{V}\mathbb{A}]_{2,\mathrm{ps}}}
\end{tikzcd}$$

where $i\colon [0]\sqcup[0]\to E_{\mathrm{adj}}$ denotes the inclusion 2-functor. This 2-functor i is a cofibration in 2Cat, by Corollary 6.2.3, and since every 2-category is fibrant, we get by monoidality of the model structure on 2Cat (see Theorem 6.3.5) that the 2-functors $[i,\mathbf{H}\mathbb{A}]_{2,\mathrm{ps}}$ and $[i,\mathcal{V}\mathbb{A}]_{2,\mathrm{ps}}$ are Lack fibrations. It follows by the commutativity of the above diagrams that $\mathbf{H}P$ and $\mathcal{V}P$ are also Lack fibrations. This shows that $(\mathbf{H},\mathcal{V})P$ is a fibration in $2\mathrm{Cat}\times 2\mathrm{Cat}$. □

This allows us to prove the existence of the desired right-induced model structure.

Theorem 7.1.3. *Consider the adjunction*

$$\begin{tikzcd}
\mathrm{DblCat} \arrow[r, bend right, "(\mathbf{H},\mathcal{V})"'] \arrow[r, phantom, "\perp"] & 2\mathrm{Cat}\times 2\mathrm{Cat}\,, \arrow[l, bend right, "\mathbb{H}\sqcup\mathbb{L}"']
\end{tikzcd}$$

where each copy of 2Cat *is endowed with the model structure of Theorem 6.1.8. Then the right-induced model structure on* DblCat *exists.*

Proof. First recall that the categories 2Cat and DblCat are locally presentable by Propositions 2.1.6 and 3.1.6. Then, by Theorem 6.2.6, the model category 2Cat is combinatorial and every 2-category is fibrant. Therefore, we can apply Corollary 5.1.5 and Proposition 7.1.2 verifies the required conditions for the path object $\mathrm{Path}(\mathbb{A})$ of Definition 7.1.1 for every double category $\mathbb{A}$. This proves the existence of the right-induced model structure on DblCat. □

Remark 7.1.4. Note that every double category is fibrant in the model structure on DblCat of Theorem 7.1.3. Indeed, this follows from the facts that every 2-category is fibrant, and that a double category $\mathbb{A}$ is fibrant in the right-induced model structure along $(\mathbf{H},\mathcal{V})$ if and only if the 2-categories $\mathbf{H}\mathbb{A}$ and $\mathcal{V}\mathbb{A}$ are fibrant.

7.2. **Double biequivalences and double fibrations.** In this section, we characterize the weak equivalences, fibrations, and trivial fibrations in the model structure on DblCat of Theorem 7.1.3. We show that these admit characterizations in terms of conditions analogous to the ones that biequivalences, Lack fibrations, or trivial fibrations in 2Cat satisfy. First, we show that the weak equivalences in DblCat are precisely the double functors which are *double biequivalences.*

Definition 7.2.1. Let $\mathbb{A}$ and $\mathbb{B}$ be double categories. A double functor $F\colon\mathbb{A}\to\mathbb{B}$ is a **double biequivalence** if

(db1) for every object $B\in\mathbb{B}$, there is an object $A\in\mathbb{A}$ together with a horizontal equivalence $B\xrightarrow{\simeq} FA$ in $\mathbb{B}$,

(db2) for every pair of objects $A,C\in\mathbb{A}$ and every horizontal morphism $b\colon FA\to FC$ in $\mathbb{B}$, there is a horizontal morphism $a\colon A\to C$ together with a vertically invertible square in $\mathbb{B}$ of the form

$$\begin{tikzcd} FA \arrow[r, "b"] \arrow[d, equal, "\bullet" marking] \arrow[dr, phantom, "\cong"] & FC \arrow[d, equal, "\bullet" marking] \\ FA \arrow[r, "Fa"'] & FC \end{tikzcd},$$

(db3) for every vertical morphism $v\colon B \to B'$ in $\mathbb{B}$, there is a vertical morphism $u\colon A \to A'$ in $\mathbb{A}$ together with a weakly horizontally invertible square in $\mathbb{B}$ of the form

$$\begin{tikzcd} B \arrow[r, "\simeq"] \arrow[d, "v"', "\bullet" marking] \arrow[dr, phantom, "\simeq"] & FA \arrow[d, "Fu", "\bullet" marking] \\ B' \arrow[r, "\simeq"'] & FA' \end{tikzcd},$$

(db4) for every pair of horizontal morphisms $a\colon A \to C$ and $a'\colon A' \to C'$ in $\mathbb{A}$, every pair of vertical morphisms $u\colon A \to A'$ and $w\colon C \to C'$ in $\mathbb{A}$, and every square β in $\mathbb{B}$ as depicted below left, there is a unique square α in $\mathbb{A}$ as depicted below right such that $\beta = F\alpha$.

$$\begin{tikzcd} FA \arrow[r, "Fa"] \arrow[d, "Fu"', "\bullet" marking] \arrow[dr, phantom, "\beta"] & FC \arrow[d, "Fw", "\bullet" marking] \\ FA' \arrow[r, "Fa'"'] & FC' \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "a"] \arrow[d, "u"', "\bullet" marking] \arrow[dr, phantom, "\alpha"] & C \arrow[d, "w", "\bullet" marking] \\ A' \arrow[r, "a'"'] & C' \end{tikzcd}$$

We can see that conditions (db1-2) and (db4) are analogous to the ones that a biequivalence of 2-categories satisfy. In particular, a biequivalence G in 2Cat induces a double biequivalence $\mathbb{H}G$ in DblCat. To see that a double functor F is a double biequivalence in DblCat if and only if the 2-functors $\mathbf{H}F$ and $\mathcal{V}F$ are biequivalences in 2Cat, we study what conditions (b1-3) of Definition 6.1.6 applied to the 2-functors $\mathbf{H}F$ and $\mathcal{V}F$ mean for F.

Remark 7.2.2. Let $F\colon \mathbb{A} \to \mathbb{B}$ be a double functor. The 2-functor $\mathbf{H}F\colon \mathbf{H}\mathbb{A} \to \mathbf{H}\mathbb{B}$ is a biequivalence if and only if the double functor F satisfies (db1-2) of Definition 7.2.1 and the following condition:

(hb3) for every pair of horizontal morphisms $a, c\colon A \to C$ in $\mathbb{A}$, and every square β in $\mathbb{B}$ as depicted below left, there is a unique square α in $\mathbb{A}$ as depicted below right such that $\beta = F\alpha$.

$$\begin{tikzcd} FA \arrow[r, "Fa"] \arrow[d, equal, "\bullet" marking] \arrow[dr, phantom, "\beta"] & FC \arrow[d, equal, "\bullet" marking] \\ FA \arrow[r, "Fc"'] & FC \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "a"] \arrow[d, equal, "\bullet" marking] \arrow[dr, phantom, "\alpha"] & C \arrow[d, equal, "\bullet" marking] \\ A \arrow[r, "c"'] & C \end{tikzcd}$$

Remark 7.2.3. Let $F\colon \mathbb{A} \to \mathbb{B}$ be a double functor. The 2-functor $\mathcal{V}F\colon \mathcal{V}\mathbb{A} \to \mathcal{V}\mathbb{B}$ is a biequivalence if and only if the double functor F satisfies (db3) of Definition 7.2.1 and the following conditions:

(vb2) for every pair of vertical morphisms $u\colon A \to A'$ and $w\colon C \to C'$ in $\mathbb{A}$, and every square $\beta\colon (Fu \,{}^{b}_{b'}\, Fw)$ in $\mathbb{B}$, there is a square $\alpha\colon (u \,{}^{a}_{a'}\, w)$ in $\mathbb{A}$ together with vertically invertible squares in $\mathbb{B}$ as in the following pasting equality,

$$\begin{tikzcd} FA \arrow[r, "b"] \arrow[d, equal, "\bullet" marking] \arrow[dr, phantom, "\cong"] & FC \arrow[d, equal, "\bullet" marking] \\ FA \arrow[r, "Fa"] \arrow[d, "Fu"', "\bullet" marking] \arrow[dr, phantom, "F\alpha"] & FC \arrow[d, "Fw", "\bullet" marking] \\ FA' \arrow[r, "Fa'"'] & FC' \end{tikzcd} = \begin{tikzcd} FA \arrow[r, "b"] \arrow[d, "Fu"', "\bullet" marking] \arrow[dr, phantom, "\beta"] & FC \arrow[d, "Fw", "\bullet" marking] \\ FA' \arrow[r, "b'"'] \arrow[d, equal, "\bullet" marking] \arrow[dr, phantom, "\cong"] & FC' \arrow[d, equal, "\bullet" marking] \\ FA' \arrow[r, "Fa'"'] & FC' \end{tikzcd}$$

(vb3) for every pair of squares $\alpha\colon (u\ {}^{a}_{a'}\ w)$ and $\gamma\colon (u\ {}^{c}_{c'}\ w)$ in $\mathbb{A}$, and every pair of squares τ_0 and τ_1 in $\mathbb{B}$ as in the below left pasting equality, there are unique squares σ_0 and σ_1 in $\mathbb{A}$ as in the below right pasting equality such that $\tau_0 = F\sigma_0$ and $\tau_1 = F\sigma_1$.

$$\begin{tikzcd} FA \arrow[r, "Fa"] \arrow[d, equal, "\bullet" marking] \arrow[dr, phantom, "\tau_0"] & FC \arrow[d, equal, "\bullet" marking] \\ FA \arrow[r, "Fc"] \arrow[d, "Fu"', "\bullet" marking] \arrow[dr, phantom, "F\gamma"] & FC \arrow[d, "Fw", "\bullet" marking] \\ FA' \arrow[r, "Fc'"'] & FC' \end{tikzcd} = \begin{tikzcd} FA \arrow[r, "Fa"] \arrow[d, "Fu"', "\bullet" marking] \arrow[dr, phantom, "F\alpha"] & FC \arrow[d, "Fw", "\bullet" marking] \\ FA' \arrow[r, "Fa'"'] \arrow[d, equal, "\bullet" marking] \arrow[dr, phantom, "\tau_1"] & FC' \arrow[d, equal, "\bullet" marking] \\ FA' \arrow[r, "Fc'"'] & FC' \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "a"] \arrow[d, equal, "\bullet" marking] \arrow[dr, phantom, "\sigma_0"] & C \arrow[d, equal, "\bullet" marking] \\ A \arrow[r, "c"] \arrow[d, "u"', "\bullet" marking] \arrow[dr, phantom, "\gamma"] & C \arrow[d, "w", "\bullet" marking] \\ A' \arrow[r, "c'"'] & C' \end{tikzcd} = \begin{tikzcd} A \arrow[r, "a"] \arrow[d, "u"', "\bullet" marking] \arrow[dr, phantom, "\alpha"] & C \arrow[d, "w", "\bullet" marking] \\ A' \arrow[r, "a'"'] \arrow[d, equal, "\bullet" marking] \arrow[dr, phantom, "\sigma_1"] & C' \arrow[d, equal, "\bullet" marking] \\ A' \arrow[r, "c'"'] & C' \end{tikzcd}$$

Note that (db4) is the only condition of Definition 7.2.1 for F that does not appear in Remarks 7.2.2 and 7.2.3. We first show that (hb3) and (vb2-3) of the above remarks actually imply (db4).

Lemma 7.2.4. *Let $F\colon \mathbb{A} \to \mathbb{B}$ be a double functor satisfying (hb3) of Remark 7.2.2 and (vb2-3) of Remark 7.2.3. Then the double functor F satisfies (db4) of Definition 7.2.1.*

Proof. Let $a\colon A \to C$ and $a'\colon A' \to C'$ be horizontal morphisms in $\mathbb{A}$, and let $u\colon A \mathrel{\bullet\!\!\to} A'$ and $w\colon C \mathrel{\bullet\!\!\to} C'$ be vertical morphisms in $\mathbb{A}$. Suppose we have a square β in $\mathbb{B}$ as depicted below left.

$$\begin{tikzcd} FA \arrow[r, "Fa"] \arrow[d, "Fu"', "\bullet" marking] \arrow[dr, phantom, "\beta"] & FC \arrow[d, "Fw", "\bullet" marking] \\ FA' \arrow[r, "Fa'"'] & FC' \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "a"] \arrow[d, "u"', "\bullet" marking] \arrow[dr, phantom, "\alpha"] & C \arrow[d, "w", "\bullet" marking] \\ A' \arrow[r, "a'"'] & C' \end{tikzcd}$$

We want to show that there is a unique square α in $\mathbb{A}$ as depicted above right such that $\beta = F\alpha$. By (vb2) of Remark 7.2.3, there is a square $\gamma\colon (u\ {}^{c}_{c'}\ w)$ in $\mathbb{A}$ together with vertically invertible squares τ_0 and τ_1 in $\mathbb{B}$ as in the following pasting equality.

$$\begin{tikzcd} FA \arrow[r, "Fa"] \arrow[d, equal, "\bullet" marking] \arrow[dr, phantom, "\tau_0\ \cong"] & FC \arrow[d, equal, "\bullet" marking] \\ FA \arrow[r, "Fc"] \arrow[d, "Fu"', "\bullet" marking] \arrow[dr, phantom, "F\gamma"] & FC \arrow[d, "Fw", "\bullet" marking] \\ FA' \arrow[r, "Fc'"'] & FC' \end{tikzcd} = \begin{tikzcd} FA \arrow[r, "Fa"] \arrow[d, "Fu"', "\bullet" marking] \arrow[dr, phantom, "\beta"] & FC \arrow[d, "Fw", "\bullet" marking] \\ FA' \arrow[r, "Fa'"'] \arrow[d, equal, "\bullet" marking] \arrow[dr, phantom, "\tau_1\ \cong"] & FC' \arrow[d, equal, "\bullet" marking] \\ FA' \arrow[r, "Fc'"'] & FC' \end{tikzcd}$$

By (hb3) applied to the vertically invertible squares τ_0 and τ_1, there are unique squares σ_0 and σ_1 in $\mathbb{A}$ as depicted below such that $\tau_0 = F\sigma_0$ and $\tau_1 = F\sigma_1$.

$$
\begin{tikzcd}
A \arrow[r, "a"] \arrow[d, equal, "\bullet" marking] \arrow[dr, phantom, "\sigma_0"] & C \arrow[d, equal, "\bullet" marking] \\
A \arrow[r, "c"'] & C
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
A' \arrow[r, "a'"] \arrow[d, equal, "\bullet" marking] \arrow[dr, phantom, "\sigma_1"] & C' \arrow[d, equal, "\bullet" marking] \\
A' \arrow[r, "c'"'] & C'
\end{tikzcd}
$$

By Lemma 6.1.10 (i) applied to $\mathbf{H}F$, the squares σ_0 and σ_1 are vertically invertible, as 2-isomorphisms in $\mathbf{H}\mathbb{A}$ and $\mathbf{H}\mathbb{B}$ are precisely vertically invertible squares of this form. We set α to be the following pasting composite

$$
\begin{tikzcd}
A \arrow[r, "a"] \arrow[d, "u"', "\bullet" marking] \arrow[dr, phantom, "\alpha"] & C \arrow[d, "w", "\bullet" marking] \\
A' \arrow[r, "a'"'] & C'
\end{tikzcd}
=
\begin{tikzcd}
A \arrow[r, "a"] \arrow[d, equal, "\bullet" marking] \arrow[dr, phantom, "\sigma_0 \ \wr\Vert"] & C \arrow[d, equal, "\bullet" marking] \\
A \arrow[r, "c"] \arrow[d, "u"', "\bullet" marking] \arrow[dr, phantom, "\gamma"] & C \arrow[d, "w", "\bullet" marking] \\
A' \arrow[r, "c'"'] \arrow[d, equal, "\bullet" marking] \arrow[dr, phantom, "\sigma_1^{-1} \wr\Vert"] & C' \arrow[d, equal, "\bullet" marking] \\
A' \arrow[r, "a'"'] & C'
\end{tikzcd}
$$

which is such that $\beta = F\alpha$. This proves existence.

Now suppose that we have two squares α and γ in $\mathbb{A}$ as follows

$$
\begin{tikzcd}
A \arrow[r, "a"] \arrow[d, "u"', "\bullet" marking] \arrow[dr, phantom, "\alpha"] & C \arrow[d, "w", "\bullet" marking] \\
A' \arrow[r, "a'"'] & C'
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
A \arrow[r, "a"] \arrow[d, "u"', "\bullet" marking] \arrow[dr, phantom, "\gamma"] & C \arrow[d, "w", "\bullet" marking] \\
A' \arrow[r, "a'"'] & C'
\end{tikzcd}
$$

such that $F\alpha = F\gamma$. Take $\tau_0 = e_{Fa}$ and $\tau_1 = e_{Fa'}$ in (vb3) of Remark 7.2.3. Then there are unique squares σ_0 and σ_1 in $\mathbb{A}$ as in the following pasting equality

$$
\begin{tikzcd}
A \arrow[r, "a"] \arrow[d, equal, "\bullet" marking] \arrow[dr, phantom, "\sigma_0"] & C \arrow[d, equal, "\bullet" marking] \\
A \arrow[r, "a"] \arrow[d, "u"', "\bullet" marking] \arrow[dr, phantom, "\gamma"] & C \arrow[d, "w", "\bullet" marking] \\
A' \arrow[r, "a'"'] & C'
\end{tikzcd}
=
\begin{tikzcd}
A \arrow[r, "a"] \arrow[d, "u"', "\bullet" marking] \arrow[dr, phantom, "\alpha"] & C \arrow[d, "w", "\bullet" marking] \\
A' \arrow[r, "a'"'] \arrow[d, equal, "\bullet" marking] \arrow[dr, phantom, "\sigma_1"] & C' \arrow[d, equal, "\bullet" marking] \\
A' \arrow[r, "a'"'] & C'
\end{tikzcd}
$$

such that $e_{Fa} = F\sigma_0$ and $e_{Fa'} = F\sigma_1$. Since the vertical identity squares e_a and $e_{a'}$ satisfy this condition, by unicity of such squares, we must have $\sigma_0 = e_a$ and $\sigma_1 = e_{a'}$. By the above pasting equality, we get that $\alpha = \gamma$, which proves unicity. Therefore, the double functor F satisfies (db4) of Definition 7.2.1. $\square$

We are now ready to prove that the weak equivalences in the right-induced model structure on DblCat are precisely the double biequivalences.

Proposition 7.2.5. *Let $\mathbb{A}$ and $\mathbb{B}$ be double categories. A double functor $F\colon \mathbb{A} \to \mathbb{B}$ is a weak equivalence in the model structure on* DblCat *of Theorem 7.1.3 if and only if it is a double biequivalence.*

Proof. Suppose that $F\colon \mathbb{A} \to \mathbb{B}$ is a weak equivalence in DblCat, i.e., the 2-functors $\mathbf{H}F$ and $\mathcal{V}F$ are biequivalences. We show that F satisfies (db1-4) of Definition 7.2.1. By Remark 7.2.2, the double functor F satisfies (db1-2) and, by Remark 7.2.3, it satisfies (db3). Finally, (db4) holds by Lemma 7.2.4. This shows that F is a double biequivalence.

Now suppose that $F\colon \mathbb{A} \to \mathbb{B}$ is a double biequivalence. To prove that $\mathbf{H}F$ and $\mathcal{V}F$ are biequivalences, it suffices to show (hb3) of Remark 7.2.2 and (vb2-3) of Remark 7.2.3. First, note that (hb3) is a special case of (db4), where the vertical morphisms are identities. Therefore, the double functor F satisfies (hb3). Now let $u\colon A \mathrel{\mathrlap{\rightarrow}\mkern1mu\bullet} A'$ and $w\colon C \mathrel{\mathrlap{\rightarrow}\mkern1mu\bullet} C'$ be vertical morphisms in $\mathbb{A}$ and let $\beta\colon (Fu \; {}^{b}_{b'} \; Fw)$ be a square in $\mathbb{B}$. By (db2), there are horizontal morphisms $a\colon A \to C$ and $a'\colon A' \to C'$ in $\mathbb{A}$ together with vertically invertible squares τ_0 and τ_1 in $\mathbb{B}$ as follows.

$$
\begin{tikzcd}
FA \arrow[r, "b"] \arrow[d, equal, "\bullet" marking] \arrow[dr, phantom, "\tau_0 \ \cong"] & FC \arrow[d, equal, "\bullet" marking] \\
FA \arrow[r, "Fa"'] & FC
\end{tikzcd}
\qquad
\begin{tikzcd}
FA' \arrow[r, "b'"] \arrow[d, equal, "\bullet" marking] \arrow[dr, phantom, "\tau_1 \ \cong"] & FC' \arrow[d, equal, "\bullet" marking] \\
FA' \arrow[r, "Fa'"'] & FC'
\end{tikzcd}
$$

Let δ in $\mathbb{B}$ be the square given by the following pasting composite.

$$
\begin{tikzcd}
FA \arrow[r, "Fa"] \arrow[d, "Fu"', "\bullet" marking] \arrow[dr, phantom, "\delta"] & FC \arrow[d, "Fw", "\bullet" marking] \\
FA' \arrow[r, "Fa'"'] & FC'
\end{tikzcd}
=
\begin{tikzcd}
FA \arrow[r, "Fa"] \arrow[d, equal, "\bullet" marking] \arrow[dr, phantom, "\tau_0^{-1} \cong"] & FC \arrow[d, equal, "\bullet" marking] \\
FA \arrow[r, "b"] \arrow[d, "Fu"', "\bullet" marking] \arrow[dr, phantom, "\beta"] & FC \arrow[d, "Fw", "\bullet" marking] \\
FA' \arrow[r, "b'"'] \arrow[d, equal, "\bullet" marking] \arrow[dr, phantom, "\tau_1 \ \cong"] & FC' \arrow[d, equal, "\bullet" marking] \\
FA' \arrow[r, "Fa'"'] & FC'
\end{tikzcd}
$$

Then, by (db4), there is a unique square $\alpha\colon (u \; {}^{a}_{a'} \; w)$ in $\mathbb{A}$ such that $\delta = F\alpha$. Then the square α together with the vertically invertible squares τ_0 and τ_1 give the desired data of (vb2), which shows that F satisfies this condition. Finally, let $\alpha\colon (u \; {}^{a}_{a'} \; w)$ and $\gamma\colon (u \; {}^{c}_{c'} \; w)$ be squares in $\mathbb{A}$, and τ_0 and τ_1 be squares in $\mathbb{B}$ as in the following pasting equality.

$$
\begin{tikzcd}
FA \arrow[r, "Fa"] \arrow[d, equal, "\bullet" marking] \arrow[dr, phantom, "\tau_0"] & FC \arrow[d, equal, "\bullet" marking] \\
FA \arrow[r, "Fc"] \arrow[d, "Fu"', "\bullet" marking] \arrow[dr, phantom, "F\gamma"] & FC \arrow[d, "Fw", "\bullet" marking] \\
FA' \arrow[r, "Fc'"'] & FC'
\end{tikzcd}
=
\begin{tikzcd}
FA \arrow[r, "Fa"] \arrow[d, "Fu"', "\bullet" marking] \arrow[dr, phantom, "F\alpha"] & FC \arrow[d, "Fw", "\bullet" marking] \\
FA' \arrow[r, "Fa'"'] \arrow[d, equal, "\bullet" marking] \arrow[dr, phantom, "\tau_1"] & FC' \arrow[d, equal, "\bullet" marking] \\
FA' \arrow[r, "Fc'"'] & FC'
\end{tikzcd}
$$

By (db4) applied to τ_0 and τ_1, there are unique squares $\sigma_0\colon (e_A \; {}^{a}_{c} \; e_C)$ and $\sigma_1\colon (e_{A'} \; {}^{a'}_{c'} \; e_{C'})$ in $\mathbb{A}$ such that $\tau_0 = F\sigma_0$ and $\tau_1 = F\sigma_1$. Moreover, by unicity in (db4), we get that the following pasting equality holds

$$
\begin{tikzcd}
A \arrow[r, "a"] \arrow[d, equal, "\bullet" marking] \arrow[dr, phantom, "\sigma_0"] & C \arrow[d, equal, "\bullet" marking] \\
A \arrow[r, "a"] \arrow[d, "u"', "\bullet" marking] \arrow[dr, phantom, "\gamma"] & C \arrow[d, "w", "\bullet" marking] \\
A' \arrow[r, "a'"'] & C'
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
A \arrow[r, "a"] \arrow[d, "u"', "\bullet" marking] \arrow[dr, phantom, "\alpha"] & C \arrow[d, "w", "\bullet" marking] \\
A' \arrow[r, "a'"'] \arrow[d, equal, "\bullet" marking] \arrow[dr, phantom, "\sigma_1"] & C' \arrow[d, equal, "\bullet" marking] \\
A' \arrow[r, "a'"'] & C'
\end{tikzcd},
$$

since applying F to each vertical composite yields the same square in $\mathbb{B}$. This shows (vb3), and we conclude that F is a weak equivalence in DblCat. □

We now turn our attention to the fibrations in DblCat and show that these are precisely the *double fibrations*.

Definition 7.2.6. Let $\mathbb{A}$ and $\mathbb{B}$ be double categories. A double functor $F\colon \mathbb{A} \to \mathbb{B}$ is a **double fibration** if

(df1) for every object $C \in \mathbb{A}$ and every horizontal equivalence $b\colon B \xrightarrow{\simeq} FC$ in $\mathbb{B}$, there is a horizontal equivalence $a\colon A \xrightarrow{\simeq} C$ in $\mathbb{A}$ such that $b = Fa$.

(df2) for every horizontal morphism $c\colon A \to C$ in $\mathbb{A}$ and every vertically invertible square β in $\mathbb{B}$ as depicted below left, there is a vertically invertible square α in $\mathbb{A}$ as depicted below right such that $\beta = F\alpha$,

$$
\begin{tikzcd}
FA \arrow[r, "b"] \arrow[d, equal, "\bullet" marking] \arrow[dr, phantom, "\beta\ \cong"] & FC \arrow[d, equal, "\bullet" marking] \\
FA \arrow[r, "Fc"'] & FC
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
A \arrow[r, "a"] \arrow[d, equal, "\bullet" marking] \arrow[dr, phantom, "\alpha\ \cong"] & C \arrow[d, equal, "\bullet" marking] \\
A \arrow[r, "c"'] & C
\end{tikzcd}
$$

(df3) for every vertical morphism $w\colon C \to C'$ in $\mathbb{A}$ and every weakly horizontally invertible square β in $\mathbb{B}$ as depicted below left, there is a weakly horizontally invertible square α in $\mathbb{A}$ as depicted below right such that $\beta = F\alpha$.

$$
\begin{tikzcd}
B \arrow[r, "b", "\simeq"'] \arrow[d, "v"', "\bullet" marking] \arrow[dr, phantom, "\beta\ \simeq"] & FC \arrow[d, "Fw", "\bullet" marking] \\
B' \arrow[r, "\simeq", "b'"'] & FC'
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
A \arrow[r, "a", "\simeq"'] \arrow[d, "u"', "\bullet" marking] \arrow[dr, phantom, "\alpha\ \simeq"] & C \arrow[d, "w", "\bullet" marking] \\
A' \arrow[r, "\simeq", "a'"'] & C'
\end{tikzcd}
$$

Again we can see that conditions (df1-2) are analogous to the ones that a Lack fibration of 2-categories satisfy. In particular, a Lack fibration G in 2Cat induces a double fibration $\mathbb{H}G$ in DblCat. To see that a double functor F is a double fibration in DblCat if and only if the 2-functors $\mathbf{H}F$ and $\mathcal{V}F$ are Lack fibrations in 2Cat, we study what conditions (f1-2) of Definition 6.1.7 applied to the 2-functors $\mathbf{H}F$ and $\mathcal{V}F$ mean for F.

Remark 7.2.7. Let $F\colon \mathbb{A} \to \mathbb{B}$ be a double functor. The 2-functor $\mathbf{H}F\colon \mathbf{H}\mathbb{A} \to \mathbf{H}\mathbb{B}$ is a Lack fibration if and only if the double functor F satisfies (df1-2) of Definition 7.2.6.

Remark 7.2.8. Let $F\colon \mathbb{A} \to \mathbb{B}$ be a double functor. The 2-functor $\mathcal{V}F\colon \mathcal{V}\mathbb{A} \to \mathcal{V}\mathbb{B}$ is a Lack fibration if and only if the double functor F satisfies (df3) of Definition 7.2.6 and the following condition:

(vf2) for every square $\gamma\colon (u \, {}^{c}_{c'} \, w)$ in $\mathbb{A}$ and every square β in $\mathbb{B}$ together with vertically invertible squares τ_0 and τ_1 in $\mathbb{B}$ as in the below left pasting equality, there is a square α in $\mathbb{A}$ together with vertically invertible squares σ_0 and σ_1 in $\mathbb{A}$ as in the below right pasting equality such that $\beta = F\alpha$, $\tau_0 = F\sigma_0$, and $\tau_1 = F\sigma_1$.

$$
\begin{tikzcd}
FA \arrow[r, "b"] \arrow[d, equal, "\bullet" marking] \arrow[dr, phantom, "\tau_0 \ \cong"] & FC \arrow[d, equal, "\bullet" marking] \\
FA \arrow[r, "Fc"] \arrow[d, "Fu"', "\bullet" marking] \arrow[dr, phantom, "F\gamma"] & FC \arrow[d, "Fw", "\bullet" marking] \\
FA' \arrow[r, "Fc'"'] & FC'
\end{tikzcd}
=
\begin{tikzcd}
FA \arrow[r, "b"] \arrow[d, "Fu"', "\bullet" marking] \arrow[dr, phantom, "\beta"] & FC \arrow[d, "Fw", "\bullet" marking] \\
FA' \arrow[r, "b'"'] \arrow[d, equal, "\bullet" marking] \arrow[dr, phantom, "\tau_1 \ \cong"] & FC' \arrow[d, equal, "\bullet" marking] \\
FA' \arrow[r, "Fc'"'] & FC'
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[r, "a"] \arrow[d, equal, "\bullet" marking] \arrow[dr, phantom, "\sigma_0 \ \cong"] & C \arrow[d, equal, "\bullet" marking] \\
A \arrow[r, "c"] \arrow[d, "u"', "\bullet" marking] \arrow[dr, phantom, "\gamma"] & C \arrow[d, "w", "\bullet" marking] \\
A' \arrow[r, "c'"'] & C'
\end{tikzcd}
=
\begin{tikzcd}
A \arrow[r, "a"] \arrow[d, "u"', "\bullet" marking] \arrow[dr, phantom, "\alpha"] & C \arrow[d, "w", "\bullet" marking] \\
A' \arrow[r, "a'"'] \arrow[d, equal, "\bullet" marking] \arrow[dr, phantom, "\sigma_1 \ \cong"] & C' \arrow[d, equal, "\bullet" marking] \\
A' \arrow[r, "c'"'] & C'
\end{tikzcd}
$$

We are now ready to prove the desired characterization.

Proposition 7.2.9. *Let $\mathbb{A}$ and $\mathbb{B}$ be double categories. A double functor $F\colon \mathbb{A} \to \mathbb{B}$ is a fibration in the model structure on* DblCat *of Theorem 7.1.3 if and only if it is a double fibration.*

Proof. Suppose that $F\colon \mathbb{A} \to \mathbb{B}$ is a fibration in DblCat, i.e., the 2-functors $\mathbf{H}F$ and $\mathcal{V}F$ are Lack fibrations. By Remarks 7.2.7 and 7.2.8, we directly see that F satisfies (df1-3) of Definition 7.2.6. Hence F is a double fibration.

Now suppose that $F\colon \mathbb{A} \to \mathbb{B}$ is a double fibration. From Remark 7.2.7, we directly have that $\mathbf{H}F$ is a Lack fibration. To prove that $\mathcal{V}F$ is a Lack fibration, it suffices to show (vf2) of Remark 7.2.8. Let $\gamma\colon (u \,{}^{c}_{c'}\, w)$ be a square in $\mathbb{A}$, β be a square in $\mathbb{B}$, and τ_0 and τ_1 be vertically invertible squares τ_0 and τ_1 in $\mathbb{B}$ such that the following pasting equality holds.

$$
\begin{tikzcd}
FA \arrow[r, "b"] \arrow[d, equal, "\bullet" marking] \arrow[dr, phantom, "\tau_0 \ \cong"] & FC \arrow[d, equal, "\bullet" marking] \\
FA \arrow[r, "Fc"] \arrow[d, "Fu"', "\bullet" marking] \arrow[dr, phantom, "F\gamma"] & FC \arrow[d, "Fw", "\bullet" marking] \\
FA' \arrow[r, "Fc'"'] & FC'
\end{tikzcd}
=
\begin{tikzcd}
FA \arrow[r, "b"] \arrow[d, "Fu"', "\bullet" marking] \arrow[dr, phantom, "\beta"] & FC \arrow[d, "Fw", "\bullet" marking] \\
FA' \arrow[r, "b'"'] \arrow[d, equal, "\bullet" marking] \arrow[dr, phantom, "\tau_1 \ \cong"] & FC' \arrow[d, equal, "\bullet" marking] \\
FA' \arrow[r, "Fc'"'] & FC'
\end{tikzcd}
$$

By (df2) of Definition 7.2.6, there are vertically invertible squares σ_0 and σ_1 in $\mathbb{A}$

$$
\begin{tikzcd}
A \arrow[r, "a"] \arrow[d, equal, "\bullet" marking] \arrow[dr, phantom, "\sigma_0 \ \cong"] & C \arrow[d, equal, "\bullet" marking] \\
A \arrow[r, "c"'] & C
\end{tikzcd}
\qquad
\begin{tikzcd}
A' \arrow[r, "a'"] \arrow[d, equal, "\bullet" marking] \arrow[dr, phantom, "\sigma_1 \ \cong"] & C' \arrow[d, equal, "\bullet" marking] \\
A' \arrow[r, "c'"'] & C'
\end{tikzcd}
$$

such that $\tau_0 = F\sigma_0$ and $\tau_1 = F\sigma_1$. We set α to be the following pasting composite.

$$
\begin{tikzcd}
A \arrow[r, "a"] \arrow[d, "u"', "\bullet" marking] \arrow[dr, phantom, "\alpha"] & C \arrow[d, "w", "\bullet" marking] \\
A' \arrow[r, "a'"'] & C'
\end{tikzcd}
=
\begin{tikzcd}
A \arrow[r, "a"] \arrow[d, equal, "\bullet" marking] \arrow[dr, phantom, "\sigma_0 \ \cong"] & C \arrow[d, equal, "\bullet" marking] \\
A \arrow[r, "c"] \arrow[d, "u"', "\bullet" marking] \arrow[dr, phantom, "\gamma"] & C \arrow[d, "w", "\bullet" marking] \\
A' \arrow[r, "c'"'] \arrow[d, equal, "\bullet" marking] \arrow[dr, phantom, "\sigma_1^{-1} \cong"] & C' \arrow[d, equal, "\bullet" marking] \\
A' \arrow[r, "a'"'] & C'
\end{tikzcd}
$$

Then $(\alpha, \sigma_0, \sigma_1)$ gives the desired data, which shows (vf2). We conclude that F is a fibration in DblCat. $\square$

Finally, by using these characterizations of fibrations and weak equivalences in DblCat, we can see that trivial fibrations are precisely the double functors which are surjective on objects, full on horizontal morphisms, surjective on vertical morphisms, and fully faithful on squares. Again, these conditions resemble the conditions that a trivial fibration in 2Cat satisfies.

Proposition 7.2.10. *Let $\mathbb{A}$ and $\mathbb{B}$ be double categories. A double functor $F\colon \mathbb{A} \to \mathbb{B}$ is a trivial fibration in the model structure on* DblCat *of Theorem 7.1.3 if and only if it is surjective on objects, full on horizontal morphisms, surjective on vertical morphisms, and fully faithful on squares.*

Proof. A double functor $F\colon \mathbb{A} \to \mathbb{B}$ is a trivial fibration in DblCat if and only if the 2-functors $\mathbf{H}F$ and $\mathcal{V}F$ are trivial fibrations in 2Cat, which is the case if and only if the 2-functors $\mathbf{H}F$ and $\mathcal{V}F$ are surjective on objects, full on morphisms, and fully faithful on 2-morphisms by Proposition 6.1.11. Another characterization says that $F\colon \mathbb{A} \to \mathbb{B}$ is a trivial fibration in DblCat if and only if F is both a weak equivalence and a fibration in DblCat, which is the case if and only if F is both a double biequivalence and a double fibration by Propositions 7.2.5 and 7.2.9. To show the result, we use these two different ways of characterizing trivial fibrations in DblCat depending on which one is more accurate.

Suppose that $F\colon \mathbb{A} \to \mathbb{B}$ is a trivial fibration in DblCat. Surjectivity on objects and fullness on morphisms of $\mathbf{H}F$ give surjectivity on objects and fullness on horizontal morphisms for F. Surjectivity on objects of $\mathcal{V}F$ gives surjectivity on vertical morphisms for F. Finally, fully faithfulness on squares follows from the fact that F is in particular a double biequivalence and hence satisfies (db4) of Definition 7.2.1.

Now suppose that $F\colon \mathbb{A} \to \mathbb{B}$ is surjective on objects, full on horizontal morphisms, surjective on vertical morphisms, and fully faithful on squares. We show that $\mathbf{H}F$ and $\mathcal{V}F$ are surjective on objects, full on morphisms, and fully faithful on 2-morphisms. Surjectivity on objects and fullness on morphisms for $\mathbf{H}F$ is precisely surjectivity on objects and fullness on horizontal morphisms of F. Fully faithfulness on 2-morphisms for $\mathbf{H}F$ follows from the fully faithfulness on squares of F, since 2-morphisms in $\mathbf{H}\mathbb{A}$ and $\mathbf{H}\mathbb{B}$ are squares in $\mathbb{A}$ and $\mathbb{B}$ with trivial vertical boundaries. Surjectivity on objects for $\mathcal{V}F$ is precisely surjectivity on vertical morphisms for F. We show that $\mathcal{V}F$ is full on horizontal morphisms. Let $u\colon A \to\!\!\!\!\!\bullet\;\; A'$ and $w\colon C \to\!\!\!\!\!\bullet\;\; C'$ be vertical morphisms in $\mathbb{A}$. A morphism $\beta\colon Fu \to Fw$ in $\mathcal{V}\mathbb{B}$ is a square β in $\mathbb{B}$ of the form

$$\begin{tikzcd}
FA \arrow[r, "b"] \arrow[d, "Fu"', "\bullet" marking] \arrow[dr, phantom, "\beta"] & FC \arrow[d, "Fw", "\bullet" marking] \\
FA' \arrow[r, "b'"'] & FC' .
\end{tikzcd}$$

By fullness on horizontal morphisms of F, there are horizontal morphisms $a\colon A \to C$ and $a'\colon A' \to C'$ in $\mathbb{A}$ such that $b = Fa$ and $b' = Fa'$. Then, by fully faithfulness on squares of F, there is a unique square $\alpha\colon (u \,{}^{a}_{a'}\, w)$ in $\mathbb{A}$ such that $\beta = F\alpha$. This gives a morphism $\alpha\colon u \to w$ in $\mathcal{V}\mathbb{A}$ such that $\beta = (\mathcal{V}F)\alpha$ as desired. Finally, fully faithfulness on 2-morphisms for $\mathcal{V}F$ follows directly from fully faithfulness on squares of F. This shows that F is a trivial fibration in DblCat. $\square$

7.3. **Cofibrations, cofibrant objects, and generating sets.** By Theorem 6.2.6, Lack's model structure on 2Cat is cofibrantly generated. Therefore, by Proposition 5.1.6, the right-induced model structure on DblCat is also cofibrantly generated. This proposition actually gives us sets of generating cofibrations and generating trivial cofibrations for the model structure on DblCat, which depends on sets of generating cofibrations and generating trivial cofibrations for the model structure on 2Cat; e.g. one can choose the sets $\mathcal{I}_2$ and $\mathcal{J}_2$ of 2-functors described in Notation 6.2.5.

Proposition 7.3.1. *The model structure on* DblCat *of Theorem 7.1.3 is cofibrantly generated with generating sets of cofibrations and trivial cofibrations given by*

$$\mathcal{I}' = \{\mathbb{H}i, \mathbb{L}i = \mathbb{H}i \times \mathbb{V}[1] \mid i \in \mathcal{I}_2\} \quad \text{and} \quad \mathcal{J}' = \{\mathbb{H}j, \mathbb{L}j = \mathbb{H}j \times \mathbb{V}[1] \mid j \in \mathcal{J}_2\},$$

where $\mathcal{I}_2$ and $\mathcal{J}_2$ are generating sets of cofibrations and trivial cofibrations for the model structure on 2Cat *of Theorem 6.1.8.*

Proof. By Theorem 6.2.6, the model structure on 2Cat is cofibrantly generated. Let $\mathcal{I}_2$ and $\mathcal{J}_2$ be generating sets of cofibrations and trivial cofibrations in 2Cat, respectively. By Proposition 5.1.6, since the model structure on DblCat is right-induced along the adjunction $\mathbb{H} \sqcup \mathbb{L} \dashv (\mathbf{H}, \mathcal{V})$ from $2\mathrm{Cat} \times 2\mathrm{Cat}$, generating sets of cofibrations and trivial cofibrations in DblCat are given by

$$(\mathbb{H} \sqcup \mathbb{L})(\mathcal{I}_2 \times \mathcal{I}_2) = \{\mathbb{H}i \sqcup \mathbb{L}i' \mid i, i' \in \mathcal{I}_2\}, \quad (\mathbb{H} \sqcup \mathbb{L})(\mathcal{J}_2 \times \mathcal{J}_2) = \{\mathbb{H}j \sqcup \mathbb{L}j' \mid j, j' \in \mathcal{J}_2\}.$$

We show that $\mathcal{I}'$ is a set of generating cofibrations in DblCat. For $i \in \mathcal{I}_2$, the double functors $\mathbb{H}i$ and $\mathbb{L}i$ are cofibrations in DblCat, since $\mathbb{H} \sqcup \mathbb{L}$ is left Quillen by Proposition 5.1.3, and we have that $(\mathbb{H} \sqcup \mathbb{L})(i, \mathrm{id}_\emptyset) = \mathbb{H}i$ and $(\mathbb{H} \sqcup \mathbb{L})(\mathrm{id}_\emptyset, i) = \mathbb{L}i$, where $(i, \mathrm{id}_\emptyset)$ and $(\mathrm{id}_\emptyset, i)$ are cofibrations in $2\mathrm{Cat} \times 2\mathrm{Cat}$. This shows that $\mathcal{I}'$ is contained in the class of cofibrations. Now, given $i, i' \in \mathcal{I}_2$, the cofibration $\mathbb{H}i \sqcup \mathbb{L}i'$ of $(\mathbb{H} \sqcup \mathbb{L})(\mathcal{I}_2 \times \mathcal{I}_2)$ can be obtained as a coproduct of $\mathbb{H}i$ and $\mathbb{L}i'$ in $\mathcal{I}'$. This shows that $\mathcal{I}'$ generates $(\mathbb{H} \sqcup \mathbb{L})(\mathcal{I}_2 \times \mathcal{I}_2)$, and since this latter generates all cofibrations in DblCat, it follows that $\mathcal{I}'$ is a generating set of cofibrations.

Similarly, one can show that $\mathcal{J}'$ is a generating set of trivial cofibrations. □

We now want to give smaller and more explicit sets of generating cofibrations and generating trivial cofibrations. For this, we first aim to characterize cofibrations and cofibrant objects in DblCat. In analogy with the characterization of cofibrations in 2Cat in terms of their underlying functor, cofibrations in DblCat can be characterized in terms of their underlying horizontal and vertical functors.

Remark 7.3.2. The functor $U\mathbf{H}\colon \mathrm{DblCat} \to \mathrm{Cat}$, which sends a double category to its underlying horizontal category, has a right adjoint $\mathbb{R}_h\colon \mathrm{Cat} \to \mathrm{DblCat}$. It sends a category $\mathcal{C}$ to the double category $\mathbb{R}_h\mathcal{C}$ with the same objects as $\mathcal{C}$, horizontal morphisms the morphisms of $\mathcal{C}$, a unique vertical morphism between each pair of objects, and a unique square $!\colon (!\, {}^{f}_{g}\, !)$ for each pair of morphisms f and g in $\mathcal{C}$. A functor $F\colon \mathcal{C} \to \mathcal{D}$ is sent to the double functor $\mathbb{R}_hF\colon \mathbb{R}_h\mathcal{C} \to \mathbb{R}_h\mathcal{D}$ which acts as F on objects and horizontal morphisms, sends the unique vertical morphism $!\colon C \twoheadrightarrow C'$ in $\mathbb{R}_h\mathcal{C}$ to the unique vertical morphism $!\colon FC \twoheadrightarrow FC'$ in $\mathbb{R}_h\mathcal{D}$, for all objects $C, C' \in \mathcal{C}$, and the unique square $!\colon (!\, {}^{f}_{g}\, !)$ in $\mathbb{R}_h\mathcal{C}$ to the unique square $!\colon (!\, {}^{Ff}_{Fg}\, !)$ in $\mathbb{R}_h\mathcal{D}$, for all morphisms f, g in $\mathcal{C}$. Note that the double functor $\mathbb{R}_hF$ is fully faithful on vertical morphisms and squares.

Remark 7.3.3. The functor $U\mathbf{V}\colon \mathrm{DblCat} \to \mathrm{Cat}$, which sends a double category to its underlying vertical category, has a right adjoint $\mathbb{R}_v\colon \mathrm{Cat} \to \mathrm{DblCat}$. It sends a category $\mathcal{C}$ to the double category $\mathbb{R}_v\mathcal{C}$ with the same objects as $\mathcal{C}$, a unique horizontal morphism between each pair of objects, vertical morphisms the morphisms of $\mathcal{C}$, and a unique square $!\colon (f \,{}^{!}_{!}\, g)$ for each pair of morphisms f and g in $\mathcal{C}$. A functor $F\colon \mathcal{C} \to \mathcal{D}$ is sent to the double functor $\mathbb{R}_vF\colon \mathbb{R}_v\mathcal{C} \to \mathbb{R}_v\mathcal{D}$ which acts as F on objects and vertical morphisms, sends the unique horizontal morphism $!\colon C \to D$ in $\mathbb{R}_v\mathcal{C}$ to the unique horizontal morphism

$!\colon FC \to FD$ in $\mathbb{R}_v\mathcal{D}$, for all objects $C, D \in \mathcal{C}$, and the unique square $!\colon (f\ {}^{!}_{!}\ g)$ in $\mathbb{R}_v\mathcal{C}$ to the unique square $!\colon (Ff\ {}^{!}_{!}\ Fg)$ in $\mathbb{R}_v\mathcal{D}$, for all morphisms f, g in $\mathcal{C}$. Note that the double functor $\mathbb{R}_hF$ is fully faithful on horizontal morphisms and squares.

Theorem 7.3.4. *Let $\mathbb{A}$ and $\mathbb{B}$ be double categories. A double functor $F\colon \mathbb{A} \to \mathbb{B}$ is a cofibration in the model structure on* DblCat *of Theorem 7.1.3 if and only if*

(i) its underlying horizontal functor $U\mathbf{H}F\colon U\mathbf{H}\mathbb{A} \to U\mathbf{H}\mathbb{B}$ has the left lifting property with respect to all surjective on objects and full functors, and
(ii) its underlying vertical functor $U\mathbf{V}F\colon U\mathbf{V}\mathbb{A} \to U\mathbf{V}\mathbb{B}$ has the left lifting property with respect to all surjective on objects and surjective on morphisms functors.

Proof. Suppose that $F\colon \mathbb{A} \to \mathbb{B}$ is a cofibration in DblCat. Let $G\colon \mathcal{C} \to \mathcal{D}$ be a surjective on objects and full functor in Cat. We need to show that there is a lift in every commutative diagram as below left.

$$\begin{tikzcd} U\mathbf{H}\mathbb{A} \arrow[r] \arrow[d, "U\mathbf{H}F"'] & \mathcal{C} \arrow[d, "G"] \\ U\mathbf{H}\mathbb{B} \arrow[r] \arrow[ur, dashed] & \mathcal{D} \end{tikzcd} \qquad \begin{tikzcd} \mathbb{A} \arrow[r] \arrow[d, "F"'] & \mathbb{R}_h\mathcal{C} \arrow[d, "\mathbb{R}_hG"] \\ \mathbb{B} \arrow[r] \arrow[ur, dashed] & \mathbb{R}_h\mathcal{D} \end{tikzcd}$$

By the adjunction $U\mathbf{H} \dashv \mathbb{R}_h$, such a lift exists if and only if there is a lift in the above right commutative diagram. This is indeed the case since $\mathbb{R}_hG\colon \mathbb{R}_h\mathcal{C} \to \mathbb{R}_h\mathcal{D}$ is a trivial fibration in DblCat: it is surjective on objects and full on horizontal morphisms since G is surjective on objects and full on morphisms, and fully faithful on vertical morphisms and squares by Remark 7.3.2; see Proposition 7.2.10. This shows (i).

Now, let $G\colon \mathcal{C} \to \mathcal{D}$ be a surjective on objects and surjective on morphisms functor. As before, using the adjunction $U\mathbf{V} \dashv \mathbb{R}_v$, we can show that $U\mathbf{V}F\colon U\mathbf{V}\mathbb{A} \to U\mathbf{H}\mathbb{B}$ has the left lifting property with respect to G, since $\mathbb{R}_vG$ is a trivial fibration in DblCat: it is surjective on objects and surjective on vertical morphisms since G is surjective on objects and surjective on morphisms, and fully faithful on horizontal morphisms and squares by Remark 7.3.3; see Proposition 7.2.10. This shows (ii).

Now suppose that F is such that (i) and (ii) hold. Let $P\colon \mathbb{X} \to \mathbb{Y}$ be a trivial fibration in DblCat. We need to show that there is a lift in every commutative diagram as below left.

$$\begin{tikzcd} \mathbb{A} \arrow[r] \arrow[d, "F"'] & \mathbb{X} \arrow[d, "P"] \\ \mathbb{B} \arrow[r, "Q"'] \arrow[ur, dashed, "L"] & \mathbb{Y} \end{tikzcd} \qquad \begin{tikzcd} U\mathbf{V}\mathbb{A} \arrow[r] \arrow[d, "U\mathbf{V}F"'] & U\mathbf{V}\mathbb{X} \arrow[d, "U\mathbf{V}P"] \\ U\mathbf{V}\mathbb{B} \arrow[r, "U\mathbf{V}Q"'] \arrow[ur, dashed, "L_v"] & U\mathbf{V}\mathbb{Y} \end{tikzcd} \qquad \begin{tikzcd} U\mathbf{H}\mathbb{A} \arrow[r] \arrow[d, "U\mathbf{H}F"'] & U\mathbf{H}\mathbb{X} \arrow[d, "U\mathbf{H}P"] \\ U\mathbf{H}\mathbb{B} \arrow[r, "U\mathbf{H}Q"'] \arrow[ur, dashed, "L_h"] & U\mathbf{H}\mathbb{Y} \end{tikzcd}$$

Since $U\mathbf{V}P\colon U\mathbf{V}\mathbb{X} \to U\mathbf{V}\mathbb{X}$ is surjective on objects and morphisms as P is surjective on objects and vertical morphisms by Proposition 7.2.10, there is a lift $L_v\colon U\mathbf{V}\mathbb{B} \to U\mathbf{V}\mathbb{X}$ in the diagram above middle. Furthermore, since $U\mathbf{H}P\colon U\mathbf{H}\mathbb{X} \to U\mathbf{H}\mathbb{Y}$ is surjective on objects and full on morphisms as P is surjective on objects and full on horizontal morphisms by Proposition 7.2.10, there is a lift $L_h\colon U\mathbf{H}\mathbb{B} \to U\mathbf{H}\mathbb{X}$ in the above right diagram and we can choose this lift to be such that L_h and L_v coincide on objects, by fullness of $U\mathbf{H}P$. This defines the double functor $L\colon \mathbb{B} \to \mathbb{X}$ on objects, horizontal morphisms, and vertical morphisms. Given a square $\beta\colon (u\ {}^{b}_{b'}\ v)$ in $\mathbb{B}$, by fully faithfulness of P on squares (see Proposition 7.2.10), there is a unique square $\chi\colon (Lu\ {}^{Lb}_{Lb'}\ Lv)$ in $\mathbb{X}$ such that $Q\beta = P\chi$. We set $L\beta = \chi$ and this gives a well-defined double functor making the diagram above left commute. Hence F is a cofibration in DblCat. $\square$

Corollary 7.3.5. *Let $\mathbb{A}$ and $\mathbb{B}$ be double categories. A double functor $F\colon \mathbb{A} \to \mathbb{B}$ is a cofibration in the model structure on* DblCat *of Theorem 7.1.3 if and only if*

(i) it is injective on objects, and faithful on horizontal and vertical morphisms,
(ii) the underlying horizontal category $U\mathbf{H}\mathbb{B}$ is a retract of a category obtained from the image of $U\mathbf{H}\mathbb{A}$ under $U\mathbf{H}F$ by freely adjoining objects and then morphisms between objects, and
(iii) the underlying vertical category $U\mathbf{V}\mathbb{B}$ is a retract of a category obtained from the image of $U\mathbf{V}\mathbb{A}$ under $U\mathbf{V}F$ by freely adjoining objects and morphisms.

Proof. By Theorem 7.3.4, we have that $F\colon \mathbb{A} \to \mathbb{B}$ is a cofibration in DblCat if and only if the functor $U\mathbf{H}F$ has the left lifting property with respect to surjective on objects and full functors, and the functor $U\mathbf{V}F$ has the left lifting property with respect to surjective on objects and surjective on morphisms functors. As in the proof of Corollary 6.2.3, we can see that $U\mathbf{H}F$ has the left lifting property with respect to surjective on objects and full functors if and only if F is injective on objects, faithful on horizontal morphisms, and (ii) is satisfied. It remains to show that $U\mathbf{V}F$ has the left lifting property with respect to surjective on objects and surjective on morphisms functors if and only if F is injective on objects, faithful on vertical morphisms, and (iii) is satisfied.

There is a cofibrantly generated weak factorization system $(\mathcal{L}, \mathcal{R})$ on Cat, where the right class $\mathcal{R}$ contains all surjective on objects and surjective on morphisms functors. A generating set of morphisms is given by the set containing the unique functor $\emptyset \to [0]$ and the unique functor $\emptyset \to [1]$. Then the left class $\mathcal{L}$ contains all functors $G\colon \mathcal{C} \to \mathcal{D}$ which are injective on objects and faithful on morphisms, and such that $\mathcal{D}$ is a retract of a category obtained from the image of $\mathcal{C}$ under G by freely adjoining objects and morphisms. This shows the desired result. $\square$

By applying this result to a double functor whose source is the empty double category $\emptyset$, we get the following characterization of cofibrant double categories.

Theorem 7.3.6. *A double category $\mathbb{A}$ is cofibrant in the model structure on* DblCat *of Theorem 7.1.3 if and only if*

(i) its underlying horizontal category $U\mathbf{H}\mathbb{A}$ is free, and
(ii) its underlying vertical category $U\mathbf{V}\mathbb{A}$ is a disjoint union of copies of $[0]$ *and* $[1]$.

Proof. By Corollary 7.3.5, the double functor $\emptyset \to \mathbb{A}$ is cofibrant if and only if the category $U\mathbf{H}\mathbb{A}$ is a retract of a free category and the category $U\mathbf{V}\mathbb{A}$ is a retract of a category which is a disjoint union of copies of $[0]$ and $[1]$. However, a retract of a free category is itself free, and a retract of a category which is a disjoint union of copies of $[0]$ and $[1]$ is itself a disjoint union of copies of $[0]$ and $[1]$. $\square$

By looking at the characterizations of fibrations and trivial fibrations in DblCat given in Propositions 7.2.9 and 7.2.10, we can describe new sets of generating cofibrations and generating trivial cofibrations by studying the lifting properties. These can be described as follows.

Notation 7.3.7. Let $\mathbb{S} = \mathbb{H}[1] \times \mathbb{V}[1]$ be the double category free on a square, $\delta\mathbb{S}$ be its boundary, and $\mathbb{S}_2$ be the double category free on two squares with the same boundaries.

$$\mathbb{S} = \begin{tikzcd} 0 \arrow[r] \arrow[d, "\bullet" marking] \arrow[dr, phantom, "\alpha"] & 1 \arrow[d, "\bullet" marking] \\ 0' \arrow[r] & 1' \end{tikzcd} \; ; \; \delta\mathbb{S} = \begin{tikzcd} 0 \arrow[r] \arrow[d, "\bullet" marking] & 1 \arrow[d, "\bullet" marking] \\ 0' \arrow[r] & 1' \end{tikzcd} \; ; \; \mathbb{S}_2 = \begin{tikzcd} 0 \arrow[r] \arrow[d, "\bullet" marking] \arrow[dr, phantom, "\alpha_0 \;\; \alpha_1"] & 1 \arrow[d, "\bullet" marking] \\ 0' \arrow[r] & 1' \end{tikzcd} .$$

Let $\mathcal{I}$ denote the set containing the following double functors:

(i) the unique double functor $I_1\colon \emptyset \to [0]$,

(ii) the inclusion double functor $I_2\colon [0] \sqcup [0] \to \mathbb{H}[1]$,
(iii) the inclusion double functor $I_3\colon \emptyset \to \mathbb{V}[1]$,
(iv) the inclusion double functor $I_4\colon \delta\mathbb{S} \to \mathbb{S}$,
(v) the double functor $I_5\colon \mathbb{S}_2 \to \mathbb{S}$ sending the two non trivial squares in $\mathbb{S}_2$ to the non trivial square of $\mathbb{S}$.

Let $\mathcal{J}$ denote the set containing the following double functors:

(i) the inclusion double functor $J_1\colon [0] \to \mathbb{H}E_{\mathrm{adj}}$, where the 2-category E_{adj} is the "free-living adjoint equivalence",
(ii) the inclusion double functor $J_2\colon \mathbb{H}[1] \to \mathbb{H}C_{\mathrm{inv}}$, where the 2-category C_{inv} is the "free-living 2-isomorphism",
(iii) the inclusion double functor $J_3\colon \mathbb{V}[1] \to \mathbb{H}E_{\mathrm{adj}} \times \mathbb{V}[1]$, where $\mathbb{H}E_{\mathrm{adj}} \times \mathbb{V}[1]$ can be described as the "free-living weakly horizontally invertible square with horizontal adjoint equivalence data".

Proposition 7.3.8. *The model structure on* DblCat *of Theorem 7.1.3 is cofibrantly generated, and sets of generating cofibrations and generating trivial cofibrations can be given by the sets $\mathcal{I}$ and $\mathcal{J}$, respectively, of Notation 7.3.7.*

Proof. Using the characterization of cofibrations of Corollary 7.3.5, it is straightforward to see that the double functors in $\mathcal{I}$ are cofibrations in DblCat. A direct computation shows that a double functor has the right lifting property with respect to each double functor in $\mathcal{I}$ precisely when it is surjective on objects, full on horizontal morphisms, surjective on vertical morphisms, and fully faithful on squares. By Proposition 7.2.10, such a double functor is precisely a trivial fibration in DblCat.

Similarly, one can check that the double functors in $\mathcal{J}$ are trivial cofibrations in DblCat and that a double functor has the right lifting property with respect to each double functor in $\mathcal{J}$ precisely when it satisfies (df1-3) of Definition 7.2.6. By Proposition 7.2.9, such a double functor is precisely a fibration in DblCat. □

7.4. **Quillen pairs between** DblCat **and** 2Cat**.** In this section, we show that the horizontal embedding $\mathbb{H}\colon 2\mathrm{Cat} \to \mathrm{DblCat}$ is both left and right Quillen, and that it is homotopically fully faithful. This shows that the homotopy theory of 2-categories is embedded into that of double categories in a reflective and co-reflective way. The relation is however even stronger: the model structure on 2Cat is both left- and right-induced along $\mathbb{H}$ from that on DblCat. This says that the functor $\mathbb{H}$ preserves and reflects the whole homotopical structure. Therefore, the model structure on DblCat is as compatible as possible with respect to the horizontal embedding $\mathbb{H}$.

Recall that $\mathbb{H}$ has both adjoints by Proposition 3.4.5. We first show that the functor $\mathbb{H}$ is a Quillen co-reflection (see Definition 4.4.8).

Theorem 7.4.1. *The adjunction*

$$\begin{tikzcd} \mathrm{DblCat} \arrow[r, bend right, "\mathbf{H}"'] \arrow[r, phantom, "\perp"] & 2\mathrm{Cat} \arrow[l, bend right, "\mathbb{H}"'] \end{tikzcd}$$

is a Quillen co-reflection, where 2Cat *is endowed with the model structure of Theorem 6.1.8 and* DblCat *is endowed with the model structure of Theorem 7.1.3.*

Proof. Since a double functor F is a weak equivalence (resp. fibration) in DblCat if and only if the 2-functors $\mathbf{H}F$ and $\mathcal{V}F$ are biequivalences (resp. Lack fibrations) in 2Cat, it is straightforward to see that $\mathbf{H}$ preserves weak equivalences and fibrations. Therefore, the functor $\mathbf{H}$ is right Quillen. Now let $\mathcal{A}$ be a 2-category. Since $\mathbf{H}\mathbb{H}\mathcal{A} = \mathcal{A}$, the unit $\eta_{\mathcal{A}}$ is given by the identity at $\mathcal{A}$. Since all objects are fibrant in DblCat, the components of the derived unit are given by the components of the unit at cofibrant 2-categories, which are identities. This shows that $\mathbb{H} \dashv \mathbf{H}$ is a Quillen co-reflection. $\square$

We do not expect the model structures on 2Cat and DblCat to be Quillen equivalent, as the homotopy theory of double categories should be richer than that of 2-categories. The following remark shows that this is indeed not the case.

Remark 7.4.2. The components of the derived counit of the adjunction $\mathbb{H} \dashv \mathbf{H}$ are not double biequivalences in general. To see this, consider the double category $\mathbb{V}[1]$ free on a vertical morphism. Then $\mathbb{V}[1]$ is fibrant, as every double category is fibrant, and we have that $\mathbf{H}\mathbb{V}[1] = [0] \sqcup [0]$. Since $[0] \sqcup [0]$ is cofibrant in 2Cat, the component of the derived counit at $\mathbb{V}[1]$ is given by the inclusion double functor $\mathbb{H}([0] \sqcup [0]) = [0] \sqcup [0] \to \mathbb{V}[1]$, which is not a double biequivalence, as it is does not satisfy (db3) of Definition 7.2.1.

Remark 7.4.3. Similarly, the adjunction $\mathbb{L} \dashv \mathcal{V}$ is also a Quillen pair between DblCat and 2Cat, as shown in [MSV20a, Proposition 6.15]. However, it is neither a Quillen co-reflection nor a Quillen reflection by [MSV20a, Remarks 6.16 and 6.17].

We now show that the horizontal embedding $\mathbb{H}$ is also a Quillen reflection.

Theorem 7.4.4. *The adjunction*

$$\begin{tikzcd} \mathrm{2Cat} \arrow[r, bend right, "\mathbb{H}"'] \arrow[r, phantom, "\perp"] & \mathrm{DblCat} \arrow[l, bend right, "L"'] \end{tikzcd}$$

is a Quillen reflection, where 2Cat *is endowed with the model structure of Theorem 6.1.8 and* DblCat *is endowed with the model structure of Theorem 7.1.3.*

Proof. We show that $\mathbb{H}\colon \mathrm{2Cat} \to \mathrm{DblCat}$ preserves fibrations and trivial fibrations. Let $F\colon \mathcal{A} \to \mathcal{B}$ be a Lack fibration in 2Cat. We show that $\mathbb{H}F\colon \mathbb{H}\mathcal{A} \to \mathbb{H}\mathcal{B}$ is a fibration in DblCat by proving that it satisfies (df1-3) of Definition 7.2.6 (see Proposition 7.2.9). Since $\mathbf{H}\mathbb{H}F = F$ is a Lack fibration, then $\mathbb{H}F$ satisfies (df1-2) by Remark 7.2.7. It remains to prove (df3). Let $C \in \mathcal{A}$ be an object and suppose we have a weakly horizontally invertible square β in $\mathbb{H}\mathcal{B}$ as follows.

$$\begin{tikzcd} B \arrow[r, "b", "\simeq"'] \arrow[d, equal] \arrow[dr, phantom, "\beta \simeq"] & FC \arrow[d, equal] \\ B \arrow[r, "\simeq", "d"'] & FC \end{tikzcd}$$

By Proposition 3.6.7, the weakly horizontally invertible square β in $\mathbb{H}\mathcal{B}$ corresponds to a 2-isomorphism $\beta\colon b \cong d$. By (f1) of Definition 6.1.7, there is an equivalence $c\colon A \xrightarrow{\simeq} C$ in $\mathcal{A}$ such that $d = Fc$. Then, by (f2) of Definition 6.1.7 applied to the 2-isomorphism $\beta\colon b \cong Fc$, there is a 2-isomorphism $\alpha\colon a \cong c$ in $\mathcal{A}$ such that $\beta = F\alpha$. By Proposition 3.6.7, this equivalently gives a weakly horizontally invertible square α in $\mathbb{H}\mathcal{A}$

$$\begin{tikzcd} A \arrow[r, "a", "\simeq"'] \arrow[d, equal] \arrow[dr, phantom, "\alpha \simeq"] & C \arrow[d, equal] \\ A \arrow[r, "\simeq", "c"'] & C \end{tikzcd}$$

such that $\beta = (\mathbb{H}F)\alpha$, which proves (df3). This shows that $\mathbb{H}F$ is a double fibration in DblCat and that $\mathbb{H}$ preserves fibrations.

Now, let $F\colon \mathcal{A} \to \mathcal{B}$ be a trivial fibration in 2Cat. By Proposition 6.1.11, it is surjective on objects, full on morphisms, and fully faithful on 2-morphisms. Then the double functor $\mathbb{H}F\colon \mathbb{H}\mathcal{A} \to \mathbb{H}\mathcal{B}$ is surjective on objects, full on horizontal morphisms, and fully faithful on squares, as these are given by the objects, morphisms, and 2-morphisms in $\mathcal{A}$ and $\mathcal{B}$, respectively. It is further surjective on vertical morphisms, as $\mathbb{H}\mathcal{A}$ and $\mathbb{H}\mathcal{B}$ have only trivial vertical morphisms, and F is surjective on objects. By Proposition 7.2.10, this shows that $\mathbb{H}F$ is a trivial fibration in DblCat. Therefore, the functor $\mathbb{H}$ is right Quillen.

Let $\mathcal{A}$ be a 2-category. Since the functor $\mathbb{H}$ is right Quillen and all 2-categories are fibrant, it preserves weak equivalences by Corollary 4.4.7. Moreover, by Theorem 7.4.1, the functor $\mathbb{H}$ is left Quillen and therefore preserves cofibrant objects. This shows that a cofibrant replacement of $\mathbb{H}\mathcal{A}$ in DblCat can be given by $\mathbb{H}(q_{\mathcal{A}})\colon \mathbb{H}(\mathcal{A}^c) \to \mathbb{H}\mathcal{A}$, where $q_{\mathcal{A}}\colon \mathcal{A}^c \to \mathcal{A}$ is a cofibrant replacement of $\mathcal{A}$ in 2Cat. Furthermore, the counit $\epsilon_{\mathcal{A}}$ of the adjunction $L \dashv \mathbb{H}$ at any 2-category $\mathcal{A}$ is an isomorphism by Proposition 3.4.5. Therefore, we have a commutative square in 2Cat

$$\begin{tikzcd} L\mathbb{H}(\mathcal{A}^c) \arrow[r, "L\mathbb{H}(q_{\mathcal{A}})"] \arrow[d, "\epsilon_{\mathcal{A}^c}"', "\cong"] & L\mathbb{H}\mathcal{A} \arrow[d, "\epsilon_{\mathcal{A}}", "\cong"'] \\ \mathcal{A}^c \arrow[r, "\sim", "q_{\mathcal{A}}"'] & \mathcal{A} \end{tikzcd}$$

and, by 2-out-of-3, we conclude that the derived counit $\epsilon_{\mathcal{A}} \circ L\mathbb{H}(q_{\mathcal{A}})$ is a biequivalence. This shows that $L \dashv \mathbb{H}$ is a Quillen reflection. $\square$

As before, the following remark shows that the adjunction $L \dashv \mathbb{H}$ is not a Quillen equivalence.

Remark 7.4.5. The components of the derived unit of the adjunction $L \dashv \mathbb{H}$ are not double biequivalences in general. To see this, consider the double category $\mathbb{V}[1]$ free on a vertical morphism. Then $\mathbb{V}[1]$ is cofibrant in DblCat by Theorem 7.3.6, and we have that $\mathbb{H}L\mathbb{V}[1] = [0]$ since the functor L collapses the vertical direction (see Definition 3.4.4). Since all objects are fibrant in 2Cat, the component of the derived unit at $\mathbb{V}[1]$ is given by the unique double functor $\mathbb{V}[1] \to [0]$. This is not a double biequivalence, as it identifies the two objects of $\mathbb{V}[1]$ which are not related by a horizontal equivalence and therefore does not satisfy (db2) of Definition 7.2.1.

Theorems 7.4.1 and 7.4.4 imply that the functor $\mathbb{H}\colon \text{2Cat} \to \text{DblCat}$ preserves cofibrations, fibrations, and weak equivalences (since every 2-category is fibrant). We now show that $\mathbb{H}$ also reflects cofibrations, fibrations, and weak equivalences.

Theorem 7.4.6. *The model structure on* 2Cat *of Theorem 6.1.8 is right-induced along the adjunction*

$$\begin{tikzcd} \text{2Cat} \arrow[r, bend right, "\mathbb{H}"'] \arrow[r, phantom, "\perp"] & \text{DblCat} \arrow[l, bend right, "L"'] \end{tikzcd}$$

from the model structure on DblCat *of Theorem 7.1.3.*

Proof. We need to show that a 2-functor F is a biequivalence (resp. Lack fibration) in 2Cat if and only if the double functor $\mathbb{H}F$ is a weak equivalence (resp. fibration) in DblCat. Since the functor $\mathbb{H}$ is right Quillen by Theorem 7.4.4, it preserves fibrations and, since all objects in 2Cat are fibrant, it preserves weak equivalences by Corollary 4.4.7. This shows that, if F is a biequivalence (resp. Lack fibration) in 2Cat, then $\mathbb{H}F$ is a weak equivalence (resp. fibration) in DblCat. Conversely, if $\mathbb{H}F$ is a weak equivalence (resp. fibration) in DblCat, then $\mathbf{H}\mathbb{H}F = F$ is a biequivalence (resp. Lack fibration) in 2Cat by definition of the weak equivalences (resp. fibrations) in the right-induced model structure on DblCat of Theorem 7.1.3. Therefore, the model structure on 2Cat is right-induced along $\mathbb{H}$ from that on DblCat. ☐

Theorem 7.4.7. *The model structure on* 2Cat *of Theorem 6.1.8 is left-induced along the adjunction*

$$\begin{tikzcd} \mathrm{DblCat} \arrow[r, bend right, "\mathbf{H}"'] \arrow[r, phantom, "\perp"] & 2\mathrm{Cat} \arrow[l, bend right, "\mathbb{H}"'] \end{tikzcd}$$

from the model structure on DblCat *of Theorem 7.1.3.*

Proof. By Theorem 7.4.6, a 2-functor F is a biequivalence in 2Cat if and only if the double functor $\mathbb{H}F$ is a weak equivalence in DblCat. Therefore, it remains to show that a 2-functor F is a cofibration in 2Cat if and only if the double functor $\mathbb{H}F$ is a cofibration in DblCat. By Theorem 7.4.1, the functor $\mathbb{H}$ is left Quillen, and therefore, if F is a cofibration in 2Cat, then $\mathbb{H}F$ is a cofibration in DblCat. Now suppose that $F\colon \mathcal{A} \to \mathcal{B}$ is a 2-functor such that $\mathbb{H}F\colon \mathbb{H}\mathcal{A} \to \mathbb{H}\mathcal{B}$ is a cofibration in DblCat. Let $P\colon \mathcal{X} \to \mathcal{Y}$ be a trivial fibration in 2Cat. We need to show that there is a lift in every commutative diagram as below left.

$$\begin{tikzcd} \mathcal{A} \arrow[r] \arrow[d, "F"'] & \mathcal{X} = \mathbf{H}\mathbb{H}\mathcal{X} \arrow[d, "P = \mathbf{H}\mathbb{H}P"] \\ \mathcal{B} \arrow[r] \arrow[ur, dashed] & \mathcal{Y} = \mathbf{H}\mathbb{H}\mathcal{Y} \end{tikzcd} \qquad \begin{tikzcd} \mathbb{H}\mathcal{A} \arrow[r] \arrow[d, "\mathbb{H}F"'] & \mathbb{H}\mathcal{X} \arrow[d, "\mathbb{H}P"] \\ \mathbb{H}\mathcal{B} \arrow[r] \arrow[ur, dashed] & \mathbb{H}\mathcal{Y} \end{tikzcd}$$

By the adjunction $\mathbb{H} \dashv \mathbf{H}$, such a lift exists if and only if there is a lift in the above right commutative diagram. Since $\mathbb{H}$ is right Quillen by Theorem 7.4.4, then $\mathbb{H}P$ is a trivial fibration in DblCat. Therefore, there is a lift in the above right diagram as $\mathbb{H}F$ is a cofibration. This shows that F is a cofibration in 2Cat. We conclude that the model structure on 2Cat is left-induced along $\mathbb{H}$ from that on DblCat. ☐

We now turn our attention to the relation between the canonical model structure on Cat and the model structure on DblCat. Recall the Quillen reflection $P \dashv D$ between Cat and 2Cat (see Theorem 6.1.14). By composing with the Quillen reflection $L \dashv \mathbb{H}$ between 2Cat and DblCat, we get the following result.

Corollary 7.4.8. *The adjunction*

$$\begin{tikzcd} \mathrm{Cat} \arrow[r, bend right, "\mathbb{H}D"'] \arrow[r, phantom, "\perp"] & \mathrm{DblCat} \arrow[l, bend right, "PL"'] \end{tikzcd}$$

is a Quillen reflection, where Cat *is endowed with the model structure of Theorem 6.1.3 and* DblCat *is endowed with the model structure of Theorem 7.1.3.*

Proof. This follows directly from Theorems 7.4.4 and 6.1.14. ☐

As the canonical model structure on Cat is right-induced along D from the model structure on 2Cat and 2Cat is right-induced along $\mathbb{H}$ from the model structure on DblCat, we get the following result.

Corollary 7.4.9. *The model structure on* Cat *of Theorem 6.1.3 is right-induced along the adjunction*

$$\begin{tikzcd} \mathrm{Cat} \arrow[r, bend right, "\mathbb{H}D"'] \arrow[r, phantom, "\perp"] & \mathrm{DblCat} \arrow[l, bend right, "PL"'] \end{tikzcd}$$

from the model structure on DblCat *of Theorem 7.1.3.*

Proof. This follows directly from Theorems 7.4.6 and 6.1.16. □

7.5. 2Cat-**enrichment.** In analogy with the model structure on 2Cat, the model structure on DblCat is not monoidal with respect to the cartesian product. However, it is not monoidal with respect to the Gray tensor product $\otimes_{\mathrm{Gr}}$ of Proposition 3.3.5 on DblCat either. This is a consequence of the fact that the model structure on DblCat is not symmetric between the horizontal and vertical directions, while the Gray tensor product is. To remedy this issue, we consider a less symmetric version of the Gray tensor product by restricting along the horizontal embedding in one variable, i.e., we consider the tensoring functor $\otimes\colon \mathrm{DblCat}\times 2\mathrm{Cat}\to \mathrm{DblCat}$ defined in Definition 3.5.1. We show that the model structure on DblCat is 2Cat-enriched with respect to $\otimes$, where the 2Cat-enrichment is given by the underlying horizontal 2-categories $\mathbf{H}[-,-]_{\mathrm{ps}}$ of the pseudo-hom double categories (see Definition 3.3.4).

We first show that the model structure on DblCat is not cartesian closed, by using a similar argument than that of Remark 6.3.1.

Remark 7.5.1. The model structure on DblCat of Theorem 7.1.3 is not monoidal with respect to the cartesian product. Given the generating cofibration $I_2\colon [0]\sqcup[0]\to \mathbb{H}[1]$, then the pushout-product

$$I_2\,\square_\times\, I_2\colon \mathbb{H}[1]\sqcup\mathbb{H}[1] \coprod_{[0]\sqcup[0]\sqcup[0]\sqcup[0]} \mathbb{H}[1]\sqcup\mathbb{H}[1]\to \mathbb{H}[1]\times\mathbb{H}[1]$$

is not a cofibration in DblCat since it is not faithful on horizontal morphism (see Corollary 7.3.5). Indeed, it sends the two horizontal composites $(1,f)(f,0)$ and $(f,1)(0,f)$ of the domain to the same horizontal morphism $(1,f)(f,0)=(f,1)(0,f)$ in the codomain. See Remark 6.3.1 for more details.

While the model structure on 2Cat is monoidal with respect to the Gray tensor product $\otimes_2$ (see Theorem 6.3.5), it is not the case that the model structure on DblCat is monoidal with respect to the double categorical analogue $\otimes_{\mathrm{Gr}}$ of the Gray tensor product, defined in Proposition 3.3.5.

Remark 7.5.2. The model structure on DblCat of Theorem 7.1.3 is not monoidal with respect to the Gray tensor product $\otimes_{\mathrm{Gr}}$. Given the generating cofibration $I_3\colon \emptyset\to\mathbb{V}[1]$, the pushout-product

$$I_3\,\square_{\otimes_{\mathrm{Gr}}}\, I_3\colon \emptyset\to \mathbb{V}[1]\otimes_{\mathrm{Gr}}\mathbb{V}[1]$$

is not a cofibration in DblCat. To see this, note that the underlying vertical category $\mathbb{V}[1]\otimes_{\mathrm{Gr}}\mathbb{V}[1]$ is the free category on a non-commutative square of morphisms. However, this is not a disjoint union of copies of $[0]$ and $[1]$, and therefore $\mathbb{V}[1]\otimes_{\mathrm{Gr}}\mathbb{V}[1]$ is not cofibrant by Theorem 7.3.6.

The rest of the section is devoted to the proof of the 2Cat-enrichment of the model structure on DblCat, as given in the following theorem.

Theorem 7.5.3. *The model structure on* DblCat *of Theorem 7.1.3 is* 2Cat-*enriched, where the enrichment is given by* $\mathbf{H}[-,-]_{\mathrm{ps}}$.

We show this result by showing that the pushout-product $I\,\square_\otimes\, i$ of a cofibration I in DblCat with a cofibration i in 2Cat is a cofibration in DblCat, which is trivial if one of I or i is. By Remark 4.5.7, it is enough to show this result when I is a generating cofibration or a generating trivial cofibration in DblCat. Recall the sets

$$\mathcal{I}' = \{\mathbb{H}i, \mathbb{L}i = \mathbb{H}i \times \mathbb{V}[1] \mid i \in \mathcal{I}_2\} \text{ and } \mathcal{J}' = \{\mathbb{H}j, \mathbb{L}j = \mathbb{H}j \times \mathbb{V}[1] \mid j \in \mathcal{J}_2\},$$

of generating cofibrations and generating trivial cofibrations in DblCat given in Proposition 7.3.1, where $\mathcal{I}_2$ and $\mathcal{J}_2$ denote sets of generating cofibrations and generating trivial cofibrations in 2Cat. We first prove the following technical result, which allows us to compute the pushout-product $I \,\square_\otimes\, i$ of a double functor I of the form $\mathbb{H}i'$ or $\mathbb{L}i'$, for i' a 2-functor, with a 2-functor i.

Lemma 7.5.4. *Let $i\colon \mathcal{A} \to \mathcal{B}$ and $i'\colon \mathcal{A}' \to \mathcal{B}'$ be 2-functors. Then*

(i) there is an isomorphism $\mathbb{H}i' \,\square_\otimes\, i \cong \mathbb{H}(i' \,\square_{\otimes_2}\, i)$ in the category of morphisms $\mathrm{DblCat}^{[1]}$,

(ii) there is an isomorphism $\mathbb{L}i' \,\square_\otimes\, i \cong \mathbb{H}(i' \,\square_{\otimes_2}\, i) \times \mathbb{V}[1] = \mathbb{L}(i' \,\square_{\otimes_2}\, i)$ in the category of morphisms $\mathrm{DblCat}^{[1]}$.

Proof. Since the functor $\mathbb{H}$ is a left adjoint, it preserves pushouts. Moreover, by Corollary 3.5.7, we have isomorphisms $\mathbb{H}\mathcal{A} \otimes \mathcal{C} \cong \mathbb{H}(\mathcal{A} \otimes_2 \mathcal{C})$ natural in $\mathcal{A}$ and $\mathcal{C}$, for every pair of 2-categories $\mathcal{A}$ and $\mathcal{C}$. Therefore, we directly get the isomorphism $\mathbb{H}i' \,\square_\otimes\, i \cong \mathbb{H}(i' \,\square_{\otimes_2}\, i)$, which proves (i). Then (ii) follows from the following sequence of isomorphisms

$$\begin{aligned}
\mathbb{L}i' \,\square_\otimes\, i &\cong (\mathbb{V}[1] \times \mathbb{H}i') \,\square_\otimes\, i \cong (\mathbb{V}[1] \otimes i') \,\square_\otimes\, i \\
&= (\mathbb{V}[1] \otimes_{\mathrm{Gr}} \mathbb{H}i') \,\square_\otimes\, i \cong \mathbb{V}[1] \otimes_{\mathrm{Gr}} (\mathbb{H}i' \,\square_\otimes\, i) \\
&\cong \mathbb{V}[1] \otimes_{\mathrm{Gr}} \mathbb{H}(i' \,\square_{\otimes_2}\, i) = \mathbb{V}[1] \otimes (i' \,\square_{\otimes_2}\, i) \\
&\cong \mathbb{V}[1] \times \mathbb{H}(i' \,\square_{\otimes_2}\, i) \cong \mathbb{L}(i' \,\square_{\otimes_2}\, i),
\end{aligned}$$

which hold by definition of $\mathbb{L} = \mathbb{H}(-) \times \mathbb{V}[1] \cong \mathbb{V}[1] \times \mathbb{H}(-)$, by Remark 3.5.5, which says that there is an isomorphism $\mathcal{C} \otimes \mathbb{V}[1] \cong \mathbb{H}\mathcal{C} \times \mathbb{V}[1]$ natural in $\mathcal{C}$, for every 2-category $\mathcal{C}$, by associativity and symmetry of the Gray tensor product $\otimes_{\mathrm{Gr}}$, by definition of the tensor $\otimes$, and by point (i). □

Proof of Theorem 7.5.3. We need to show that the pushout-product $I \,\square_\otimes\, i$ of a generating cofibration $I \in \mathcal{I}'$ with a cofibration i in 2Cat is a cofibration in DblCat, which is trivial if either $I \in \mathcal{J}'$ is a generating trivial cofibration or i is a biequivalence, where $\mathcal{I}'$ and $\mathcal{J}'$ are the generating sets of cofibrations and trivial cofibrations of Proposition 7.3.1 in DblCat. Suppose first that $I \in \mathcal{I}'$. Then I is of the form $\mathbb{H}i'$ or $\mathbb{L}i'$ for some generating cofibration i' in 2Cat. By Lemma 7.5.4, we have isomorphisms

$$\mathbb{H}i' \,\square_\otimes\, i \cong \mathbb{H}(i' \,\square_{\otimes_2}\, i) \text{ and } \mathbb{L}i' \,\square_\otimes\, i \cong \mathbb{L}(i' \,\square_{\otimes_2}\, i).$$

Since the model structure on 2Cat is monoidal with respect to $\otimes_2$ by Theorem 6.3.5, the pushout-product $i' \,\square_{\otimes_2}\, i$ is a cofibration in 2Cat, which is trivial when i is a biequivalence. By Theorem 7.4.1 and Remark 7.4.3, the functors $\mathbb{H}$ and $\mathbb{L}$ are left Quillen, and therefore preserve cofibrations and trivial cofibrations. This shows that $I \,\square_\otimes\, i$ is a cofibration in DblCat, which is trivial when i is a biequivalence, for all $I \in \mathcal{I}'$. Now suppose that $I \in \mathcal{J}'$. Then I is of the form $\mathbb{H}j$ or $\mathbb{L}j$ for some generating trivial cofibration j in 2Cat. Similarly, we can show that $\mathbb{H}j \,\square_\otimes\, i \cong \mathbb{H}(j \,\square_{\otimes_2}\, i)$ and $\mathbb{L}j \,\square_\otimes\, i \cong \mathbb{L}(j \,\square_{\otimes_2}\, i)$ are trivial cofibrations in DblCat. This shows that the model structure on DblCat is 2Cat-enriched. □

8. The second model structure for double categories

The asymmetry between the horizontal and vertical direction in the first model structure on DblCat constructed above can be noticed using the fact that the inclusion $[0] \sqcup [0] \to \mathbb{H}[1]$ of the two end-points in the horizontal morphism is a cofibration, while the transposed inclusion $[0] \sqcup [0] \to \mathbb{V}[1]$ is not. We therefore want to add this latter to our class of cofibrations. Furthermore, as mentioned in the introduction, we aim to get a model structure on DblCat in which the fibrant objects are precisely the weakly horizontally invariant double categories. Hence, we construct a weakly horizontally invariant replacement of double categories, and define a double functor to be a weak equivalence if it induces a double biequivalence between weakly horizontally invariant replacements. The new class of cofibrations, generated by the set $\mathcal{I}$ of cofibrations of Notation 7.3.7 and the additional inclusion $[0] \sqcup [0] \to \mathbb{V}[1]$, together with the class of weak equivalences described above exists. Its proof is the content of Sections 8.1 to 8.3.

In Section 8.1, we first describe the cofibrations, trivial fibrations, and weak equivalences of the desired model structure. In particular, the cofibrant objects are now the double categories which have free underlying horizontal *and* vertical categories, which shows that this model structure is more symmetric than the first one. We also show that the class of double biequivalences is contained in the class of weak equivalences of this model structure. Then, in Section 8.2, we introduce a set $\mathcal{J}_w$ of trivial cofibrations, which is such that the trivial cofibrations and fibrations between weakly horizontally invariant double categories are precisely the $\mathcal{J}_w$-cofibrations and $\mathcal{J}_w$-injectives, respectively. This result is useful in Section 8.3 to prove that the classes of trivial cofibrations and fibrations in the proposed model structure on DblCat form a weak factorization system. We further show that the fibrant objects are precisely the weakly horizontally invariant double categories, as desired.

Then, in Section 8.4, we compare this new model structure with the model structures previously introduced in Sections 6 and 7. We show that the identity on DblCat gives a homotopically full embedding of the model structure for weakly horizontally invariant double categories into the model structure for double categories constructed in Theorem 7.1.3. The horizontal embedding $\mathbb{H}\colon 2\mathrm{Cat} \to \mathrm{DblCat}$ is still left Quillen from Lack's model structure to the model structure for weakly horizontally invariant double categories, however it is not right Quillen anymore. This is due for instance to the fact that the horizontal double category associated to a 2-category is not weakly horizontally invariant in general, and hence that $\mathbb{H}$ does not preserve fibrant objects. Instead, we consider the homotopical version $\mathbb{H}^{\simeq}\colon 2\mathrm{Cat} \to \mathrm{DblCat}$ of the horizontal embedding introduced in Definition 3.4.11, and we show that it gives the desired homotopically full embedding of 2Cat into DblCat. In particular, for every 2-category $\mathcal{A}$, the double category $\mathbb{H}^{\simeq}\mathcal{A}$ provides a fibrant replacement of the horizontal double category $\mathbb{H}\mathcal{A}$. We also show that Lack's model structure on 2Cat is right-induced along $\mathbb{H}^{\simeq}$ from the model structure on DblCat for weakly horizontally invariant double categories.

Finally, in Section 8.5, we show that this model structure is monoidal with respect to the Gray tensor product for double categories. In comparison with the first model structure on DblCat constructed in Section 7, the cofibrations of this new model structure can be characterized by a symmetric condition on their underlying horizontal and vertical functors, and hence the arguments showing that the first model structure is not monoidal, do not hold anymore. In particular, it is also 2Cat-enriched for the enrichment of DblCat of Proposition 3.5.2.

The results presented here are joint work with Maru Sarazola and Paula Verdugo, and will appear in a forthcoming version of the paper [MSV20b].

8.1. **Description of the model structure.** We now want to define a model structure on DblCat, whose fibrant objects are the weakly horizontally invariant double categories, since these are the double categories whose nerve is fibrant. For this, we first determine the class of cofibrations, to which we add the inclusion $[0]\sqcup[0] \to \mathbb{V}[1]$ of the two end-points into the vertical morphism. This yields the following set of generating cofibrations.

Notation 8.1.1. Recall from Notation 7.3.7 that $\mathbb{S} = \mathbb{H}[1] \times \mathbb{V}[1]$ denotes the double category free on a square, $\delta\mathbb{S}$ denotes its boundary, and $\mathbb{S}_2$ denotes the double category free on two squares with the same boundaries. Let $\mathcal{I}_w$ denote the set containing the following double functors:

(i) the unique double functor $I_1 \colon \emptyset \to [0]$,
(ii) the inclusion double functor $I_2 \colon [0] \sqcup [0] \to \mathbb{H}[1]$,
(iii) the inclusion double functor $I_3' \colon [0] \sqcup [0] \to \mathbb{V}[1]$,
(iv) the inclusion double functor $I_4 \colon \delta\mathbb{S} \to \mathbb{S}$,
(v) the double functor $I_5 \colon \mathbb{S}_2 \to \mathbb{S}$ sending the two non trivial squares in $\mathbb{S}_2$ to the non trivial square of $\mathbb{S}$.

Having this set of generating cofibrations in hands, we already know that the cofibrations in our model structure are going to be the $\mathcal{I}_w$-cofibrations, and that the trivial fibrations are going to be the $\mathcal{I}_w$-injectives. Let us first give a description of these classes of double functors.

Proposition 8.1.2. *A double functor $F \colon \mathbb{A} \to \mathbb{B}$ is in $\mathcal{I}_w-\mathrm{inj}$ if and only if it is surjective on objects, full on horizontal morphisms, full on vertical morphisms, and fully faithful on squares.*

Proof. A double functor F has the right lifting property with respect to $I_1 \colon \emptyset \to [0]$ if and only if it is surjective on objects; with respect to $I_2 \colon [0] \sqcup [0] \to \mathbb{H}[1]$ if and only if it is full on horizontal morphisms; with respect to $I_3' \colon [0] \sqcup [0] \to \mathbb{V}[1]$ if and only if it is full on vertical morphisms; and with respect to $I_4 \colon \delta\mathbb{S} \to \mathbb{S}$ and $I_5 \colon \mathbb{S}_2 \to \mathbb{S}$ if and only if it is full and faithful on squares, respectively. This shows the result. ☐

Remark 8.1.3. It is straightforward to see that every double functor in $\mathcal{I}_w-\mathrm{inj}$ trivially satisfies (db1-4) of Definition 7.2.1, using this description. Hence every double functor in $\mathcal{I}_w-\mathrm{inj}$ is a double biequivalence.

Theorem 8.1.4. *A double functor $F \colon \mathbb{A} \to \mathbb{B}$ is in $\mathcal{I}_w-\mathrm{cof}$ if and only if its underlying horizontal and vertical functors $U\mathbf{H}F \colon U\mathbf{H}\mathbb{A} \to U\mathbf{H}\mathbb{B}$ and $U\mathbf{V}F \colon U\mathbf{V}\mathbb{A} \to U\mathbf{V}\mathbb{B}$ have the left lifting property with respect to surjective on objects and full functors.*

Proof. The proof works as in Theorem 7.3.4, except that now a trivial fibration is full on vertical morphism instead of surjective, and therefore the underlying vertical functor $U\mathbf{V}F$ has the left lifting property with respect to surjective on objects and full functors instead of surjective on objects and surjective on morphisms functors. ☐

Corollary 8.1.5. *A double functor $F \colon \mathbb{A} \to \mathbb{B}$ is in $\mathcal{I}_w-\mathrm{cof}$ if and only if*

(i) it is injective on objects, and faithful on horizontal and vertical morphisms,
(ii) the underlying horizontal category $U\mathbf{H}\mathbb{B}$ is a retract of a category obtained from the image of $U\mathbf{H}\mathbb{A}$ under $U\mathbf{H}F$ by freely adjoining objects and then morphisms between objects, and
(iii) the underlying vertical category $U\mathbf{V}\mathbb{B}$ is a retract of a category obtained from the image of $U\mathbf{V}\mathbb{A}$ under $U\mathbf{V}F$ by freely adjoining objects and then morphisms between objects.

Proof. The proof works as in Corollary 7.3.5, with the modifications imposed by Theorem 8.1.4. ☐

As a direct consequence of this result, we get that the cofibrant objects are precisely the double categories whose underlying horizontal *and* vertical categories are free. In comparison with the cofibrant objects of the first model structure, characterized in Theorem 7.3.6, the underlying vertical category of a cofibrant double category can now contain free composites of morphisms.

Corollary 8.1.6. *A double category $\mathbb{A}$ is cofibrant if and only if its underlying horizontal and vertical categories $U\mathbf{H}\mathbb{A}$ and $U\mathbf{V}\mathbb{A}$ are free.*

Proof. By Corollary 8.1.5, the double functor $\emptyset \to \mathbb{A}$ is cofibrant if and only if the categories $U\mathbf{H}\mathbb{A}$ and $U\mathbf{V}\mathbb{A}$ are retracts of free categories. However, a retract of a free category is itself free. ☐

It remains to introduce the class of weak equivalences in order to determine a model structure on DblCat. We define a weak equivalence to be a double functor that induces a double biequivalence between weakly horizontally invariant replacements. To construct a weakly horizontally invariant double category from a double category $\mathbb{A}$, we attach $\mathbb{H}^{\simeq}E_{\mathrm{adj}}$-data freely to every horizontal adjoint equivalence in $\mathbb{A}$. Let us first give a precise description of the double category $\mathbb{H}^{\simeq}E_{\mathrm{adj}}$.

Description 8.1.7. We first describe the double category $\mathbb{H}^{\simeq}E_{\mathrm{adj}}$. It contains a horizontal adjoint equivalence (f, g, η, ϵ), where $f\colon 0 \to 1$ and $g\colon 1 \to 0$ are horizontal morphisms, and η and ϵ are vertically invertible squares, as in Definition 3.6.1, which represents the unit and counit of the horizontal adjoint equivalence. It also contains two vertical morphisms $u\colon 0 \to 1$ and $v\colon 1 \to 0$. The squares are generated by η and ϵ, as well as the two following weakly horizontally invertible squares.

$$
\begin{tikzcd}
0 \arrow[r, "f", "\simeq"'] \arrow[d, "u"', "\bullet" marking] \arrow[dr, phantom, "\alpha \simeq"] & 1 \arrow[d, equals, "\bullet" marking] \\
1 \arrow[r, equals] & 1
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
1 \arrow[r, "g", "\simeq"'] \arrow[d, "v"', "\bullet" marking] \arrow[dr, phantom, "\gamma \simeq"] & 0 \arrow[d, equals, "\bullet" marking] \\
0 \arrow[r, equals] & 0
\end{tikzcd}
$$

Let us denote by α' and γ' their weak inverses, given by Proposition 3.6.6, with respect to the identity horizontal adjoint equivalence and the horizontal adjoint equivalence (f, g, η, ϵ). Then, we can form the following weakly horizontally invertible squares β and δ, and we also denote by β' and δ' their weak inverses, given by Proposition 3.6.6, with respect to the identity horizontal adjoint equivalence and the horizontal adjoint equivalence (f, g, η, ϵ).

$$
\begin{tikzcd}
0 \arrow[r, equals] \arrow[d, equals, "\bullet" marking] \arrow[dr, phantom, "\beta \simeq"] & 0 \arrow[d, "u", "\bullet" marking] \\
0 \arrow[r, "\simeq", "f"'] & 1
\end{tikzcd}
=
\begin{tikzcd}
0 \arrow[rr, equals] \arrow[d, equals, "\bullet" marking] & \arrow[d, phantom, "\eta \ \cong"] & 0 \arrow[d, equals, "\bullet" marking] \\
0 \arrow[r, "f", "\simeq"'] \arrow[d, equals, "\bullet" marking] \arrow[dr, phantom, "e_f"] & 1 \arrow[r, "g", "\simeq"'] \arrow[d, equals, "\bullet" marking] \arrow[dr, phantom, "\alpha' \simeq"] & 0 \arrow[d, "u", "\bullet" marking] \\
0 \arrow[r, "\simeq", "f"'] & 1 \arrow[r, equals] & 1
\end{tikzcd}
\qquad
\begin{tikzcd}
1 \arrow[r, equals] \arrow[d, equals, "\bullet" marking] \arrow[dr, phantom, "\delta \simeq"] & 1 \arrow[d, "v", "\bullet" marking] \\
1 \arrow[r, "\simeq", "g"'] & 0
\end{tikzcd}
=
\begin{tikzcd}
1 \arrow[rr, equals] \arrow[d, equals, "\bullet" marking] & \arrow[d, phantom, "\epsilon^{-1} \cong"] & 1 \arrow[d, equals, "\bullet" marking] \\
1 \arrow[r, "g", "\simeq"'] \arrow[d, equals, "\bullet" marking] \arrow[dr, phantom, "e_g"] & 0 \arrow[r, "f", "\simeq"'] \arrow[d, equals, "\bullet" marking] \arrow[dr, phantom, "\gamma' \simeq"] & 1 \arrow[d, "v", "\bullet" marking] \\
1 \arrow[r, "\simeq", "g"'] & 0 \arrow[r, equals] & 0
\end{tikzcd}
$$

Moreover, the horizontal composite of β with α is the vertical identity square e_f at f, and the vertical composite of β with α is the horizontal identity square id_u at u. In other words, this says that (f, u, α, β) is the data of an *orthogonal companion pair*; see [Gra20, §4.1.1]. On the other hand, the horizontal composite of α' with β' is the vertical identity square e_g at g, and the vertical composite of β' with α' is the horizontal identity square id_u at u. In other words, this says that (g, u, α', β') is the data of an *orthogonal adjoint pair*; see [Gra20, §4.1.2]. Similarly, (g, v, γ, δ) is the data of an orthogonal companion pair, and (f, v, γ', δ') is the data of an orthogonal adjoint pair.

The pair of vertical morphisms (u, v) further forms a vertical adjoint equivalence, i.e., an adjoint equivalence in the underlying vertical 2-category $\mathbf{V}\mathbb{H}^{\simeq}E_{\mathrm{adj}}$, with unit η' given by the vertical composite of β with γ', and counit ϵ' given by the vertical composite of δ' with α. In particular, all the squares in $\mathbb{H}^{\simeq}E_{\mathrm{adj}}$ are also *weakly vertically invertible* – the transposed notion of weakly horizontally invertible – with vertical weak inverses given by the obvious square.

Note that there is an inclusion $J_4 \colon \mathbb{H}E_{\mathrm{adj}} \to \mathbb{H}^{\simeq}E_{\mathrm{adj}}$ which sends the horizontal adjoint equivalence in $\mathbb{H}E_{\mathrm{adj}}$ to the horizontal adjoint equivalence (f,g,η,ϵ) in $\mathbb{H}^{\simeq}E_{\mathrm{adj}}$.

Remark 8.1.8. Let $\mathbb{A}$ be a double category. Given a double functor $G \colon \mathbb{H}^{\simeq}E_{\mathrm{adj}} \to \mathbb{A}$, its data is completely determined by the image $(Gf, Gg, G\eta, G\epsilon)$ of the horizontal adjoint equivalence (f,g,η,ϵ) of $\mathbb{H}^{\simeq}E_{\mathrm{adj}}$, and the images $G\alpha$ and $G\gamma$ of the squares α and γ in $\mathbb{H}^{\simeq}E_{\mathrm{adj}}$. This follows from the fact that the weak inverse of a weakly horizontally invertible square with respect to fixed horizontal adjoint equivalence data is unique, by Proposition 3.6.6, and from the fact that, given a data (f,u,α) as above, a square β such that the data (f,u,α,β) is that of an orthogonal companion pair is also uniquely determined. The latter can be easily proven by using the relations between α and β.

We are now ready to introduce weakly horizontally invariant replacements.

Construction 8.1.9. Let $\mathbb{A}$ be a double category and let $\mathrm{HorEq}(\mathbb{A})$ denote the class of all horizontal adjoint equivalence data in $\mathbb{A}$. Then each horizontal adjoint equivalence (a,c,η,ϵ) in $\mathbb{A}$ defines a double functor $\mathbb{H}E_{\mathrm{adj}} \to \mathbb{A}$ and we define $\mathbb{A}^{\mathrm{whi}}$ to be the pushout

$$\begin{tikzcd}
\bigsqcup_{\mathrm{HorEq}(\mathbb{A})} \mathbb{H}E_{\mathrm{adj}} \arrow[r] \arrow[d, "\bigsqcup_{\mathrm{HorEq}(\mathbb{A})} J_4"'] & \mathbb{A} \arrow[d, "j_{\mathbb{A}}"] \\
\bigsqcup_{\mathrm{HorEq}(\mathbb{A})} \mathbb{H}^{\simeq}E_{\mathrm{adj}} \arrow[r] & \mathbb{A}^{\mathrm{whi}} \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd} .$$

Now let $F \colon \mathbb{A} \to \mathbb{B}$ be a double functor. Then F induces double functors

$$F_0 \colon \bigsqcup_{\mathrm{HorEq}(\mathbb{A})} \mathbb{H}E_{\mathrm{adj}} \to \bigsqcup_{\mathrm{HorEq}(\mathbb{B})} \mathbb{H}E_{\mathrm{adj}} \quad (\text{resp. } F_1 \colon \bigsqcup_{\mathrm{HorEq}(\mathbb{A})} \mathbb{H}^{\simeq}E_{\mathrm{adj}} \to \bigsqcup_{\mathrm{HorEq}(\mathbb{B})} \mathbb{H}^{\simeq}E_{\mathrm{adj}})$$

by sending the copy of $\mathbb{H}E_{\mathrm{adj}}$ (resp. $\mathbb{H}^{\simeq}E_{\mathrm{adj}}$) at a horizontal adjoint equivalence (a,c,η,ϵ) in $\mathrm{HorEq}(\mathbb{A})$ to the copy of $\mathbb{H}E_{\mathrm{adj}}$ (resp. $\mathbb{H}^{\simeq}E_{\mathrm{adj}}$) at the horizontal adjoint equivalence $(Fa, Fc, F\eta, F\epsilon)$ in $\mathrm{HorEq}(\mathbb{B})$. Then there is a unique double functor $F^{\mathrm{whi}} \colon \mathbb{A}^{\mathrm{whi}} \to \mathbb{B}^{\mathrm{whi}}$ making the following diagram commute.

$$\begin{tikzcd}
\bigsqcup_{\mathrm{HorEq}(\mathbb{A})} \mathbb{H}E_{\mathrm{adj}} \arrow[rr] \arrow[dr, "F_0"] \arrow[dd, "\bigsqcup_{\mathrm{HorEq}(\mathbb{A})} J_4"'] & & \mathbb{A} \arrow[dr, "F"] \arrow[dd, "j_{\mathbb{A}}"' near start] & \\
& \bigsqcup_{\mathrm{HorEq}(\mathbb{B})} \mathbb{H}E_{\mathrm{adj}} \arrow[rr] \arrow[dd, "\bigsqcup_{\mathrm{HorEq}(\mathbb{B})} J_4"' near start] & & \mathbb{B} \arrow[dd, "j_{\mathbb{B}}"] \\
\bigsqcup_{\mathrm{HorEq}(\mathbb{A})} \mathbb{H}^{\simeq}E_{\mathrm{adj}} \arrow[rr] \arrow[dr, "F_1"'] & & \mathbb{A}^{\mathrm{whi}} \arrow[dr, dashed, "F^{\mathrm{whi}}"] & \\
& \bigsqcup_{\mathrm{HorEq}(\mathbb{B})} \mathbb{H}^{\simeq}E_{\mathrm{adj}} \arrow[rr] & & \mathbb{B}^{\mathrm{whi}}
\end{tikzcd}$$

This defines a functor $(-)^{\mathrm{whi}} \colon \mathrm{DblCat} \to \mathrm{DblCat}^{[1]}$ sending a double category $\mathbb{A}$ to the double functor $j_{\mathbb{A}} \colon \mathbb{A} \to \mathbb{A}^{\mathrm{whi}}$ and a double functor $F \colon \mathbb{A} \to \mathbb{B}$ to the following commutative square in DblCat.

$$\begin{tikzcd}
\mathbb{A} \arrow[r, "F"] \arrow[d, "j_{\mathbb{A}}"'] & \mathbb{B} \arrow[d, "j_{\mathbb{B}}"] \\
\mathbb{A}^{\mathrm{whi}} \arrow[r, "F^{\mathrm{whi}}"'] & \mathbb{B}^{\mathrm{whi}}
\end{tikzcd}$$

This is functorial by construction.

Remark 8.1.10. Since the double functor $j_{\mathbb{A}} \colon \mathbb{A} \to \mathbb{A}^{\mathrm{whi}}$ is a pushout of coproducts of the double functor $J_4 \colon \mathbb{H}E_{\mathrm{adj}} \to \mathbb{H}^{\simeq}E_{\mathrm{adj}}$, it is the identity on underlying horizontal categories and it is fully faithful on squares, for every double category $\mathbb{A}$. Hence a double functor $F\colon \mathbb{A} \to \mathbb{B}$ coincides with $F^{\mathrm{whi}}\colon \mathbb{A}^{\mathrm{whi}} \to \mathbb{B}^{\mathrm{whi}}$ on underlying horizontal categories.

Remark 8.1.11. The construction $j_{\mathbb{A}}\colon \mathbb{A} \to \mathbb{A}^{\mathrm{whi}}$ adds to each horizontal adjoint equivalence (a, c, η, ϵ) in $\mathbb{A}$ $\mathbb{H}^{\simeq}E_{\mathrm{adj}}$-data in $\mathbb{A}^{\mathrm{whi}}$, as detailed in Description 8.1.7, extending the given horizontal adjoint equivalence. In particular, by studying this $\mathbb{H}^{\simeq}E_{\mathrm{adj}}$-data which extends (a, c, η, ϵ), we can see that two vertical morphisms u and v were freely added in $\mathbb{A}^{\mathrm{whi}}$ together with weakly horizontally and vertically invertible squares as in Description 8.1.7. To refer to these freely added vertical morphisms u and v, we often say that the morphisms u and v were *added using the horizontal adjoint equivalence data* (a, c, η, ϵ) *in* $\mathbb{A}$.

In particular, we check that the double category $\mathbb{A}^{\mathrm{whi}}$ constructed this way is indeed weakly horizontally invariant (see Definition 3.6.5).

Proposition 8.1.12. *For every double category* $\mathbb{A}$*, the double category* $\mathbb{A}^{\mathrm{whi}}$ *is weakly horizontally invariant.*

Proof. Let $a\colon A \xrightarrow{\simeq} C$ and $a'\colon A' \xrightarrow{\simeq} C'$ be horizontal equivalences in $\mathbb{A}$ and $w\colon C \to C'$ be a vertical morphism in $\mathbb{A}^{\mathrm{whi}}$. Let (a, c, η, ϵ) and $(a', c', \eta', \epsilon')$ be horizontal adjoint equivalence data in $\mathbb{A}$ for a and a'. By construction of $\mathbb{A}^{\mathrm{whi}}$, there are four vertical morphisms in $\mathbb{A}^{\mathrm{whi}}$ that were added using the horizontal adjoint equivalences (a, c, η, ϵ) and $(a', c', \eta', \epsilon')$ and, in particular, there are vertical morphisms $u\colon A \to C$ and $v\colon C' \to A'$ in $\mathbb{A}^{\mathrm{whi}}$ together with weakly horizontally invertible squares α and γ' in $\mathbb{A}^{\mathrm{whi}}$ as depicted below (see Description 8.1.7).

$$
\begin{tikzcd}
A \arrow[r, "a", "\simeq"'] \arrow[d, "u"', "\bullet" marking] \arrow[dr, phantom, "\alpha\ \simeq"] & C \arrow[d, equal, "\bullet" marking] \\
C \arrow[r, equal] & C
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
C' \arrow[r, equal] \arrow[d, "v"', "\bullet" marking] \arrow[dr, phantom, "\gamma'\ \simeq"] & C' \arrow[d, equal, "\bullet" marking] \\
A' \arrow[r, "\simeq", "a'"'] & C'
\end{tikzcd}
$$

Then the composite of vertical morphisms $vwu\colon A \to A'$ together with the following pasting of weakly horizontally invertible squares

$$
\begin{tikzcd}
A \arrow[r, "a", "\simeq"'] \arrow[d, "u"', "\bullet" marking] \arrow[dr, phantom, "\alpha\ \simeq"] & C \arrow[d, equal, "\bullet" marking] \\
C \arrow[r, equal] \arrow[d, "w"', "\bullet" marking] \arrow[dr, phantom, "\mathrm{id}_w"] & C \arrow[d, "w", "\bullet" marking] \\
C' \arrow[r, equal] \arrow[d, "v"', "\bullet" marking] \arrow[dr, phantom, "\gamma'\ \simeq"] & C' \arrow[d, equal, "\bullet" marking] \\
A' \arrow[r, "\simeq", "a'"'] & C'
\end{tikzcd}
$$

gives the required data, showing that $\mathbb{A}^{\mathrm{whi}}$ is weakly horizontally invariant. $\square$

This construction allows us to define the class of weak equivalences in our model structure to be the class of double functors which induce a double biequivalence between weakly horizontally invariant replacements.

Definition 8.1.13. We define $\mathcal{W}$ to be the class of double functors $F\colon \mathbb{A} \to \mathbb{B}$ such that the induced double functor $F^{\mathrm{whi}}\colon \mathbb{A}^{\mathrm{whi}} \to \mathbb{B}^{\mathrm{whi}}$ is a double biequivalence (see Definition 7.2.1).

In particular, since double biequivalences are the weak equivalences in the model structure on DblCat of Theorem 7.1.3, they satisfy 2-out-of-3 and are closed under retracts. As a direct consequence, we can prove that the class $\mathcal{W}$ also has these properties.

Proposition 8.1.14. *The class $\mathcal{W}$ of Definition 8.1.13 satisfies 2-out-of-3, and is closed under retracts.*

Proof. Recall that the class of double biequivalences is the class of weak equivalences in the model structure on DblCat of Theorem 7.1.3. Hence it satisfies 2-out-of-3, and it is closed under retracts. Since the replacement $(-)^{\mathrm{whi}}$ is functorial, the fact that the class $\mathcal{W}$ of Definition 8.1.13 satisfies 2-out-of-3, and is closed under retracts follows directly from the fact that double biequivalences satisfy these properties. □

By taking cofibrations to be $\mathcal{I}_w$-cofibrations and weak equivalences to be double functors in $\mathcal{W}$, we obtain the desired model structure on the category DblCat, as stated in the theorem below. Since we need several additional technical results proven in Section 8.2 in order to show that the classes of trivial cofibrations and fibrations form a weak factorization system, the proof of this results is reported to Section 8.3. We also report to Section 8.3 the characterization of fibrant objects as weakly horizontally invariant double categories.

Theorem 8.1.15. *There is a model structure $(\mathcal{C}, \mathcal{F}, \mathcal{W})$ on* DblCat *such that*

(i) the class $\mathcal{C}$ of cofibrations is given by $\mathcal{C} := \mathcal{I}_w-\mathrm{cof}$, where $\mathcal{I}_w$ is the set described in Notation 8.1.1,
(ii) the class $\mathcal{W}$ of weak equivalences is the class $\mathcal{W}$ as described in Definition 8.1.13,
(iii) the class $\mathcal{F}$ of fibrations is given by $\mathcal{F} := (\mathcal{C} \cap \mathcal{W})^{\boxslash}$, and
(iv) the fibrant objects are the weakly horizontally invariant double categories.

Proof. By Proposition 8.1.14, we already know that the class $\mathcal{W}$ of weak equivalences satisfy the 2-out-of-3 property. Furthermore, by Proposition 8.1.17 below, we have that $\mathcal{F} \cap \mathcal{W} = \mathcal{I}_w-\mathrm{inj}$, and hence the pair $(\mathcal{C}, \mathcal{F} \cap \mathcal{W}) = (\mathcal{I}_w-\mathrm{cof}, \mathcal{I}_w-\mathrm{inj})$ is the weak factorization system generated by the set $\mathcal{I}_w$ of Notation 8.1.1. It remains to show that the pair $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$ forms a weak factorization system, which is the content of Theorem 8.3.5 and Corollary 8.3.6. The proof that the fibrant objects are precisely the weakly horizontally invariant double categories is in Theorem 8.3.1. □

Remark 8.1.16. In particular, since $\mathcal{C} = \mathcal{I}_w-\mathrm{cof}$ and $\mathcal{W}$ are closed under retracts (see Proposition 8.1.14), then the class $\mathcal{C} \cap \mathcal{W}$ is also closed under retracts.

As a first check to see if this indeed defines a model structure, we verify that an $\mathcal{I}_w$-injective is precisely a fibration which is also a weak equivalence.

Proposition 8.1.17. *We have that $\mathcal{F} \cap \mathcal{W} = \mathcal{I}_w-\mathrm{inj}$.*

Proof. We first prove that $\mathcal{I}_w-\mathrm{inj} \subseteq \mathcal{F} \cap \mathcal{W}$. Since $\mathcal{C} \cap \mathcal{W} \subseteq \mathcal{C} = \mathcal{I}_w-\mathrm{cof}$, it follows that $\mathcal{I}_w-\mathrm{inj} = \mathcal{I}_w-\mathrm{cof}^{\boxslash} \subseteq (\mathcal{C} \cap \mathcal{W})^{\boxslash} = \mathcal{F}$. Hence it remains to show that $\mathcal{I}_w-\mathrm{inj} \subseteq \mathcal{W}$. Let $Q\colon \mathbb{A} \to \mathbb{B}$ be a double functor in $\mathcal{I}_w-\mathrm{inj}$. We show that the induced double functor $Q^{\mathrm{whi}}\colon \mathbb{A}^{\mathrm{whi}} \to \mathbb{B}^{\mathrm{whi}}$ is in $\mathcal{I}_w-\mathrm{inj}$ using Proposition 8.1.2. By Remark 8.1.3, this shows that Q^{whi} is a double biequivalence and hence that $Q \in \mathcal{W}$.

By Remark 8.1.10, the double functors Q and Q^{whi} coincide on underlying horizontal categories and the double functors $j_{\mathbb{A}}\colon \mathbb{A} \to \mathbb{A}^{\mathrm{whi}}$ and $j_{\mathbb{B}}\colon \mathbb{B} \to \mathbb{B}^{\mathrm{whi}}$ are fully faithful on squares. It follows that Q^{whi} is surjective on objects and full on horizontal morphisms as Q is so, and it is fully faithful on squares since Q, $j_{\mathbb{A}}$ and $j_{\mathbb{B}}$ are so and $Q^{\mathrm{whi}} j_{\mathbb{A}} = j_{\mathbb{B}} Q$. It remains to prove that Q^{whi} is full on vertical morphisms. Let A, A' be objects in $\mathbb{A}$ and $v\colon QA \to QA'$ be a vertical morphism in $\mathbb{B}^{\mathrm{whi}}$. If $v \in \mathbb{B}$, then there is a vertical morphism $u\colon A \to A'$ in $\mathbb{A}$ such that $Qu = v$ since Q is full on vertical morphisms. Now suppose that v was freely added in $\mathbb{B}^{\mathrm{whi}}$ using a horizontal adjoint equivalence (b,d,η,ϵ) between the objects QA and QA' in $\mathbb{B}$. Since Q is full on horizontal morphisms, and fully faithful on squares, there is a horizontal adjoint equivalence (a,c,η',ϵ') between the objects A and A' in $\mathbb{A}$ whose image under Q is (b,d,η,ϵ). If $u\colon A \to A'$ is the vertical morphism in $\mathbb{A}^{\mathrm{whi}}$ which was freely added using the horizontal adjoint equivalence (a,c,η',ϵ'), then we have $Q^{\mathrm{whi}}u = v$ by definition of Q^{whi}. Finally, since every vertical morphism in $\mathbb{B}^{\mathrm{whi}}$ is a composite of vertical morphisms in $\mathbb{B}$ and freely added vertical morphisms as considered above, it follows that, for every vertical morphism $v\colon QA \to QA'$ in $\mathbb{B}$, there is a vertical morphism $u\colon A \to A'$ in $\mathbb{A}^{\mathrm{whi}}$ such that $Q^{\mathrm{whi}}u = v$. This vertical morphism u is constructed by taking a lift as above for each part of the composite of v. Hence Q^{whi} is full on vertical morphisms. This shows that $\mathcal{I}_w-\mathrm{inj} \subseteq \mathcal{F}\cap\mathcal{W}$.

We now prove that $\mathcal{F}\cap\mathcal{W} \subseteq \mathcal{I}_w-\mathrm{inj}$. Let $P\colon \mathbb{A}\to\mathbb{B}$ be a double functor in $\mathcal{F}\cap\mathcal{W}$. We factor P as

$$\begin{tikzcd} \mathbb{A} \arrow[rr, "P"] \arrow[dr, "I"'] & & \mathbb{B} \\ & \mathbb{C} \arrow[ur, "Q"'] & \end{tikzcd}$$

with $I \in \mathcal{I}_w-\mathrm{cell}$ and $Q \in \mathcal{I}_w-\mathrm{inj}$. Since $Q \in \mathcal{W}$ by the above result, and $P \in \mathcal{W}$, by assumption, we get that $I \in \mathcal{W}$ by 2-out-of-3. Hence $I \in \mathcal{C}\cap\mathcal{W}$. Therefore, since $P \in \mathcal{F} = (\mathcal{C}\cap\mathcal{W})^{\boxslash}$ has the right lifting property with respect to I, by the retract argument (see Proposition 4.1.6), we have that P is a retract of $Q \in \mathcal{I}_w-\mathrm{inj}$. Finally, since $\mathcal{I}_w-\mathrm{inj}$ is closed under retracts, we get that $P \in \mathcal{I}_w-\mathrm{inj}$. This concludes the proof that $\mathcal{F}\cap\mathcal{W} = \mathcal{I}_w-\mathrm{inj}$. □

We finally prove that the class of double biequivalences is included in $\mathcal{W}$. The reversed inclusion does not hold, but, as we will see later in Proposition 8.3.4, a weak equivalence whose source is a weakly horizontally invariant double category is always a double biequivalence.

Proposition 8.1.18. *Every double biequivalence is in $\mathcal{W}$.*

Proof. Let $F\colon \mathbb{A}\to\mathbb{B}$ be a double biequivalence. We need to show that the induced double functor $F^{\mathrm{whi}}\colon \mathbb{A}^{\mathrm{whi}} \to \mathbb{B}^{\mathrm{whi}}$ is a double biequivalence. We show that F^{whi} satisfies (db1-4) of Definition 7.2.1. Since F and F^{whi} coincide on underlying horizontal categories by Remark 8.1.10 and $\mathbb{B}$ is included in $\mathbb{B}^{\mathrm{whi}}$, then F^{whi} satisfies (db1-2) since F does so. Moreover, since $j_{\mathbb{A}}$, $j_{\mathbb{B}}$, and F are fully faithful on squares and $F^{\mathrm{whi}}j_{\mathbb{A}} = Fj_{\mathbb{B}}$, then F^{whi} is also fully faithful on squares, i.e., it satisfies (db4).

It remains to show (db3). Let $v\colon B \to B'$ be a vertical morphism in $\mathbb{B}^{\mathrm{whi}}$. If $v \in \mathbb{B}$, then, by (db3) for F, there is a vertical morphism $u\colon A \to A'$ in $\mathbb{A}$ together with a weakly horizontally invertible square β in $\mathbb{B}$

$$\begin{tikzcd} FA \arrow[r, "b", "\simeq"'] \arrow[d, "v"'] \arrow[dr, phantom, "\beta\ \simeq"] & B \arrow[d, "Fu"] \\ FA' \arrow[r, "\simeq", "b'"'] & B' . \end{tikzcd}$$

Since $\mathbb{A}$ and $\mathbb{B}$ are included in $\mathbb{A}^{\mathrm{whi}}$ and $\mathbb{B}^{\mathrm{whi}}$, this gives the desired data for v in $\mathbb{B}^{\mathrm{whi}}$. Now suppose that v is a composite of freely added vertical morphisms in $\mathbb{B}^{\mathrm{whi}}$. Then, there is a horizontal adjoint equivalence (f,g,η,ϵ) in $\mathbb{B}$ between the objects B and B' together with a weakly horizontally invertible square α in $\mathbb{B}^{\mathrm{whi}}$ of the form

$$\begin{tikzcd}
B \arrow[r, "f", "\simeq"'] \arrow[d, "v"', "\bullet" marking] \arrow[dr, phantom, "\alpha \simeq"] & B' \arrow[d, equal, "\bullet" marking] \\
B' \arrow[r, equal] & B'
\end{tikzcd}$$

obtained by composing the corresponding weakly horizontally invertible squares for each freely added vertical morphism appearing in the composite of v. We now show that, given already fixed horizontal equivalences $b\colon FA \xrightarrow{\simeq} B$ and $b'\colon FA' \xrightarrow{\simeq} B'$ in $\mathbb{B}$, there is a vertical morphism $u\colon A \to A'$ in $\mathbb{A}^{\mathrm{whi}}$ together with a weakly horizontally invertible square β in $\mathbb{B}^{\mathrm{whi}}$ of the form

$$\begin{tikzcd}
FA \arrow[r, "b", "\simeq"'] \arrow[d, "F^{\mathrm{whi}}u"', "\bullet" marking] \arrow[dr, phantom, "\beta \simeq"] & B \arrow[d, "v", "\bullet" marking] \\
FA' \arrow[r, "\simeq", "b'"'] & B' .
\end{tikzcd}$$

So let $b\colon FA \xrightarrow{\simeq} B$ and $b'\colon FA' \xrightarrow{\simeq} B'$ be horizontal equivalences in $\mathbb{B}$ and let $(b', d', \eta', \epsilon')$ be horizontal adjoint equivalence data for b'. Since F^{whi} satisfies (db2) and (db4), there is a horizontal equivalence $a\colon A \xrightarrow{\simeq} A'$ in $\mathbb{A}$ together with a vertically invertible square ψ in $\mathbb{B}^{\mathrm{whi}}$ of the form

$$\begin{tikzcd}
FA \arrow[r, "b", "\simeq"'] \arrow[d, equal, "\bullet" marking] & B \arrow[r, "f", "\simeq"'] \arrow[d, phantom, "\psi \cong"] & B' \arrow[r, "d'", "\simeq"'] & FA' \arrow[d, equal, "\bullet" marking] \\
FA \arrow[rrr, "\simeq", "Fa"'] & & & FA' .
\end{tikzcd}$$

Let $u\colon A \to A'$ be the freely added vertical morphism in $\mathbb{A}^{\mathrm{whi}}$ using a horizontal adjoint equivalence data for a. We define β to be given by the following pasting

$$\begin{tikzcd}
FA \arrow[r, "b"] \arrow[d, "F^{\mathrm{whi}}u"', "\bullet" marking] \arrow[dr, phantom, "\beta \simeq"] & B \arrow[d, "v", "\bullet" marking] \\
FA' \arrow[r, "b'"'] & B'
\end{tikzcd}
=
\begin{tikzcd}
FA \arrow[r, "b"] \arrow[d, equal, "\bullet" marking] \arrow[dr, phantom, "e_b"] & B \arrow[rrrr, equal] \arrow[d, equal, "\bullet" marking] & & \eta \cong & & B \arrow[d, equal, "\bullet" marking] \\
FA \arrow[r, "b"] \arrow[d, equal, "\bullet" marking] \arrow[dr, phantom, "e_b"] & B \arrow[r, "f"] \arrow[d, equal, "\bullet" marking] \arrow[dr, phantom, "e_f"] & B' \arrow[rr, equal] \arrow[d, equal, "\bullet" marking] & (\epsilon')^{-1} \cong & B' \arrow[r, "g"] \arrow[d, equal, "\bullet" marking] \arrow[dr, phantom, "e_g"] & B \arrow[d, equal, "\bullet" marking] \\
FA \arrow[r, "b"'] \arrow[d, equal, "\bullet" marking] & B \arrow[r, "f"'] & B' \arrow[r, "d'"] \arrow[dl, phantom, "\psi \cong"] & FA' \arrow[r, "b'"] \arrow[d, equal, "\bullet" marking] \arrow[dr, phantom, "e_{b'}"] & B' \arrow[r, "g"] \arrow[d, equal, "\bullet" marking] \arrow[dr, phantom, "e_g"] & B \arrow[d, equal, "\bullet" marking] \\
FA \arrow[rrr, "Fa"'] \arrow[d, "F^{\mathrm{whi}}u"', "\bullet" marking] & & F^{\mathrm{whi}}\overline{\alpha} \simeq & FA' \arrow[r, "b'"] \arrow[d, equal, "\bullet" marking] \arrow[dr, phantom, "e_{b'}"] & B' \arrow[r, "g"] \arrow[d, equal, "\bullet" marking] \arrow[dr, phantom, "\gamma \simeq"] & B \arrow[d, "v", "\bullet" marking] \\
FA' \arrow[rrr, equal] & & & FA' \arrow[r, "b'"'] & B' \arrow[r, equal] & B' ,
\end{tikzcd}$$

where $\overline{\alpha}$ is the weakly horizontally invertible square in $\mathbb{A}^{\mathrm{whi}}$ which was freely added with u (see Description 8.1.7), and γ is the weak inverse of α in $\mathbb{B}^{\mathrm{whi}}$ with respect to the horizontal adjoint equivalence (f, g, η, ϵ) and the identity horizontal adjoint equivalence. This gives the desired weakly horizontally invertible square.

Note that every vertical morphism v in $\mathbb{B}^{\mathrm{whi}}$ can be decomposed as follows: it is the composite of a sequence of composable vertical morphism $v_\mu\colon B_\mu \to B_{\mu+1}$ such that, if

$v_\mu \in \mathbb{B}$, then $v_{\mu-1}$ and $v_{\mu+1}$ are composites of freely added vertical morphisms as considered above, and if v_μ is a composite of freely added vertical morphisms as considered above, then $v_{\mu-1}$ and $v_{\mu+1}$ are in $\mathbb{B}$, for all μ. Using this decomposition of v, we can first fix a weakly horizontally invertible square β_μ relating v_μ to a vertical morphism Fu_μ, where u_μ is a vertical morphism in $\mathbb{A}$, for each vertical morphism v_μ in $\mathbb{B}$. Then, we can choose a weakly horizontally invertible square β_μ relating v_μ to a vertical morphism $F^{\mathrm{whi}}u_\mu$, where u_μ is a vertical morphism in $\mathbb{A}^{\mathrm{whi}}$, in a such a way that the horizontal equivalence which is the source of β_μ corresponds to the horizontal equivalence which is the target of $\beta_{\mu-1}$ and the horizontal equivalence which is the target of β_μ corresponds to the horizontal equivalence which is the source of $\beta_{\mu+1}$, where $\beta_{\mu-1}$ and $\beta_{\mu+1}$ were previously fixed since $v_{\mu-1}$ and $v_{\mu+1}$ are in $\mathbb{B}$. Hence, the composite of the vertical morphisms u_μ is a vertical morphism u in $\mathbb{A}^{\mathrm{whi}}$ and the vertical composite of the squares β_μ (which is well-defined by construction) gives a weakly horizontally invertible square in $\mathbb{B}^{\mathrm{whi}}$ relating v to $F^{\mathrm{whi}}u$. This shows (db3) for F^{whi} and hence that F is in $\mathcal{W}$. □

8.2. **$\mathcal{J}_w$-cofibrations and $\mathcal{J}_w$-injectives.** In order to prove Theorem 8.1.15, we need to study more closely the trivial cofibrations and fibrations of the proposed model structure on DblCat. While we were not able to find a nice description of these double functors in general, we can prove that every trivial cofibration induces a $\mathcal{J}_w$-cofibrations between weakly horizontally invariant replacements, and that a fibration with weakly horizontally invariant target is a $\mathcal{J}_w$-injective, where $\mathcal{J}_w$ is the following set of cofibrations.

Notation 8.2.1. Let $\mathbb{W}$ denote the "free-living weakly horizontally invertible square with horizontal adjoint equivalence data", and $\mathbb{W}^-$ be its double subcategory where we remove one of the vertical morphisms.

$$\mathbb{W} = \begin{tikzcd} 0 \arrow[r, "\simeq"] \arrow[d, "\bullet" marking] \arrow[dr, phantom, "\simeq"] & 1 \arrow[d, "\bullet" marking] \\ 0' \arrow[r, "\simeq"'] & 1' \end{tikzcd} \quad ; \quad \mathbb{W}^- = \begin{tikzcd} 0 \arrow[r, "\simeq"] & 1 \arrow[d, "\bullet" marking] \\ 0' \arrow[r, "\simeq"'] & 1' \end{tikzcd} .$$

Let $\mathcal{J}_w$ denote the set containing the following double functors:

(i) the inclusion double functor $J_1\colon [0] \to \mathbb{H}E_{\mathrm{adj}}$, where the 2-category E_{adj} is the "free-living adjoint equivalence",
(ii) the inclusion double functor $J_2\colon \mathbb{H}[1] \to \mathbb{H}C_{\mathrm{inv}}$, where the 2-category C_{inv} is the "free-living 2-isomorphism",
(iii) the inclusion double functor $J_3'\colon \mathbb{W}^- \to \mathbb{W}$.

Remark 8.2.2. It is straightforward from the description of cofibrations in $\mathcal{I}_w$−cof in Corollary 8.1.5 that the double functors J_1, J_2, and J_3 are in $\mathcal{I}_w$−cof.

By definition, a weakly horizontally invariant double category is precisely a double category which has the right lifting property with respect to $J_3'\colon \mathbb{W}^- \to \mathbb{W}$. This yields the following result.

Proposition 8.2.3. *A double category $\mathbb{A}$ is weakly horizontally invariant if and only if the double functor $\mathbb{A} \to [0]$ is in $\mathcal{J}_w$−*inj.

Proof. First note that every double functor $\mathbb{A} \to [0]$ trivially lifts against $J_1\colon [0] \to \mathbb{H}E_{\mathrm{adj}}$ and $J_2\colon \mathbb{H}[1] \to \mathbb{H}C_{\mathrm{inv}}$. Then it lifts against $J_3\colon \mathbb{W}^- \to \mathbb{W}$ if and only if it is weakly horizontally invariant by definition; see Definition 3.6.5. □

By studying the lifting properties with respect to the double functors in $\mathcal{J}_w$, we can characterize the $\mathcal{J}_w$-injectives as follows.

Proposition 8.2.4. *A double functor $F\colon \mathbb{A} \to \mathbb{B}$ is in $\mathcal{J}_w-\mathrm{inj}$ if and only if it satisfies* (df1-2) *of Definition 7.2.6 and the following condition:*

(df3') *for every vertical morphism $w\colon C \to C'$ in $\mathbb{A}$, every pair of horizontal equivalences $a\colon A \xrightarrow{\simeq} C$ and $a'\colon A' \xrightarrow{\simeq} C'$ in $\mathbb{A}$, and every weakly horizontally invertible square β in $\mathbb{B}$ as depicted below left, there is a weakly horizontally invertible square α in $\mathbb{A}$ as depicted below right such that $\beta = F\alpha$.*

$$\begin{tikzcd} FA \arrow[r, "Fa", "\simeq"'] \arrow[d, "v"'] \arrow[dr, phantom, "\beta \simeq"] & FC \arrow[d, "Fw"] \\ FA' \arrow[r, "\simeq", "Fa'"'] & FC' \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "a", "\simeq"'] \arrow[d, "u"'] \arrow[dr, phantom, "\alpha \simeq"] & C \arrow[d, "w"] \\ A' \arrow[r, "\simeq", "a'"'] & C' \end{tikzcd}$$

Proof. As mentioned in Proposition 7.3.8, a double functor F has the right lifting property with respect to the double functors $J_1\colon [0] \to \mathbb{H}E_{\mathrm{adj}}$ and $J_2\colon \mathbb{H}[1] \to \mathbb{H}C_{\mathrm{inv}}$ if and only if it satisfies (df1- 2) of Definition 7.2.6. Furthermore, it has the right lifting property with respect to $J_3'\colon \mathbb{W}^- \to \mathbb{W}$ if and only if it satisfies (df3') above. □

In particular, the following useful result tells us that every $\mathcal{J}_w$-injective has the right lifting property with respect to the double functor $J_4\colon \mathbb{H}E_{\mathrm{adj}} \to \mathbb{H}^{\simeq}E_{\mathrm{adj}}$.

Proposition 8.2.5. *Let $F\colon \mathbb{A} \to \mathbb{B}$ be a double functor in $\mathcal{J}_w-\mathrm{inj}$. Then F is in $\{J_4\}-\mathrm{inj}$, where $J_4\colon \mathbb{H}E_{\mathrm{adj}} \to \mathbb{H}^{\simeq}E_{\mathrm{adj}}$ is the inclusion double functor introduced in Description 8.1.7.*

Proof. To show that F is in $\{J_4\}-\mathrm{inj}$, it is enough to show that it has the right lifting property with respect to $J_4\colon \mathbb{H}E_{\mathrm{adj}} \to \mathbb{H}^{\simeq}E_{\mathrm{adj}}$. Consider a commutative square in DblCat of the form

$$\begin{tikzcd} \mathbb{H}E_{\mathrm{adj}} \arrow[r, "{(a,c,\eta,\epsilon)}"] \arrow[d, "J_4"'] & \mathbb{A} \arrow[d, "F"] \\ \mathbb{H}^{\simeq}E_{\mathrm{adj}} \arrow[r, "G"'] \arrow[ur, dashed, "L"] & \mathbb{B}, \end{tikzcd}$$

where $a\colon A \xrightarrow{\simeq} C$ is a horizontal adjoint equivalence with data (a, c, η, ϵ). We want to find a lift $L\colon \mathbb{H}^{\simeq}E_{\mathrm{adj}} \to \mathbb{A}$ in this diagram. By Description 8.1.7, the images under G of the weakly horizontally invertible squares α and γ in $\mathbb{H}^{\simeq}E_{\mathrm{adj}}$ are weakly horizontally invertible squares in $\mathbb{B}$ as depicted below.

$$\begin{tikzcd} FA \arrow[r, "Fa", "\simeq"'] \arrow[d, "Gu"'] \arrow[dr, phantom, "G\alpha \simeq"] & FC \arrow[d, equal] \\ FC \arrow[r, equal] & FC \end{tikzcd} \qquad \begin{tikzcd} FC \arrow[r, "Fc", "\simeq"'] \arrow[d, "Gv"'] \arrow[dr, phantom, "G\gamma \simeq"] & FA \arrow[d, equal] \\ FA \arrow[r, equal] & FA \end{tikzcd}$$

By (df3') of Proposition 8.2.4, there are weakly horizontally invertible squares $\overline{\alpha}$ and $\overline{\gamma}$ in $\mathbb{A}$, as depicted below, such that $F\overline{\alpha} = G\alpha$ and $F\overline{\gamma} = G\gamma$.

$$\begin{tikzcd} A \arrow[r, "a", "\simeq"'] \arrow[d, "\overline{u}"'] \arrow[dr, phantom, "\overline{\alpha} \simeq"] & C \arrow[d, equal] \\ C \arrow[r, equal] & C \end{tikzcd} \qquad \begin{tikzcd} C \arrow[r, "c", "\simeq"'] \arrow[d, "\overline{v}"'] \arrow[dr, phantom, "\overline{\gamma} \simeq"] & A \arrow[d, equal] \\ A \arrow[r, equal] & A \end{tikzcd}$$

By Description 8.1.7, the data (a, c, η, ϵ) together with the squares $\overline{\alpha}$ and $\overline{\gamma}$ generate $\mathbb{H}^{\simeq}E_{\mathrm{adj}}$-data in $\mathbb{A}$, i.e., they determine a double functor $L\colon \mathbb{H}^{\simeq}E_{\mathrm{adj}} \to \mathbb{A}$ which extends (a, c, η, ϵ). Furthermore, since the $\mathbb{H}^{\simeq}E_{\mathrm{adj}}$-data $G\colon \mathbb{H}^{\simeq}E_{\mathrm{adj}} \to \mathbb{B}$ and $FL\colon \mathbb{H}^{\simeq}E_{\mathrm{adj}} \to \mathbb{B}$ coincide on the images of the horizontal adjoint equivalence and the squares α and γ of $\mathbb{H}^{\simeq}E_{\mathrm{adj}}$, they must be equal, by unicity of such data (see Remark 8.1.8), i.e., we have $FL = G$. Hence L gives the desired lift, and this shows that $F \in \{J_4\}$−inj. □

Remark 8.2.6. As a consequence of this result and Proposition 8.2.3, we have that, for every weakly horizontally invariant double category $\mathbb{A}$, the double functor $\mathbb{A} \to [0]$ is in $\{J_4\}$−inj.

Finally, we show that the class of $\mathcal{J}_w$-injectives which are double biequivalences is precisely the class of $\mathcal{I}_w$-injectives, i.e., the class of trivial fibrations in our proposed model structure on DblCat.

Proposition 8.2.7. *A double functor $F\colon \mathbb{A} \to \mathbb{B}$ is a $\mathcal{J}_w$-injective and a double biequivalence if and only if it is an $\mathcal{I}_w$-injective.*

Proof. Since $\mathcal{J}_w \subseteq \mathcal{I}_w$−cof by Remark 8.2.2, then $\mathcal{I}_w\text{−inj} = \mathcal{I}_w\text{−cof}^{\boxslash} \subseteq \mathcal{J}_w^{\boxslash} = \mathcal{J}_w\text{−inj}$. Furthermore, by Remark 8.1.3, an $\mathcal{I}_w$-injective is in particular a double biequivalence. This proves that, if F is an $\mathcal{I}_w$-injective, then F is a $\mathcal{J}_w$-injective and a double biequivalence.

Now suppose that F is a $\mathcal{J}_w$-injective and a double biequivalence. We prove that F is an $\mathcal{I}_w$-injective using Proposition 8.1.2. Note that, since F satisfies (db4) of Definition 7.2.1, it is fully faithful on squares. It remains to prove that F is surjective on objects, and full on horizontal and vertical morphisms.

We first show that F is surjective on objects. Let B be an object in $\mathbb{B}$. Since F satisfies (db1) of Definition 7.2.1, there is an object $C \in \mathbb{A}$ together with a horizontal equivalence $b\colon B \xrightarrow{\simeq} FC$ in $\mathbb{B}$. By (df1) of Definition 7.2.6, it follows that there is a horizontal equivalence $a\colon A \xrightarrow{\simeq} C$ in $\mathbb{A}$ such that $b = Fa$. In particular, we have $B = FA$. We now show that F is full on horizontal morphisms. Let A, C be objects in $\mathbb{A}$ and $b\colon FA \to FC$ be a horizontal morphism in $\mathbb{B}$. Since F satisfies (db2) of Definition 7.2.1, there is a horizontal morphism $c\colon A \to C$ in $\mathbb{A}$ together with a vertically invertible square β in $\mathbb{B}$ as depicted below left.

$$\begin{tikzcd} FA \arrow[r, "b"] \arrow[d, equal, "\bullet" marking] \arrow[dr, phantom, "\beta\ \cong"] & FC \arrow[d, equal, "\bullet" marking] \\ FA \arrow[r, "Fc"'] & FC \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "a"] \arrow[d, equal, "\bullet" marking] \arrow[dr, phantom, "\alpha\ \cong"] & C \arrow[d, equal, "\bullet" marking] \\ A \arrow[r, "c"'] & C \end{tikzcd}$$

By (df2) of Definition 7.2.6, there is a vertically invertible square α in $\mathbb{A}$ as above right such that $\beta = F\alpha$. In particular, we have $b = Fa$. We finally prove that F is full on vertical morphisms. Let A, A' be objects in $\mathbb{A}$ and $v\colon FA \to\!\!\!\!\bullet\ FA'$ be a vertical morphism in $\mathbb{B}$. Since F satisfies (db3) of Definition 7.2.1, there is a vertical morphism $w\colon C \to\!\!\!\!\bullet\ C'$ in $\mathbb{A}$ together with a weakly horizontally invertible square β in $\mathbb{B}$ as depicted below left.

$$\begin{tikzcd} FA \arrow[r, "b", "\simeq"'] \arrow[d, "v"', "\bullet" marking] \arrow[dr, phantom, "\beta\ \simeq"] & FC \arrow[d, "Fw", "\bullet" marking] \\ FA' \arrow[r, "\simeq", "b'"'] & FC' \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "a", "\simeq"'] \arrow[d, "u"', "\bullet" marking] \arrow[dr, phantom, "\alpha\ \simeq"] & C \arrow[d, "w", "\bullet" marking] \\ A' \arrow[r, "\simeq", "a'"'] & C' \end{tikzcd}$$

Since F is full on horizontal morphisms and fully faithful on squares, there are horizontal equivalences $a\colon A \xrightarrow{\simeq} C$ and $a'\colon A' \xrightarrow{\simeq} C'$ in $\mathbb{A}$ such that $b = Fa$ and $b' = Fa'$. Then, by (df3') of Proposition 8.2.4, there is a weakly horizontally invertible square α in $\mathbb{A}$ as depicted above right such that $\beta = F\alpha$. In particular, we have $v = Fu$. This shows that F is an $\mathcal{I}_w$-injective and concludes the proof. □

We now want to study more closely the $\mathcal{J}_w$-cofibrations. We first show that they are cofibrations, which satisfy the conditions of a double biequivalence except for the condition (db3) on vertical morphisms.

Proposition 8.2.8. *Let $J\colon \mathbb{A} \to \mathbb{B}$ be a double functor in $\mathcal{J}_w-$*cell. *Then J satisfies the following conditions:*

(i) it is injective on objects, and faithful on horizontal and vertical morphisms,
(ii) it satisfies (db1-2) *and* (db4) *of Definition 7.2.1 of double biequivalences.*

Proof. Since $\mathcal{J}_w \subseteq \mathcal{I}_w-$cof by Remark 8.2.2, then $\mathcal{J}_w-$cell $\subseteq \mathcal{I}_w-$cof and, by Corollary 8.1.5, we have that J is injective on objects, and faithful on horizontal and vertical morphisms. This proves (i).

We now prove (ii). First note that J satisfies (db4) of Definition 7.2.1. Indeed, since taking pushouts along the double functors J_1, J_2, and J_3 do not allow to add squares between existing boundaries nor to identify two squares, and since J is a transfinite composition of pushouts along these double functors, then J is fully faithful on squares. In particular, since J is also injective on objects and faithful on horizontal and vertical morphisms by (i), we have an isomorphism of double categories $\mathbb{A} \cong J(\mathbb{A})$, where $J(\mathbb{A})$ is the image of J. Let λ be an ordinal and let $\mathbb{X}\colon \lambda \to \mathrm{DblCat}$ be a transfinite composition of pushouts of double functors in $\mathcal{J}_w$ such that J is the composite

$$J\colon \mathbb{A} \cong J(\mathbb{A}) = \mathbb{X}_0 \xrightarrow{\iota_0} \mathrm{colim}_{\mu<\lambda}\mathbb{X}_\mu = \mathbb{B}.$$

We first prove that J satisfies (db1) of Definition 7.2.1. Let B be an object in $\mathbb{B}$. We prove that there is an object $A \in \mathbb{A}$ and a horizontal equivalence $b\colon JA \xrightarrow{\simeq} B$ in $\mathbb{B}$ by transfinite induction. If $B \in \mathbb{X}_0 = J(\mathbb{A})$, then there is an object $A \in \mathbb{A}$ such that $JA = B$ and we can take $b = \mathrm{id}_{JA}$. Now suppose that $B \in \mathbb{X}_{\mu+1}$ for a successor ordinal $\mu + 1 < \lambda$. If $B \in \mathbb{X}_\mu$, then we are done by induction. Otherwise, since pushouts along J_2 and J_3' do not modify the objects, the double category $\mathbb{X}_{\mu+1}$ was obtained as a pushout along J_1 of the following form

$$\begin{tikzcd} {[0]} \arrow[r, "D"] \arrow[d, "J_1"'] & \mathbb{X}_\mu \arrow[d, "i_\mu"] \\ \mathbb{H}E_{\mathrm{adj}} \arrow[r, "d"'] & \mathbb{X}_{\mu+1} \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd},$$

where D is an object in $\mathbb{X}_\mu$ and $d\colon D \xrightarrow{\simeq} B$ is a horizontal adjoint equivalence in $\mathbb{B}$. By induction, since $D \in \mathbb{X}_\mu$, there is an object $A \in \mathbb{A}$ and a horizontal equivalence $f\colon JA \xrightarrow{\simeq} D$ in $\mathbb{B}$. Then, the composite $b := df\colon JA \xrightarrow{\simeq} B$ gives the desired horizontal equivalence in $\mathbb{B}$. If $B \in \mathbb{X}_\kappa$ for a limit ordinal $\kappa < \lambda$, since $\mathbb{X}_\kappa = \mathrm{colim}_{\mu<\kappa}\mathbb{X}_\mu$, there is an ordinal $\mu < \kappa$ such that $B \in \mathbb{X}_\mu$, and we are done by induction. This shows (db1) for J.

We now prove that J satisfies (db2) of Definition 7.2.1. Let A, C be objects in $\mathbb{A}$ and $b\colon JA \to JC$ be a horizontal morphism in $\mathbb{B}$. We prove that there is a horizontal morphism $a\colon A \to C$ in $\mathbb{A}$ and a vertically invertible square β in $\mathbb{B}$ of the form

$$\begin{tikzcd} JA \arrow[r, "Ja"] \arrow[d, equal, "\bullet" marking] \arrow[dr, phantom, "\beta\ \cong"] & JC \arrow[d, equal, "\bullet" marking] \\ JA \arrow[r, "b"'] & JC \end{tikzcd},$$

by transfinite induction. If $b \in \mathbb{X}_0 = J(\mathbb{A})$, then there is a horizontal morphism $a\colon A' \to C'$ in $\mathbb{A}$ such that $Ja = b$. Then $JA = JA'$ and $JC = JC'$ and, since J is injective on objects, we have that $A = A'$ and $C = C'$. Hence $a\colon A \to C$ is such that $Ja = b$ and we can take $\beta = e_{Ja}$. Now suppose that $b \in \mathbb{X}_{\mu+1}$ for a successor ordinal $\mu+1 < \lambda$. If $b \in \mathbb{X}_\mu$, then we are done by induction. Otherwise, the double category $\mathbb{X}_{\mu+1}$ was obtained as a pushout along J_2 as depicted below left,

$$\begin{tikzcd} \mathbb{H}[1] \arrow[r, "d"] \arrow[d, "J_2"'] & \mathbb{X}_\mu \arrow[d, "i_\mu"] \\ \mathbb{H}C_{\mathrm{inv}} \arrow[r, "\delta"'] & \mathbb{X}_{\mu+1} \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd} \qquad \begin{tikzcd} B \arrow[r, "d"] \arrow[d, equal] \arrow[dr, phantom, "\delta\ \cong"] & D \arrow[d, equal] \\ B \arrow[r, "f"'] & D \end{tikzcd}$$

where $d\colon B \to D$ is a horizontal morphism in $\mathbb{X}_\mu$ and δ is a vertically invertible square in $\mathbb{B}$ as depicted above right. Then the horizontal morphism $b \in \mathbb{X}_{\mu+1}$ is a composite of horizontal morphisms in $\mathbb{X}_\mu$ and the freely added horizontal morphism f. By taking instances of the vertically invertible square δ in order to replace the instances of f in the composite of b by instances of d, we find that there is a horizontal morphism $\bar{b}\colon JA \to JC$ in $\mathbb{X}_\mu$ and a vertically invertible square φ in $\mathbb{B}$ as depicted below left.

$$\begin{tikzcd} JA \arrow[r, "\bar{b}"] \arrow[d, equal] \arrow[dr, phantom, "\varphi\ \cong"] & JC \arrow[d, equal] \\ JA \arrow[r, "b"'] & JC \end{tikzcd} \qquad \begin{tikzcd} JA \arrow[r, "Ja"] \arrow[d, equal] \arrow[dr, phantom, "\psi\ \cong"] & JC \arrow[d, equal] \\ JA \arrow[r, "\bar{b}"'] & JC \end{tikzcd}$$

Since $\bar{b} \in \mathbb{X}_\mu$, by induction, there is a horizontal morphism $a\colon A \to C$ in $\mathbb{A}$ and a vertically invertible square ψ as above left. By setting β to be the vertical composite of ψ with φ, we get the desired vertically invertible square in $\mathbb{B}$. If $b \in \mathbb{X}_\kappa$ for a limit ordinal $\kappa < \lambda$, since $\mathbb{X}_\kappa = \mathrm{colim}_{\mu<\kappa}\mathbb{X}_\mu$, there is an ordinal $\mu < \kappa$ such that $b \in \mathbb{X}_\mu$, and we are done by induction. This shows (db2) for J and concludes the proof. $\square$

Remark 8.2.9. Note that double functors in $\mathcal{J}_w-$cof also satisfy the conditions of Proposition 8.2.8 since they are retracts of double functors in $\mathcal{J}_w-$cell. We do not prove this result here since we do not need it.

When the source of a $\mathcal{J}_w$-cofibration is a weakly horizontally invariant double category, we can further show that (db3) of Definition 7.2.1 is satisfied, and hence that every such $\mathcal{J}_w$-cofibration is a double biequivalence.

Proposition 8.2.10. *Let $J\colon \mathbb{A} \to \mathbb{B}$ be a double functor in $\mathcal{J}_w-$*cell *such that $\mathbb{A}$ is weakly horizontally invariant. Then J is a double biequivalence.*

Proof. Let $J\colon \mathbb{A} \to \mathbb{B}$ be in $\mathcal{J}_w-$cell. By Proposition 8.2.8, we have that J satisfies (db1-2) and (db4) of Definition 7.2.1. It remains to show (db3) of Definition 7.2.1 for J in order to show that it is a double biequivalence. Let λ be an ordinal and let $\mathbb{X}\colon \lambda \to \mathrm{DblCat}$ be a transfinite composition of pushouts of double functors in $\mathcal{J}_w$ such that J is the composite

$$J\colon \mathbb{A} \cong J(\mathbb{A}) = \mathbb{X}_0 \xrightarrow{\iota_0} \mathrm{colim}_{\mu<\lambda}\mathbb{X}_\mu = \mathbb{B}.$$

Let $v\colon B \nrightarrow B'$ be a vertical morphism in $\mathbb{B}$. We show that there is a vertical morphism $u\colon A \nrightarrow A'$ in $\mathbb{A}$ and a weakly horizontally invertible square β in $\mathbb{B}$ of the form

$$\begin{tikzcd} JA \arrow[r, "\simeq"] \arrow[d, "Ju"', "\bullet" marking] \arrow[dr, phantom, "\beta \simeq"] & B \arrow[d, "v", "\bullet" marking] \\ JA' \arrow[r, "\simeq"'] & B', \end{tikzcd}$$

by transfinite induction. If $v \in \mathbb{X}_0 = J(\mathbb{A})$, then there is a vertical morphism $u\colon A \to A'$ in $\mathbb{A}$ such that $Ju = v$ and we can take $\beta = \mathrm{id}_{Ju}$. Now suppose that $v \in \mathbb{X}_{\mu+1}$ for a successor ordinal $\mu + 1 < \lambda$. If $v \in \mathbb{X}_\mu$, then we are done by induction. Otherwise, the double category $\mathbb{X}_{\mu+1}$ was obtained as a pushout along J_3' as depicted below left,

$$\begin{tikzcd} \mathbb{W}^- \arrow[r, "{(w,d,d')}"] \arrow[d, "J_3'"'] & \mathbb{X}_\mu \arrow[d, "i_\mu"] \\ \mathbb{W} \arrow[r, "\delta"'] & \mathbb{X}_{\mu+1} \arrow[ul, phantom, very near start, "\ulcorner"] \end{tikzcd} \qquad \begin{tikzcd} D \arrow[r, "d", "\simeq"'] \arrow[d, "w"', "\bullet" marking] \arrow[dr, phantom, "\delta \simeq"] & Y \arrow[d, "\overline{w}", "\bullet" marking] \\ D' \arrow[r, "\simeq", "d'"'] & Y' \end{tikzcd}$$

where $w\colon D \to D'$ is a vertical morphism in $\mathbb{X}_\mu$, $d\colon D \xrightarrow{\simeq} Y$ and $d'\colon D' \xrightarrow{\simeq} Y'$ are horizontal equivalences in $\mathbb{X}_\mu$, and δ is a weakly horizontally invertible square in $\mathbb{B}$ as depicted above right. Then the vertical morphism $v \in \mathbb{X}_{\mu+1}$ is a composite of vertical morphisms in $\mathbb{X}_\mu$ and the freely added vertical morphism $\overline{w}$. We prove that the result holds for a composite of the form $v = v_1\overline{w}v_0$ with $v_0\colon B \to Y$ and $v_1\colon Y' \to B'$ two vertical morphisms in $\mathbb{X}_\mu$; the other cases where $\overline{w}$ appears several time in the decomposition of v can be proven similarly. By induction, since v_0, v_1, and w are in $\mathbb{X}_\mu$, there are vertical morphisms $u_0\colon A \to C$, $u_1\colon C' \to A'$, and $t\colon X \to X'$ in $\mathbb{A}$ and weakly horizontally invertible squares β_0, β_1, and φ in $\mathbb{B}$ as depicted below.

$$\begin{tikzcd} JA \arrow[r, "b_0", "\simeq"'] \arrow[d, "Ju_0"', "\bullet" marking] \arrow[dr, phantom, "\beta_0 \simeq"] & B \arrow[d, "v_0", "\bullet" marking] \\ JC \arrow[r, "\simeq", "b_0'"'] & Y \end{tikzcd} \qquad \begin{tikzcd} JC' \arrow[r, "b_1", "\simeq"'] \arrow[d, "Ju_1"', "\bullet" marking] \arrow[dr, phantom, "\beta_1 \simeq"] & Y' \arrow[d, "v_1", "\bullet" marking] \\ JA' \arrow[r, "\simeq", "b_1'"'] & B' \end{tikzcd} \qquad \begin{tikzcd} JX \arrow[r, "f", "\simeq"'] \arrow[d, "Jt"', "\bullet" marking] \arrow[dr, phantom, "\varphi \simeq"] & D \arrow[d, "w", "\bullet" marking] \\ JX' \arrow[r, "\simeq", "f'"'] & D' \end{tikzcd}$$

Let (df, g, η, ϵ) and $(d'f', g', \eta', \epsilon')$ be horizontal adjoint equivalences in $\mathbb{B}$ for the composites $df\colon JX \xrightarrow{\simeq} Y$ and $d'f'\colon JX' \xrightarrow{\simeq} Y'$. Since J satisfies (db2) and (db4) of Definition 7.2.1, there are horizontal equivalences $a\colon C \xrightarrow{\simeq} X$ and $a'\colon C' \xrightarrow{\simeq} X'$ in $\mathbb{A}$ together with vertically invertible squares ψ and ψ' in $\mathbb{B}$ as depicted below.

$$\begin{tikzcd} JC \arrow[r, "b_0'", "\simeq"'] \arrow[d, equal, "\bullet" marking] & Y \arrow[r, "g", "\simeq"'] \arrow[d, phantom, "\psi \cong"] & JX \arrow[d, equal, "\bullet" marking] \\ JC \arrow[rr, "\simeq", "Ja"'] & & JX \end{tikzcd} \qquad \begin{tikzcd} JC' \arrow[r, "b_1", "\simeq"'] \arrow[d, equal, "\bullet" marking] & Y' \arrow[r, "g'", "\simeq"'] \arrow[d, phantom, "\psi' \cong"] & JX' \arrow[d, equal, "\bullet" marking] \\ JC' \arrow[rr, "\simeq", "Ja'"'] & & JX' \end{tikzcd}$$

Hence, since $\mathbb{A}$ is weakly horizontally invariant, the below left diagram in $\mathbb{A}$ can be filled with a vertical morphism $\overline{u}\colon C \to C'$ and a weakly horizontally invertible square α in $\mathbb{A}$ as depicted below right.

$$
\begin{tikzcd}
C \arrow[r, "a", "\simeq"'] & X \arrow[d, "t", "\bullet" marking] \\
C' \arrow[r, "\simeq", "a'"'] & X'
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
C \arrow[r, "a", "\simeq"'] \arrow[d, "\overline{u}"', "\bullet" marking] \arrow[dr, phantom, "\alpha\ \simeq"] & X \arrow[d, "t", "\bullet" marking] \\
C' \arrow[r, "\simeq", "a'"'] & X
\end{tikzcd}
$$

Finally, by setting $u \coloneqq u_1\overline{u}u_0 \colon A \nrightarrow A'$ and by considering the following pasting of squares in $\mathbb{B}$

$$
\begin{tikzcd}
JA \arrow[r, "b_0"] \arrow[d, "Ju_0"', "\bullet" marking] \arrow[dr, phantom, "\beta_0\ \simeq"] & B \arrow[rrr, equal] \arrow[d, "v_0", "\bullet" marking] \arrow[drrr, phantom, "\mathrm{id}_{v_0}"] & & & B \arrow[d, "v_0", "\bullet" marking] \\
JC \arrow[r, "b_0'"'] \arrow[d, equal, "\bullet" marking] \arrow[dr, phantom, "e_{b_0'}"] & Y \arrow[rrr, equal] \arrow[d, equal, "\bullet" marking] \arrow[drrr, phantom, "\epsilon^{-1}\ \cong"] & & & Y \arrow[d, equal, "\bullet" marking] \\
JC \arrow[r, "b_0'"'] \arrow[d, equal, "\bullet" marking] & Y \arrow[r, "g"'] \arrow[d, phantom, "\psi\ \cong"] & JX \arrow[r, "f"] \arrow[d, equal, "\bullet" marking] \arrow[dr, phantom, "e_f"] & D \arrow[r, "d"] \arrow[d, equal, "\bullet" marking] \arrow[dr, phantom, "e_d"] & Y \arrow[d, equal, "\bullet" marking] \\
JC \arrow[rr, "Ja"] \arrow[d, "J\overline{u}"', "\bullet" marking] \arrow[drr, phantom, "J\alpha\ \simeq"] & & JX \arrow[r, "f"] \arrow[d, "Jt"', "\bullet" marking] \arrow[dr, phantom, "\varphi\ \simeq"] & D \arrow[r, "d"] \arrow[d, "w"', "\bullet" marking] \arrow[dr, phantom, "\delta\ \simeq"] & Y \arrow[d, "\overline{w}", "\bullet" marking] \\
JC' \arrow[rr, "Ja'"'] \arrow[d, equal, "\bullet" marking] \arrow[drr, phantom, "(\psi')^{-1}\cong"] & & JX' \arrow[r, "f'"'] \arrow[d, equal, "\bullet" marking] \arrow[dr, phantom, "e_{f'}"] & D' \arrow[r, "d'"'] \arrow[d, equal, "\bullet" marking] \arrow[dr, phantom, "e_{d'}"] & Y' \arrow[d, equal, "\bullet" marking] \\
JC' \arrow[r, "b_1"] \arrow[d, equal, "\bullet" marking] \arrow[dr, phantom, "e_{b_1}"] & Y' \arrow[r, "g'"] \arrow[d, equal, "\bullet" marking] \arrow[drrr, phantom, "\epsilon'\ \cong"] & JX' \arrow[r, "f'"'] & D' \arrow[r, "d'"'] & Y' \arrow[d, equal, "\bullet" marking] \\
JC' \arrow[r, "b_1"] \arrow[d, "Ju_1"', "\bullet" marking] \arrow[dr, phantom, "\beta_1\ \simeq"] & Y' \arrow[rrr, equal] \arrow[d, "v_1", "\bullet" marking] \arrow[drrr, phantom, "\mathrm{id}_{v_1}"] & & & Y' \arrow[d, "v_1", "\bullet" marking] \\
JA' \arrow[r, "b_1'"'] & B' \arrow[rrr, equal] & & & B' ,
\end{tikzcd}
$$

we get a weakly horizontally invertible square of the desired form between the vertical morphisms $Ju = (Ju_1)(J\overline{u})(Ju_0)$ and $v = v_1\overline{w}v_0$. Finally, if $v \in \mathbb{X}_\kappa$ for a limit ordinal $\kappa < \lambda$, since $\mathbb{X}_\kappa = \mathrm{colim}_{\mu<\kappa}\mathbb{X}_\mu$, there is an ordinal $\mu < \kappa$ such that $v \in \mathbb{X}_\mu$, and we are done by induction. This shows (db3) for J, and proves that J is a double biequivalence. $\square$

Remark 8.2.11. If $J \colon \mathbb{A} \to \mathbb{B}$ is a double functor in $\mathcal{J}_w$−cof such that $\mathbb{A}$ is weakly horizontally invariant, then it is a retract of a double functor $K \colon \mathbb{A} \to \mathbb{C}$ in $\mathcal{J}_w$−cell by Proposition 4.2.12, whose source is also $\mathbb{A}$. Hence, by Proposition 8.2.10, the double functor K is a double biequivalence and, since double biequivalences are closed under retract, this shows that J is also a double biequivalence.

We now want to show that the fibrations in our proposed model structure, whose target is a weakly horizontally invariant double category, are precisely the $\mathcal{J}_w$-injectives. We first show that the class of fibrations is included in $\mathcal{J}_w$−inj.

Lemma 8.2.12. *We have that* $\mathcal{F} \subseteq \mathcal{J}_w$−inj.

Proof. First note that every double functor in $\mathcal{J}_w$ satisfies (db1-4) of Definition 7.2.1. Hence every double functor in $\mathcal{J}_w$ is a double biequivalence and then, by Proposition 8.1.18, they are in particular in $\mathcal{W}$. With this observation and by Remark 8.2.2, we have that $\mathcal{J}_w \subseteq \mathcal{C} \cap \mathcal{W}$. It follows that $\mathcal{F} = (\mathcal{C} \cap \mathcal{W})^{\boxslash} \subseteq \mathcal{J}_w^{\boxslash} = \mathcal{J}_w-\text{inj}$. $\square$

To prove that a $\mathcal{J}_w$-injective with weakly horizontally invariant target is a fibration, we first study the weakly horizontally invariant replacement of a trivial cofibration in our proposed model structure. We first show that it is a cofibration, and then show that it is actually a $\mathcal{J}_w$-cofibration. We then use the lifting property of our considered $\mathcal{J}_w$-injective with respect to $\mathcal{J}_w-\text{cof}$ to prove that it has the right lifting property with respect to every trivial cofibration, and hence is a fibration.

We first show that the weakly horizontally invariant replacement of a cofibration which is fully faithful on squares is a cofibration. This includes the class of trivial cofibrations, since every weak equivalence in our model structure is fully faithful on squares.

Lemma 8.2.13. *Let $I\colon \mathbb{A} \to \mathbb{B}$ be a double functor in $\mathcal{C}$ which is fully faithful on squares. Then the induced double functor $I^{\mathrm{whi}}\colon \mathbb{A}^{\mathrm{whi}} \to \mathbb{B}^{\mathrm{whi}}$ is in $\mathcal{C} = \mathcal{I}_w-\text{cof}$.*

Proof. We first prove the result when I is a double functor in $\mathcal{I}_w-\text{cell}$ which is fully faithful on squares. We use Corollary 8.1.5 to prove that I^{whi} is in $\mathcal{I}_w-\text{cell}$. Since I and I^{whi} coincide on underlying horizontal categories by Remark 8.1.10, then I^{whi} is injective on objects and faithful on horizontal morphisms, as I is so. Furthermore, the underlying horizontal category $U\mathbf{H}\mathbb{B}^{\mathrm{whi}} = U\mathbf{H}\mathbb{B}$ is obtained from the image of $U\mathbf{H}\mathbb{A}^{\mathrm{whi}} = U\mathbf{H}\mathbb{A}$ by freely adding objects and then morphisms between objects. It remains to prove that I^{whi} is faithful on vertical morphisms and that the underlying vertical category $U\mathbf{V}\mathbb{B}^{\mathrm{whi}}$ is obtained from the image of $U\mathbf{V}\mathbb{A}^{\mathrm{whi}}$ by freely adding objects and then morphisms between objects. Since I is in $\mathcal{I}_w-\text{cell}$, the underlying vertical category $U\mathbf{V}\mathbb{B}$ is obtained from the image of $U\mathbf{V}\mathbb{A}$ by freely adding objects and then morphisms between objects. Then, by construction, the categories $U\mathbf{V}\mathbb{A}^{\mathrm{whi}}$ and $U\mathbf{V}\mathbb{B}^{\mathrm{whi}}$ are obtained from the image of $U\mathbf{V}\mathbb{A}$ and $U\mathbf{V}\mathbb{B}$, respectively, by freely adding morphisms between objects. Now suppose that $v\colon B \to B'$ in $\mathbb{B}^{\mathrm{whi}}$ is freely added to $\mathbb{B}$ using a horizontal adjoint equivalence (b, d, η, ϵ) in $\mathbb{B}$. If there is a horizontal adjoint equivalence (a, c, η', ϵ') in $\mathbb{A}$ such that its image under I is (b, d, η, ϵ), then (a, c, η', ϵ') is the unique such in $\mathbb{A}$ since I is injective on objects, faithful on horizontal morphisms, and fully faithful on squares. Hence the vertical morphism freely added in $\mathbb{A}^{\mathrm{whi}}$ using this horizontal adjoint equivalence (a, c, η', ϵ') is the unique vertical morphism in $\mathbb{A}^{\mathrm{whi}}$ that is sent to v. This shows that I^{whi} is faithful on vertical morphisms. If there is no horizontal adjoint equivalence in $\mathbb{A}$ whose image under I is (b, d, η, ϵ), then v is freely added in $\mathbb{B}^{\mathrm{whi}}$ to the image of $\mathbb{A}^{\mathrm{whi}}$. Hence $U\mathbf{V}\mathbb{B}^{\mathrm{whi}}$ is obtained from the image of $U\mathbf{V}\mathbb{A}^{\mathrm{whi}}$ by freely adding objects and then morphisms between objects. This shows that I^{whi} is in $\mathcal{I}_w-\text{cell}$.

Now suppose that I is a double functor in $\mathcal{I}_w-\text{cof}$ which is fully faithful on squares. We factor I as

$$\begin{tikzcd} \mathbb{A} \arrow[rr, "I"] \arrow[dr, "K"'] & & \mathbb{B} \\ & \mathbb{C} \arrow[ur, "Q"'] & \end{tikzcd}$$

with $K \in \mathcal{I}_w-\text{cell}$ and $Q \in \mathcal{I}_w-\text{inj}$. Since both I and Q are fully faithful on squares, then K is also fully faithful on squares. Since I has the left lifting property with respect to Q, by the retract argument (see Proposition 4.1.6), we have that I is a retract of K. Since the replacement $(-)^{\mathrm{whi}}$ is functorial, then I^{whi} is also a retract of K^{whi}. By the first part of the proof, we have that K^{whi} is in $\mathcal{I}_w-\text{cell}$ and hence this shows that I^{whi} is in $\mathcal{I}_w-\text{cof}$. This concludes the proof. $\square$

We now use this result to prove that the weakly horizontally invariant replacement of a trivial cofibration is a $\mathcal{J}_w$-cofibration.

Proposition 8.2.14. *Let $I\colon \mathbb{A} \to \mathbb{B}$ be a double functor in $\mathcal{C} \cap \mathcal{W}$. Then the induced double functor $I^{\mathrm{whi}}\colon \mathbb{A}^{\mathrm{whi}} \to \mathbb{B}^{\mathrm{whi}}$ is in $\mathcal{J}_w-\mathrm{cof}$.*

Proof. Since $I \in \mathcal{W}$, the induced double functor $I^{\mathrm{whi}}\colon \mathbb{A}^{\mathrm{whi}} \to \mathbb{B}^{\mathrm{whi}}$ is a double biequivalence. We factor I^{whi} as

$$\begin{tikzcd} \mathbb{A}^{\mathrm{whi}} \arrow[rr, "I^{\mathrm{whi}}"] \arrow[dr, "J"'] & & \mathbb{B}^{\mathrm{whi}} \\ & \mathbb{C} \arrow[ur, "P"'] & \end{tikzcd}$$

with $J \in \mathcal{J}_w-\mathrm{cell}$ and $P \in \mathcal{J}_w-\mathrm{inj}$. Since $J\colon \mathbb{A}^{\mathrm{whi}} \to \mathbb{C}$ is a relative $\mathcal{J}_w$-cell complex with $\mathbb{A}^{\mathrm{whi}}$ weakly horizontally invariant, then J is a double biequivalence by Proposition 8.2.10. Since both I^{whi} and J are double biequivalences, then P is also a double biequivalence, by 2-out-of-3. Hence P is both a $\mathcal{J}_w$-injective and double biequivalence, and therefore P is an $\mathcal{I}_w$-injective by Proposition 8.2.7. Since I^{whi}, $j_{\mathbb{A}}$ and $j_{\mathbb{B}}$ are fully faithful on squares and $I^{\mathrm{whi}} j_{\mathbb{A}} = j_{\mathbb{B}} I$, then I is also fully faithful on squares. Hence, by Lemma 8.2.13, the double functor I^{whi} is in $\mathcal{I}_w-\mathrm{cof}$. It follows that I^{whi} has the left lifting property with respect to $P \in \mathcal{I}_w-\mathrm{inj}$ and, by the retract argument (see Proposition 4.1.6), we get that I^{whi} is a retract of $J \in \mathcal{J}_w-\mathrm{cell}$. This shows that I^{whi} is in $\mathcal{J}_w-\mathrm{cof}$. ☐

Finally, we prove that every $\mathcal{J}_w$-injective with weakly horizontally invariant target lifts against every trivial cofibration, by using its lifting property against the weakly horizontally invariant replacement of such a double functor.

Proposition 8.2.15. *Let $P\colon \mathbb{A} \to \mathbb{B}$ be a double functor with $\mathbb{B}$ weakly horizontally invariant. Then P is in $\mathcal{F}$ if and only if P is in $\mathcal{J}_w-\mathrm{inj}$.*

Proof. If P is in $\mathcal{F}$, then P is in $\mathcal{J}_w-\mathrm{inj}$ by Lemma 8.2.12. Now suppose that P is in $\mathcal{J}_w-\mathrm{inj}$. We show that P has the right lifting property with every double functor in $\mathcal{C}\cap\mathcal{W}$, i.e., it is in $(\mathcal{C} \cap \mathcal{W})^{\boxslash} = \mathcal{F}$. Let $I\colon \mathbb{C} \to \mathbb{D}$ be a double functor in $\mathcal{C} \cap \mathcal{W}$ and consider a commutative square in DblCat of the form

$$\begin{tikzcd} \mathbb{C} \arrow[r, "F"] \arrow[d, "I"'] & \mathbb{A} \arrow[d, "P"] \\ \mathbb{D} \arrow[r, "G"'] \arrow[ur, dashed, "L"] & \mathbb{B}\,. \end{tikzcd}$$

We want to find a lift $L\colon \mathbb{D} \to \mathbb{A}$ in this diagram. Since $\mathbb{B}$ is weakly horizontally invariant, the double functor $\mathbb{B} \to [0]$ has the right lifting property with respect to coproducts of $J_4\colon \mathbb{H}E_{\mathrm{adj}} \to \mathbb{H}^{\simeq}E_{\mathrm{adj}}$ by Remark 8.2.6. Hence there is a lift in the below left diagram.

$$\begin{tikzcd} \bigsqcup_{\mathrm{HorEq}(\mathbb{D})} \mathbb{H}E_{\mathrm{adj}} \arrow[r] \arrow[d, "\bigsqcup_{\mathrm{HorEq}(\mathbb{D})} J_4"'] & \mathbb{D} \arrow[r, "G"] & \mathbb{B} \\ \bigsqcup_{\mathrm{HorEq}(\mathbb{D})} \mathbb{H}^{\simeq}E_{\mathrm{adj}} \arrow[urr, dashed, "K"'] & & \end{tikzcd}
\qquad
\begin{tikzcd} \bigsqcup_{\mathrm{HorEq}(\mathbb{D})} \mathbb{H}E_{\mathrm{adj}} \arrow[r] \arrow[d, "\bigsqcup_{\mathrm{HorEq}(\mathbb{D})} J_4"'] & \mathbb{D} \arrow[d, "j_{\mathbb{D}}"] \arrow[ddr, bend left, "G"] & \\ \bigsqcup_{\mathrm{HorEq}(\mathbb{D})} \mathbb{H}^{\simeq}E_{\mathrm{adj}} \arrow[r, "\tau_{\mathbb{D}}"'] \arrow[drr, bend right, "K"'] & \mathbb{D}^{\mathrm{whi}} \arrow[ul, phantom, "\ulcorner"] \arrow[dr, dashed, "\hat{G}"'] & \\ & & \mathbb{B} \end{tikzcd}$$

By the universal property of the pushout, there is a unique double functor $\hat{G}\colon \mathbb{D}^{\mathrm{whi}} \to \mathbb{B}$ making the above right diagram commute. Now, since $P \in \mathcal{J}_w-\mathrm{inj}$, by Proposition 8.2.5, it has the right lifting property with respect to coproducts of $J_4\colon \mathbb{H}E_{\mathrm{adj}} \to \mathbb{H}^{\simeq}E_{\mathrm{adj}}$. Hence there is a lift in the following commutative diagram

$$\begin{tikzcd}
\bigsqcup_{\mathrm{HorEq}(\mathbb{C})} \mathbb{H}E_{\mathrm{adj}} \arrow[r] \arrow[d, "\bigsqcup_{\mathrm{HorEq}(\mathbb{C})} J_4"'] & \mathbb{C} \arrow[r, "F"] & \mathbb{A} \arrow[d, "P"] \\
\bigsqcup_{\mathrm{HorEq}(\mathbb{C})} \mathbb{H}^{\simeq}E_{\mathrm{adj}} \arrow[r, "I_1"'] \arrow[urr, dashed, "K'"] & \bigsqcup_{\mathrm{HorEq}(\mathbb{D})} \mathbb{H}^{\simeq}E_{\mathrm{adj}} \arrow[r, "K"'] & \mathbb{B},
\end{tikzcd}$$

where I_1 is the double functor induced from I as described in Construction 8.1.9. By the universal property of the pushout $\mathbb{C}^{\mathrm{whi}}$, there is a unique double functor $\hat{F}\colon \mathbb{C}^{\mathrm{whi}} \to \mathbb{A}$ making the following diagram commute.

$$\begin{tikzcd}
\bigsqcup_{\mathrm{HorEq}(\mathbb{C})} \mathbb{H}E_{\mathrm{adj}} \arrow[r] \arrow[d, "\bigsqcup_{\mathrm{HorEq}(\mathbb{C})} J_4"'] & \mathbb{C} \arrow[d, "j_{\mathbb{C}}"] \arrow[ddr, bend left, "F"] & \\
\bigsqcup_{\mathrm{HorEq}(\mathbb{C})} \mathbb{H}^{\simeq}E_{\mathrm{adj}} \arrow[r, "\tau_{\mathbb{C}}"'] \arrow[drr, bend right, "K'"'] & \mathbb{C}^{\mathrm{whi}} \arrow[dr, dashed, "\hat{F}"'] & \\
 & & \mathbb{A}
\end{tikzcd}$$

Since $I^{\mathrm{whi}}j_{\mathbb{C}} = j_{\mathbb{D}}I$ and $I^{\mathrm{whi}}\tau_{\mathbb{C}} = \tau_{\mathbb{D}}I_1$, we have $\hat{G}I^{\mathrm{whi}}j_{\mathbb{C}} = \hat{G}j_{\mathbb{D}}I = GI = PF = P\hat{F}j_{\mathbb{C}}$ and $\hat{G}I^{\mathrm{whi}}\tau_{\mathbb{C}} = \hat{G}\tau_{\mathbb{D}}I_1 = KI_1 = PK' = P\hat{F}\tau_{\mathbb{C}}$. Since such a double functor is unique by the universal property of the pushout $\mathbb{C}^{\mathrm{whi}}$, we get that $\hat{G}I^{\mathrm{whi}} = P\hat{F}$ and hence the following diagram commutes.

$$\begin{tikzcd}
\mathbb{C} \arrow[r, "j_{\mathbb{C}}"] \arrow[d, "I"'] \arrow[rr, bend left, "F"] & \mathbb{C}^{\mathrm{whi}} \arrow[r, "\hat{F}"] \arrow[d, "I^{\mathrm{whi}}"'] & \mathbb{A} \arrow[d, "P"] \\
\mathbb{D} \arrow[r, "j_{\mathbb{D}}"'] \arrow[rr, bend right, "G"'] & \mathbb{D}^{\mathrm{whi}} \arrow[r, "\hat{G}"'] \arrow[ur, dashed, "\hat{L}"] & \mathbb{B}
\end{tikzcd}$$

Since $I \in \mathcal{C} \cap \mathcal{W}$, then I^{whi} is in $\mathcal{J}_w$−cof by Proposition 8.2.14. Since $P \in \mathcal{J}_w$−inj, it has the right lifting property with respect to I^{whi} and there is a lift $\hat{L}\colon \mathbb{D}^{\mathrm{whi}} \to \mathbb{A}$ in the right-hand square of the above diagram. Then the composite $L \coloneqq \hat{L}j_{\mathbb{D}}\colon \mathbb{D} \to \mathbb{A}$ gives the desired lift since $PL = P\hat{L}j_{\mathbb{D}} = \hat{G}j_{\mathbb{D}} = G$ and $LI = \hat{L}j_{\mathbb{D}}I = \hat{L}I^{\mathrm{whi}}j_{\mathbb{C}} = \hat{F}j_{\mathbb{C}} = F$. This shows that $P \in \mathcal{F}$ and concludes the proof. □

8.3. **Remaining of the proof of Theorem 8.1.15.** We now aim to give the remaining of the proof of Theorem 8.1.15. In other words, we need to show that the pair $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$ forms a weak factorization system and that the fibrant objects in DblCat are precisely the weakly horizontally invariant double categories. This second result is a direct consequence of Proposition 8.2.15.

Theorem 8.3.1. *A double category $\mathbb{A}$ is fibrant, i.e., it is such that the double functor $\mathbb{A} \to [0]$ is in $\mathcal{F}$, if and only if it is weakly horizontally invariant.*

Proof. We recall from Proposition 8.2.3 that a double category $\mathbb{A}$ is weakly horizontally invariant if and only if the double functor $\mathbb{A} \to [0]$ is in $\mathcal{J}_w$−inj. Then, since $[0]$ is weakly horizontally invariant, the result follows directly from Proposition 8.2.15. □

We now show that the weakly horizontally invariant replacements given in Construction 8.1.9 are fibrant replacements in our model structure. As a consequence of Proposition 8.2.10, since the weakly horizontally invariant replacements $j_{\mathbb{A}}\colon \mathbb{A} \to \mathbb{A}^{\mathrm{whi}}$ are $\mathcal{J}_w$-cofibration, we first get the following result.

Proposition 8.3.2. *Let $\mathbb{A}$ be a weakly horizontally invariant double category. Then the double functor $j_{\mathbb{A}}\colon \mathbb{A} \to \mathbb{A}^{\mathrm{whi}}$ is a double biequivalence.*

Proof. Note that, by construction, $j_{\mathbb{A}}\colon \mathbb{A} \to \mathbb{A}^{\mathrm{whi}}$ is a double functor in $\{J_4\}{-}\mathrm{cof}$. Since $\mathcal{J}_w{-}\mathrm{inj} \subseteq \{J_4\}{-}\mathrm{inj}$ by Proposition 8.2.5, then we have that

$$\{J_4\}{-}\mathrm{cof} = {}^{\boxslash}\{J_4\}{-}\mathrm{inj} \subseteq {}^{\boxslash}\mathcal{J}_w{-}\mathrm{inj} = \mathcal{J}_w{-}\mathrm{cof}.$$

Hence $j_{\mathbb{A}}$ is a $\mathcal{J}_w$-cofibration with weakly horizontally invariant source, and hence it is a double biequivalence by Remark 8.2.11. □

As a corollary of this result, we get that the weakly horizontally invariant replacements $j_{\mathbb{A}}\colon \mathbb{A} \to \mathbb{A}^{\mathrm{whi}}$ are trivial cofibrations, and hence are fibrant replacements in our model structure, since the weakly horizontally invariant double categories are precisely the fibrant double categories by Theorem 8.3.1.

Corollary 8.3.3. *Let $\mathbb{A}$ be a double category. Then the double functor $j_{\mathbb{A}}\colon \mathbb{A} \to \mathbb{A}^{\mathrm{whi}}$ is in $\mathcal{C} \cap \mathcal{W}$. In particular, this gives a fibrant replacement of $\mathbb{A}$.*

Proof. Note that, using Corollary 8.1.5, one can check that $J_4\colon \mathbb{H}E_{\mathrm{adj}} \to \mathbb{H}^{\simeq}E_{\mathrm{adj}}$ is in $\mathcal{I}_w{-}\mathrm{cof}$. Since $j_{\mathbb{A}}$ is obtained as a pushout of coproducts of J_4 and $\mathcal{I}_w{-}\mathrm{cof}$ is closed under coproducts and pushouts, then $j_{\mathbb{A}} \in \mathcal{I}_w{-}\mathrm{cof} = \mathcal{C}$. It remains to show that $j_{\mathbb{A}} \in \mathcal{W}$. We want to show that $(j_{\mathbb{A}})^{\mathrm{whi}}$ is a double biequivalence. Since $(j_{\mathbb{A}})^{\mathrm{whi}}$ is defined using the universal property of pushout, by uniqueness of such a double functor, we can see that $(j_{\mathbb{A}})^{\mathrm{whi}} = j_{\mathbb{A}^{\mathrm{whi}}}$. Since $\mathbb{A}^{\mathrm{whi}}$ is weakly horizontally invariant, it follows from Proposition 8.3.2 that $j_{\mathbb{A}^{\mathrm{whi}}}$ is a double biequivalence. Hence $j_{\mathbb{A}} \in \mathcal{C} \cap \mathcal{W}$ by definition of $\mathcal{W}$. Moreover, since $j_{\mathbb{A}}$ is in $\mathcal{W}$, and $\mathbb{A}^{\mathrm{whi}}$ is fibrant, by Theorem 8.3.1, this gives a fibrant replacement of $\mathbb{A}$. □

We can also prove, using Proposition 8.3.2, that every weak equivalence with fibrant source is in particular a double biequivalence, and this gives a characterization of the weak equivalences between fibrant objects.

Proposition 8.3.4. *Let $F\colon \mathbb{A} \to \mathbb{B}$ be a double functor with $\mathbb{A}$ weakly horizontally invariant. Then F is in $\mathcal{W}$ if and only if F is a double biequivalence.*

Proof. If F is a double biequivalence, then F is in $\mathcal{W}$ by Proposition 8.1.18. Now suppose that F is in $\mathcal{W}$. By definition of $\mathcal{W}$, the induced double functor $F^{\mathrm{whi}}\colon \mathbb{A}^{\mathrm{whi}} \to \mathbb{B}^{\mathrm{whi}}$ is a double biequivalence. We prove that F satisfies (db1-4) of Definition 7.2.1. Since F and F^{whi} coincide on underlying horizontal categories by Remark 8.1.10 and $j_{\mathbb{B}}$ is fully faithful on squares, then F satisfies (db1-2) since F does so. Moreover, since $j_{\mathbb{A}}$, $j_{\mathbb{B}}$, and F^{whi} are fully faithful on squares and $F^{\mathrm{whi}} j_{\mathbb{A}} = j_{\mathbb{B}} F$, then F satisfies (db4).

It remains to prove (db3) for F. By Proposition 8.3.2, since $\mathbb{A}$ is weakly horizontally invariant, the double functor $j_{\mathbb{A}}\colon \mathbb{A} \to \mathbb{A}^{\mathrm{whi}}$ is a double biequivalence. Hence the composite $F^{\mathrm{whi}} j_{\mathbb{A}}\colon \mathbb{A} \to \mathbb{B}^{\mathrm{whi}}$ is a double biequivalence. Let $v\colon B \arrownot\rightarrow B'$ be a vertical morphism in $\mathbb{B}$. By (db3) for $F^{\mathrm{whi}} j_{\mathbb{A}}$, there is a vertical morphism $u\colon A \arrownot\rightarrow A'$ in $\mathbb{A}$ and a weakly horizontally invertible square β in $\mathbb{B}^{\mathrm{whi}}$ as depicted below left.

$$\begin{tikzcd}
FA \arrow[r, "b", "\simeq"'] \arrow[d, "j_{\mathbb{B}}Fu = F^{\mathrm{whi}} j_{\mathbb{A}} u"'] \arrow[dr, phantom, "\beta \simeq"] & B \arrow[d, "j_{\mathbb{B}} v"] \\
FA' \arrow[r, "\simeq", "b'"'] & B'
\end{tikzcd}
\qquad
\begin{tikzcd}
FA \arrow[r, "b", "\simeq"'] \arrow[d, "Fu"'] \arrow[dr, phantom, "\beta' \simeq"] & B \arrow[d, "v"] \\
FA' \arrow[r, "\simeq", "b'"'] & B'
\end{tikzcd}$$

Since $j_{\mathbb{B}}F = F^{\mathrm{whi}} j_{\mathbb{A}}$ and $j_{\mathbb{B}}$ is fully faithful on squares, this yields a weakly horizontally invertible square β' in $\mathbb{B}$ as depicted above right. This shows (db3) for F and hence shows that F is a double biequivalence. □

We are now ready to finish the proof of Theorem 8.1.15 by proving that the classes of trivial cofibrations and fibrations form a weak factorization system. We first show that every double functor can be factored as a trivial cofibration followed by a fibration.

Theorem 8.3.5. *Let $F\colon \mathbb{A} \to \mathbb{B}$ be a double functor. Then there is a factorization of F as*

$$\begin{tikzcd} \mathbb{A} \arrow[rr, "F"] \arrow[dr, "I"'] & & \mathbb{B} \\ & \mathbb{C} \arrow[ur, "P"'] & \end{tikzcd}$$

with $I \in \mathcal{C} \cap \mathcal{W}$ and $P \in \mathcal{F}$.

Proof. Let $F\colon \mathbb{A} \to \mathbb{B}$. We consider the induced double functor $F^{\mathrm{whi}}\colon \mathbb{A}^{\mathrm{whi}} \to \mathbb{B}^{\mathrm{whi}}$ and we factor it as

$$\begin{tikzcd} \mathbb{A}^{\mathrm{whi}} \arrow[rr, "F^{\mathrm{whi}}"] \arrow[dr, "J"'] & & \mathbb{B}^{\mathrm{whi}} \\ & \mathbb{C} \arrow[ur, "P"'] & \end{tikzcd}$$

with $J \in \mathcal{J}_w{-}\mathrm{cell}$ and $P \in \mathcal{J}_w{-}\mathrm{inj}$. Then, by Proposition 8.2.15, the double functor P is also in $\mathcal{F}$ since $\mathbb{B}^{\mathrm{whi}}$ is weakly horizontally invariant. In particular, note that $\mathbb{C}$ is also weakly horizontally invariant, since $\mathbb{B}^{\mathrm{whi}}$ is fibrant and $P\colon \mathbb{C} \to \mathbb{B}^{\mathrm{whi}}$ is a fibration, by Theorem 8.3.1. We define $\mathbb{D}$ to be the pullback of P along $j_{\mathbb{B}}$, as depicted below.

$$\begin{tikzcd} \mathbb{A} \arrow[dr, dashed, "K"] \arrow[rr, bend left, "F"] \arrow[d, "j_{\mathbb{A}}"'] & & \mathbb{B} \arrow[dd, "j_{\mathbb{B}}"] \\ \mathbb{A}^{\mathrm{whi}} \arrow[dr, "J"'] & \mathbb{D} \arrow[ur, "P'"] \arrow[d, "\pi"'] \arrow[dr, phantom, "\lrcorner", very near start] & \\ & \mathbb{C} \arrow[r, "P"'] & \mathbb{B}^{\mathrm{whi}} \end{tikzcd}$$

Since P' is a pullback of $P \in \mathcal{F}$, it is also in $\mathcal{F}$.

We now show that $\pi\colon \mathbb{D} \to \mathbb{C}$ is in $\mathcal{W}$. For this, we construct a fibrant replacement $\hat{\pi}\colon \mathbb{D}^{\mathrm{whi}} \to \mathbb{C}$ for π such that $\pi = \hat{\pi} j_{\mathbb{D}}$ and then show that $\hat{\pi}$ is a double biequivalence. Since double biequivalences are contained in $\mathcal{W}$ by Proposition 8.1.18 and the double functor $j_{\mathbb{D}}\colon \mathbb{D} \to \mathbb{D}^{\mathrm{whi}}$ is in $\mathcal{W}$ by Corollary 8.3.3, this would imply that π is in $\mathcal{W}$ as well. Consider the set $\mathrm{HorEq}(\mathbb{D})$ of horizontal adjoint equivalences in $\mathbb{D}$. Then, the image under K of the set $\mathrm{HorEq}(\mathbb{A})$ of horizontal adjoint equivalences in $\mathbb{A}$, which we denote by S, is contained in $\mathrm{HorEq}(\mathbb{D})$. Let $T = \mathrm{HorEq}(\mathbb{D}) \setminus S$ be the complement of S. Then, since $\mathbb{C}$ is weakly horizontally invariant, by Remark 8.2.6, the double functor $\mathbb{C} \to [0]$ has the right lifting property with respect to coproducts of $J_4\colon \mathbb{H}E_{\mathrm{adj}} \to \mathbb{H}^{\simeq}E_{\mathrm{adj}}$ and the identity on $\mathbb{H}^{\simeq}E_{\mathrm{adj}}$. Hence there is a lift in the following diagram.

$$\begin{tikzcd} (\bigsqcup_S \mathbb{H}^{\simeq}E_{\mathrm{adj}}) \bigsqcup (\bigsqcup_T \mathbb{H}E_{\mathrm{adj}}) \arrow[r] \arrow[d, "(\bigsqcup_S \mathrm{id}_{\mathbb{H}^{\simeq}E_{\mathrm{adj}}}) \bigsqcup (\bigsqcup_T J_4)"'] & \mathbb{A}^{\mathrm{whi}} \bigsqcup \mathbb{D} \arrow[r, "J \bigsqcup \pi"] & \mathbb{C} \\ \bigsqcup_{S \cup T = \mathrm{HorEq}(\mathbb{D})} \mathbb{H}^{\simeq}E_{\mathrm{adj}} \arrow[urr, dashed, "L"'] & & \end{tikzcd}$$

Then, by the universal property of the pushout $\mathbb{D}^{\mathrm{whi}}$, there is a unique double functor $\hat{\pi}\colon \mathbb{D}^{\mathrm{whi}} \to \mathbb{C}$ making the following diagram commute,

$$\begin{tikzcd}
\bigsqcup_{\mathrm{HorEq}(\mathbb{D})} \mathbb{H}E_{\mathrm{adj}} \arrow[r] \arrow[d, "\bigsqcup_{\mathrm{HorEq}(\mathbb{D})} J_4"'] & \mathbb{D} \arrow[d, "j_{\mathbb{D}}"] \arrow[ddr, bend left, "\pi"] & \\
\bigsqcup_{\mathrm{HorEq}(\mathbb{D})} \mathbb{H}^{\simeq}E_{\mathrm{adj}} \arrow[r, "\tau_{\mathbb{D}}"'] \arrow[drr, bend right, "L"'] & \mathbb{D}^{\mathrm{whi}} \arrow[dr, dashed, "\hat{\pi}"'] & \\
 & & \mathbb{C}
\end{tikzcd}$$

where the outer square commutes by definition of L and since $\pi K = J j_{\mathbb{A}}$. Then, since $K^{\mathrm{whi}} j_{\mathbb{A}} = j_{\mathbb{D}} K$ and $K^{\mathrm{whi}} \tau_{\mathbb{A}} = \tau_{\mathbb{D}} K_1$, we have that $\hat{\pi} K^{\mathrm{whi}} j_{\mathbb{A}} = \hat{\pi} j_{\mathbb{D}} K = \pi K = J j_{\mathbb{A}}$ and $\hat{\pi} K^{\mathrm{whi}} \tau_{\mathbb{A}} = \hat{\pi} \tau_{\mathbb{D}} K_1 = L K_1 = J \tau_{\mathbb{A}}$. Since such a double functor is unique by the universal property of the pushout $\mathbb{A}^{\mathrm{whi}}$, we have the relation $\hat{\pi} K^{\mathrm{whi}} = J$.

We now show that $\hat{\pi}$ satisfies (db1-4) of Definition 7.2.1 of a double biequivalence. First note that π is fully faithful on squares, as it is a pullback of the double functor $j_{\mathbb{B}}\colon \mathbb{B} \to \mathbb{B}^{\mathrm{whi}}$ which is fully faithful on squares, and double functors which are fully faithful on squares are precisely the double functors with the right lifting property with respect to $I_4\colon \delta\mathbb{S} \to \mathbb{S}$ and $I_5\colon \mathbb{S}_2 \to \mathbb{S}$ of Notation 8.1.1. Hence, since $\hat{\pi} j_{\mathbb{D}} = \pi$ and $j_{\mathbb{D}}$, π are fully faithful on squares, then so is $\hat{\pi}$, i.e., it satisfies (db4). Then, since $J\colon \mathbb{A}^{\mathrm{whi}} \to \mathbb{C}$ is in $\mathcal{J}_w-$cell and $\mathbb{A}^{\mathrm{whi}}$ is weakly horizontally invariant, then J is a double biequivalence by Proposition 8.2.10, and (db1-3) for $\hat{\pi}$ follow from the fact that J satisfies (db1-3). Indeed, if C is an object in $\mathbb{C}$, then, by (db1) for J, there is an object $A \in \mathbb{A}^{\mathrm{whi}}$ together with a horizontal equivalence $c\colon C \xrightarrow{\simeq} JA = \hat{\pi} K^{\mathrm{whi}} A$ in $\mathbb{C}$, and this gives (db1) for $\hat{\pi}$. We can prove (db2-3) for $\hat{\pi}$ in a similar manner. This shows that $\hat{\pi}$ is a double biequivalence, and hence that π is in $\mathcal{W}$.

Finally, since J is a double biequivalence, it is in $\mathcal{W}$ by Proposition 8.1.18. Moreover, the double functor $j_{\mathbb{A}}\colon \mathbb{A} \to \mathbb{A}^{\mathrm{whi}}$ is in $\mathcal{W}$ by Corollary 8.3.3. Hence, since $j_{\mathbb{A}}$, J, and π are in $\mathcal{W}$ and $\pi K = J j_{\mathbb{A}}$, then K is also in $\mathcal{W}$ by 2-out-of-3. We factor K as

$$\begin{tikzcd}
\mathbb{A} \arrow[rr, "K"] \arrow[dr, "I"'] & & \mathbb{D} \\
 & \mathbb{E} \arrow[ur, "Q"'] &
\end{tikzcd}$$

with $I \in \mathcal{I}_w-\mathrm{cell} \subseteq \mathcal{C}$ and $Q \in \mathcal{I}_w-\mathrm{inj}$. Then $Q \in \mathcal{W}$ by Proposition 8.1.17 and, by 2-out-of-3, we have that $I \in \mathcal{C} \cap \mathcal{W}$. Since Q is also in $\mathcal{F}$ by Proposition 8.1.17 and $F = P'K$, this gives a factorization of F as

$$\begin{tikzcd}
\mathbb{A} \arrow[rr, "F"] \arrow[dr, "I"'] & & \mathbb{B} \\
 & \mathbb{E} \arrow[ur, "P'Q"'] &
\end{tikzcd}$$

with $I \in \mathcal{C} \cap \mathcal{W}$ and $P'Q \in \mathcal{F}$, which concludes the proof. $\square$

As a direct consequence of this result, we get that the trivial cofibrations are precisely the double functors which have the left lifting property with respect to all fibrations.

Corollary 8.3.6. *We have that* $\mathcal{C} \cap \mathcal{W} = {}^{\boxslash}\mathcal{F}$.

Proof. By definition of $\mathcal{F}$, we already know that $\mathcal{C} \cap \mathcal{W} \subseteq {}^{\boxslash}\mathcal{F}$. It remains to show that ${}^{\boxslash}\mathcal{F} \subseteq \mathcal{C} \cap \mathcal{W}$. Let $F\colon \mathbb{A} \to \mathbb{B}$ be a double functor in ${}^{\boxslash}\mathcal{F}$. By Theorem 8.3.5, there is a factorization of F as

$$\begin{tikzcd} \mathbb{A} \arrow[rr, "F"] \arrow[dr, "I"'] & & \mathbb{B} \\ & \mathbb{C} \arrow[ur, "P"'] & \end{tikzcd}$$

with $I \in \mathcal{C} \cap \mathcal{W}$ and $P \in \mathcal{F}$. Since F has the left lifting property with respect to P, by the retract argument (see Proposition 4.1.6), we get that F is a retract of $I \in \mathcal{C} \cap \mathcal{W}$ and, since $\mathcal{C} \cap \mathcal{W}$ is closed under retracts by Remark 8.1.16, then $F \in \mathcal{C} \cap \mathcal{W}$. This concludes the proof that $\mathcal{C} \cap \mathcal{W} = {}^{\boxslash}\mathcal{F}$. □

Remark 8.3.7. This shows that $\mathcal{J}_w$−cof $\subseteq \mathcal{C} \cap \mathcal{W}$. Indeed, we have that $\mathcal{F} \subseteq \mathcal{J}_w$−inj by Lemma 8.2.12, and hence we get that $\mathcal{J}_w$−cof $= {}^{\boxslash}\mathcal{J}_w$−inj $\subseteq {}^{\boxslash}\mathcal{F} = \mathcal{C} \cap \mathcal{W}$.

8.4. **Quillen pairs with** 2Cat **and the first model structure on** DblCat**.** We now turn our attention to the relation of this new model structure on DblCat with the first model structure on DblCat constructed in Theorem 7.1.3 and with Lack's model structure on 2Cat. The identity functor on DblCat induces a Quillen pair between our two model structures on DblCat, and further shows that the homotopy theory of weakly horizontally invariant double categories is included into that of double categories. As for its relation with Lack's model structure on 2Cat, while the horizontal embedding $\mathbb{H}\colon 2\mathrm{Cat} \to \mathrm{DblCat}$ is still left Quillen, it is not right Quillen anymore. However, its more homotopical version $\mathbb{H}^{\simeq}\colon 2\mathrm{Cat} \to \mathrm{DblCat}$ (see Definition 3.4.11) provides such a right Quillen functor. In particular, the double category $\mathbb{H}^{\simeq}\mathcal{A}$ associated to a 2-category $\mathcal{A}$ is weakly horizontally invariant and provides a fibrant replacement of the horizontal double category $\mathbb{H}\mathcal{A}$.

We first show that the identity adjunction is a Quillen reflection, embedding the homotopy of weakly horizontally invariant double categories into that of double categories.

Theorem 8.4.1. *The identity adjunction*

$$\begin{tikzcd} \mathrm{DblCat}_{\mathrm{whi}} \arrow[r, bend right, "\mathrm{id}"'] \arrow[r, phantom, "\perp"] & \mathrm{DblCat} \arrow[l, bend right, "\mathrm{id}"'] \end{tikzcd}$$

is a Quillen reflection, where DblCat *is endowed with the model structure of Theorem 7.1.3 and* $\mathrm{DblCat}_{\mathrm{whi}}$ *is endowed with the model structure of Theorem 8.1.15 for weakly horizontally invariant double categories.*

Proof. We first show that the sets $\mathcal{I}$ and $\mathcal{J}$ of generating cofibrations and trivial cofibrations of DblCat introduced in Notation 7.3.7 are included in $\mathcal{I}_w$−cof and $\mathcal{J}_w$−cof, respectively, where the sets $\mathcal{I}_w$ and $\mathcal{J}_w$ are the sets given in Notations 8.1.1 and 8.2.1, respectively. First note that the double functors I_1, I_2, I_4, and I_5 of $\mathcal{I}$ are also elements of $\mathcal{I}_w$. Then the double functor $I_3\colon \emptyset \to \mathbb{V}2$ in $\mathcal{I}$ can be decomposed as

$$\emptyset = \emptyset \sqcup \emptyset \xrightarrow{I_1 \sqcup I_1} [0] \sqcup [0] \xrightarrow{I_3'} \mathbb{V}[1]\,,$$

where $I_1 \sqcup I_1$ and I_3' are in $\mathcal{I}_w$−cof, since $\mathcal{I}_w$−cof is closed under coproducts and I_1, I_3' are in $\mathcal{I}_w$, by definition. This shows that $I_3 \in \mathcal{I}_w$−cof, and hence that $\mathcal{I} \subseteq \mathcal{I}_w$−cof. On the other hand, the double functors J_1 and J_2 of $\mathcal{J}$ are also elements of $\mathcal{J}_w$. Then the double functor $J_3\colon \mathbb{V}[1] \to \mathbb{W} = \mathbb{H}E_{\mathrm{adj}} \times \mathbb{V}[1]$ in $\mathcal{J}$ can be obtained as a retract of $J_3'\colon \mathbb{W}^- \to \mathbb{W}$ of $\mathcal{J}_w$ as follows,

$$\begin{tikzcd}
\mathbb{V}[1] \arrow[r, "I"] \arrow[rr, bend left, equal] \arrow[d, "J_3"'] & \mathbb{W}^- \arrow[r, "R"] \arrow[d, "J_3'"'] & \mathbb{V}[1] \arrow[d, "J_3"] \\
\mathbb{W} \arrow[r, equal] & \mathbb{W} \arrow[r, equal] & \mathbb{W}
\end{tikzcd}$$

where the double functor $I\colon \mathbb{V}[1] \to \mathbb{W}^-$ sends the non trivial vertical morphism of $\mathbb{V}[1]$ to the non trivial vertical morphism in $\mathbb{W}^-$, and the double functor $R\colon \mathbb{W}^- \to \mathbb{V}[1]$ sends the non trivial vertical morphism of $\mathbb{W}^-$ to the non trivial vertical morphism in $\mathbb{V}[1]$ and the two horizontal adjoint equivalences to identities. This shows that $J_3 \in \mathcal{J}_w{-}\mathrm{cof}$, and hence that $\mathcal{J} \subseteq \mathcal{J}_w{-}\mathrm{cof}$.

With these results, we can show that the left adjoint $\mathrm{id}\colon \mathrm{DblCat} \to \mathrm{DblCat}_{\mathrm{whi}}$ is left Quillen. Indeed, it sends every generating cofibration in $\mathcal{I}$ to a cofibration in $\mathrm{DblCat}_{\mathrm{whi}}$, i.e., a double functor in $\mathcal{C} = \mathcal{I}_w{-}\mathrm{cof}$, and it sends every generating trivial cofibration in $\mathcal{J}$ to a double functor in $\mathcal{J}_w{-}\mathrm{cof}$, which is included in the class $\mathcal{C} \cap \mathcal{W}$ of trivial cofibrations in $\mathrm{DblCat}_{\mathrm{whi}}$, by Remark 8.3.7.

It remains to show that the derived counit is level-wise a weak equivalence in $\mathrm{DblCat}_{\mathrm{whi}}$. Let $\mathbb{A}$ be a fibrant double category in $\mathrm{DblCat}_{\mathrm{whi}}$. Then the component of the derived counit at $\mathbb{A}$ is given by the cofibrant replacement $q_{\mathbb{A}}\colon \mathbb{A}^c \to \mathbb{A}$ in the first model structure on DblCat. In particular, the double functor $q_{\mathbb{A}}$ is a double biequivalence, and hence a weak equivalence in $\mathrm{DblCat}_{\mathrm{whi}}$ by Proposition 8.1.18. This shows that the identity adjunction is a Quillen reflection. ☐

However, the identity adjunction does not induce a Quillen equivalence between the two model structures on DblCat, as shown in the following remark.

Remark 8.4.2. The derived unit is not a level-wise double biequivalence. To see this, first note that the component of the derived unit at a cofibrant double category $\mathbb{A}$ in the model structure on DblCat of Theorem 7.1.3 is given by a fibrant replacement $j_{\mathbb{A}}\colon \mathbb{A} \to \mathbb{A}^f$ in the model structure on DblCat of Theorem 8.1.15 for weakly horizontally invariant double categories. In particular, we can consider the weakly horizontally invariant replacement $j_{\mathbb{A}}\colon \mathbb{A} \to \mathbb{A}^{\mathrm{whi}}$ given in Construction 8.1.9 by Corollary 8.3.3. We need to show that this weakly horizontally invariant replacement is not always a double biequivalence. To see this, let $\mathbb{A}$ be the double category spanned by the following data,

$$\begin{tikzcd}
A \arrow[d, "v"', "\bullet" marking] & \\
A' \arrow[r, "a", "\simeq"'] & B' \arrow[d, "w", "\bullet" marking] \\
& B''
\end{tikzcd}$$

where a comes with horizontal adjoint equivalence data (a, c, η, ϵ). Note that, by Theorem 7.3.6, the double category $\mathbb{A}$ is cofibrant in the model structure on DblCat of Theorem 7.1.3. The double category $\mathbb{A}^{\mathrm{whi}}$ is obtained from $\mathbb{A}$ by adding $\mathbb{H}^{\simeq}E_{\mathrm{adj}}$-data extending the horizontal adjoint equivalence (a, c, η, ϵ). In particular, a vertical morphism $u\colon A' \to B'$ gets freely added to $\mathbb{A}^{\mathrm{whi}}$. Then the composite $wuv\colon A \to B''$ in $\mathbb{A}^{\mathrm{whi}}$ does not admit a lift as required by (db3) of Definition 7.2.1, since there are no vertical morphisms between the objects A and B'' in $\mathbb{A}$. Hence $j_{\mathbb{A}}$ is not a double biequivalence, which proves the desired result.

First, as a direct consequence of the Quillen pair between the two model structures on DblCat and the fact that $\mathbb{H} \dashv \mathbf{H}$ is a Quillen pair between Lack's model structure on 2Cat and the first model structure on DblCat, we get that $\mathbb{H} \dashv \mathbf{H}$ is also a Quillen pair between 2Cat and the model structure on DblCat for weakly horizontally invariant double categories. Furthermore, it is a Quillen co-reflection.

Theorem 8.4.3. *The adjunction*

$$\begin{tikzcd} \mathrm{DblCat} \arrow[r, bend right, "\mathbf{H}"'] \arrow[r, phantom, "\perp"] & 2\mathrm{Cat} \arrow[l, bend right, "\mathbb{H}"'] \end{tikzcd}$$

is a Quillen co-reflection, where 2Cat *is endowed with the model structure of Theorem 6.1.8 and* DblCat *is endowed with the model structure of Theorem 8.1.15.*

Proof. The fact that it is a Quillen pair follows directly from Theorems 7.4.1 and 8.4.1. We now show that the derived unit is level-wise a biequivalence. Let $\mathcal{A}$ be a cofibrant 2-category. Then the component of the derived unit at $\mathcal{A}$ is given by the underlying horizontal 2-functor of a fibrant replacement $\mathbf{H}j_{\mathbb{H}\mathcal{A}}\colon \mathcal{A} = \mathbf{H}\mathbb{H}\mathcal{A} \to \mathbf{H}(\mathbb{H}\mathcal{A})^f$ of the horizontal double category $\mathbb{H}\mathcal{A}$ in DblCat. In particular, if we consider the fibrant replacement given by in Construction 8.1.9 (see also Corollary 8.3.3), it does not change the underlying horizontal 2-category of $\mathbb{H}\mathcal{A}$ by Remark 8.1.10. Hence, the 2-functor $\mathbf{H}j_{\mathbb{H}\mathcal{A}}$ is an identity, and in particular a biequivalence. This shows that $\mathbb{H} \dashv \mathbf{H}$ is a Quillen co-reflection. □

Remark 8.4.4. By the above theorem, we have that the functor $\mathbb{H}\colon 2\mathrm{Cat} \to \mathrm{DblCat}$ preserves cofibrations. Note that the functor $\mathbb{H}$ also reflects cofibrations. Indeed, given a 2-functor $F\colon \mathcal{A} \to \mathcal{B}$, if $\mathbb{H}F\colon \mathbb{H}\mathcal{A} \to \mathbb{H}\mathcal{B}$ is a cofibration in DblCat, then its underlying horizontal functor $U\mathbf{H}\mathbb{H}F$ has the left lifting property with respect to all surjective on objects and full functors by Theorem 8.1.4 and, since $U\mathbf{H}\mathbb{H}F = UF$, this proves Theorem 6.2.2 for F. Hence F is a cofibration in 2Cat.

While the functor $\mathbb{H}\colon 2\mathrm{Cat} \to \mathrm{DblCat}$ is still left Quillen with respect to this new model structure on DblCat, it is not right Quillen anymore, as not every double category is fibrant.

Remark 8.4.5. The functor $\mathbb{H}\colon 2\mathrm{Cat} \to \mathrm{DblCat}$ is not right Quillen with respect to the model structure on 2Cat of Theorem 6.1.8 and the model structure on DblCat of Theorem 8.1.15. To see this, consider the 2-category E_{adj} given by the "free-living adjoint equivalence". Then its associated horizontal double category $\mathbb{H}E_{\mathrm{adj}}$ is not weakly horizontally invariant. Indeed, there is no vertical morphism in $\mathbb{H}E_{\mathrm{adj}}$ filling the below diagram, since there are only trivial vertical morphisms in $\mathbb{H}E_{\mathrm{adj}}$.

$$\begin{tikzcd} 0 \arrow[r, "\simeq"] & 1 \arrow[d, equal, "\bullet" marking] \\ 1 \arrow[r, equal] & 1 \end{tikzcd}$$

Therefore $\mathbb{H}$ does not preserve fibrant objects, as every 2-category is fibrant, and the fibrant double categories are the weakly horizontally invariant ones by Theorem 8.3.1.

We now show that the more homotopical version $\mathbb{H}^{\simeq}$ of $\mathbb{H}$ is a right Quillen functor. For this, we first show that the double category $\mathbb{H}^{\simeq}\mathcal{A}$ associated to a 2-category $\mathcal{A}$ is weakly horizontally invariant.

Proposition 8.4.6. *Let $\mathcal{A}$ be a 2-category. Then the double category $\mathbb{H}^{\simeq}\mathcal{A}$ is weakly horizontally invariant.*

Proof. Let $a\colon A \xrightarrow{\simeq} C$ and $a'\colon A' \xrightarrow{\simeq} C'$ be horizontal equivalences in $\mathbb{H}^{\simeq}\mathcal{A}$, i.e., equivalence in $\mathcal{A}$, and let $w\colon C \twoheadrightarrow C'$ be a vertical morphism in $\mathbb{H}^{\simeq}\mathcal{A}$, i.e., an adjoint equivalence $w\colon C \xrightarrow{\simeq} C'$ in $\mathcal{A}$. Let us fix adjoint equivalence data (a, c, η, ϵ) and $(a', c', \eta', \epsilon')$ in $\mathcal{A}$ for a and a'. We set $u\colon A \twoheadrightarrow A'$ to be the vertical morphism in $\mathbb{H}^{\simeq}\mathcal{A}$ corresponding to the following composite of adjoint equivalences in $\mathcal{A}$

$$A \xrightarrow[\simeq]{a} C \xrightarrow[\simeq]{w} C' \xrightarrow[\simeq]{c'} A' \,.$$

Then, the 2-isomorphism $\alpha := \epsilon' wa\colon wa \cong a'c'wa = a'u$ induced by the counit ϵ' yields a weakly horizontally invertible square in $\mathbb{H}^{\simeq}\mathcal{A}$ as depicted below by Lemma 3.6.8.

$$\begin{tikzcd}
A \arrow[r, "a", "\simeq"'] \arrow[d, "u"', "\wr"] \arrow[dr, phantom, "\overset{\alpha}{\swarrow}\cong"] & C \arrow[d, "w", "\wr"'] \\
A' \arrow[r, "\simeq", "a'"'] & C'
\end{tikzcd}$$

This shows that $\mathbb{H}^{\simeq}\mathcal{A}$ is weakly horizontally invariant. $\square$

We now show that the double functor $\mathbb{H}^{\simeq}$ is a Quillen reflection.

Theorem 8.4.7. *The adjunction*

$$\begin{tikzcd}
\mathrm{2Cat} \arrow[r, bend right, "\mathbb{H}^{\simeq}"'] \arrow[r, phantom, "\perp"] & \mathrm{DblCat} \arrow[l, bend right, "L^{\simeq}"']
\end{tikzcd}$$

is a Quillen reflection, where 2Cat *is endowed with the model structure of Theorem 6.1.8 and* DblCat *is endowed with the model structure of Theorem 8.1.15.*

Proof. We show that $\mathbb{H}^{\simeq}\colon \mathrm{2Cat} \to \mathrm{DblCat}$ preserves fibrations and trivial fibrations. Let $F\colon \mathcal{A} \to \mathcal{B}$ be a Lack fibration in 2Cat. We show that $\mathbb{H}^{\simeq}F\colon \mathbb{H}^{\simeq}\mathcal{A} \to \mathbb{H}^{\simeq}\mathcal{B}$ is a fibration in DblCat. By Proposition 8.2.15, since $\mathbb{H}^{\simeq}\mathcal{B}$ is weakly horizontally invariant by Proposition 8.4.6, then $\mathbb{H}^{\simeq}F$ is a fibration in DblCat if and only if it is a $\mathcal{J}_w$-injective. Hence, we need to prove that $\mathbb{H}^{\simeq}F$ satisfies (df1-2) of Definition 7.2.6 and (df3') of Proposition 8.2.4. Since $\mathbf{H}\mathbb{H}^{\simeq}F = F$ is a Lack fibration in 2Cat, we have that $\mathbb{H}^{\simeq}F$ satisfies (df1-2) by Remark 7.2.7. It remains to prove (df3'). Let $w\colon C \twoheadrightarrow C'$ be a vertical morphism in $\mathbb{H}^{\simeq}\mathcal{A}$, i.e., an adjoint equivalence $w\colon C \xrightarrow{\simeq} C'$ in $\mathcal{A}$, and let $a\colon A \xrightarrow{\simeq} C$ and $a'\colon A' \xrightarrow{\simeq} C'$ be horizontal equivalences in $\mathbb{H}^{\simeq}\mathcal{A}$, i.e., equivalences in $\mathcal{A}$. Suppose that we have a weakly horizontally invertible square β in $\mathbb{H}^{\simeq}\mathcal{B}$, i.e., a 2-isomorphism in $\mathcal{B}$ by Lemma 3.6.8, as follows.

$$\begin{tikzcd}
FA \arrow[r, "Fa", "\simeq"'] \arrow[d, "v"', "\wr"] \arrow[dr, phantom, "\overset{\beta}{\swarrow}\cong"] & FC \arrow[d, "Fw", "\wr"'] \\
FA' \arrow[r, "\simeq", "Fa'"'] & FC'
\end{tikzcd}$$

Let (c', a', η, ϵ) be an adjoint equivalence data for a', and let δ be the 2-isomorphism in $\mathcal{B}$ given by the below left pasting.

$$
\begin{tikzcd}
FA \arrow[r, "Fa"] \arrow[d, "v"'] \arrow[dr, phantom, "\overset{\beta}{\Swarrow}\cong"] & FC \arrow[d, "Fw"] \\
FA' \arrow[r, "Fa'"] \arrow[dr, equal] \arrow[dr, phantom, shift left=3, "\overset{F\epsilon}{\Swarrow}\cong"] & FC' \arrow[d, "Fc'"] \\
& FA'
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
A \arrow[r, "a"] \arrow[dd, "u"'] \arrow[ddr, phantom, shift left=2, "\overset{\overline{\alpha}}{\Swarrow}\cong"] & C \arrow[d, "w"] \\
& C' \arrow[dl, "c'"'] \arrow[d, equal] \arrow[dl, phantom, shift left=3, "\overset{\eta}{\Swarrow}\cong"] \\
A' \arrow[r, "a'"'] & C'
\end{tikzcd}
$$

By (f2) of Definition 6.1.7 for F applied to δ, we get a morphism $u\colon A \to A'$ in $\mathcal{A}$ and a 2-isomorphism $\overline{\alpha}\colon c'wa \cong u$ in $\mathcal{A}$ such that $\delta = F\overline{\alpha}$. Note that u is an equivalence in $\mathcal{A}$ since u is isomorphic to the equivalence $c'wa$. Therefore, it induces a vertical morphism $u\colon A \to\!\!\!\!\bullet\; A'$ in $\mathbb{H}^{\simeq}\mathcal{A}$ by choosing an adjoint equivalence data for u. We set $\alpha\colon wa \cong a'u$ to be the above right pasting composite. Then, by the triangle identities for (η, ϵ) and the fact that $\delta = F\overline{\alpha}$, we get that $\beta = F\alpha$. Hence α gives a weakly horizontally invertible square in $\mathbb{H}^{\simeq}\mathcal{A}$ such that $\beta = (\mathbb{H}^{\simeq}F)\alpha$, as desired. This shows that $\mathbb{H}^{\simeq}F$ is a fibration in DblCat.

Now let $F\colon \mathcal{A} \to \mathcal{B}$ be a trivial fibration in 2Cat. By Proposition 6.1.11, we directly get that $\mathbb{H}^{\simeq}F$ is surjective on objects, full on horizontal morphisms, and fully faithful on squares, as these latter are just coming from 2-morphisms in $\mathcal{A}$ and $\mathcal{B}$. Fullness on vertical morphisms follows from the fact that F is full on morphisms and fully faithful on squares, and therefore, by Lemma 6.1.10, a lift of an adjoint equivalence is an adjoint equivalence. This shows that $\mathbb{H}^{\simeq}F$ is a trivial fibration in DblCat by Proposition 8.1.2, and hence that $\mathbb{H}^{\simeq}$ is right Quillen.

It remains to show that the derived counit is level-wise a biequivalence. Let $\mathcal{A}$ be a 2-category. Let $q_{\mathbb{H}^{\simeq}\mathcal{A}}\colon (\mathbb{H}^{\simeq}\mathcal{A})^c \to \mathbb{H}^{\simeq}\mathcal{A}$ denote the cofibrant replacement of $\mathbb{H}^{\simeq}\mathcal{A}$ constructed as follows. The double category $(\mathbb{H}^{\simeq}\mathcal{A})^c$ has the same objects as $\mathcal{A}$; it has a copy $\overline{a}$ for each morphism a in $\mathcal{A}$ and horizontal morphisms in $(\mathbb{H}^{\simeq}\mathcal{A})^c$ are given by free composites of $\overline{a}$'s; it has a copy $\overline{u}$ for each adjoint equivalence u in $\mathcal{A}$ and vertical morphisms in $(\mathbb{H}^{\simeq}\mathcal{A})^c$ are given by free composites of $\overline{u}$'s; and squares in $(\mathbb{H}^{\simeq}\mathcal{A})^c$ are given by squares of $\mathbb{H}^{\simeq}\mathcal{A}$ whose boundaries are the actual composites in $\mathbb{H}^{\simeq}\mathcal{A}$ of the representative of the free composites. This double category is indeed cofibrant, as its underlying horizontal and vertical categories are free, and the projection $q_{\mathbb{H}^{\simeq}\mathcal{A}}\colon (\mathbb{H}^{\simeq}\mathcal{A})^c \to \mathbb{H}^{\simeq}\mathcal{A}$ is a trivial fibration in DblCat since it clearly satisfies the conditions of Proposition 8.1.2. Then, the derived counit

$$L^{\simeq}(\mathbb{H}^{\simeq}\mathcal{A})^c \xrightarrow{L^{\simeq}q_{\mathbb{H}^{\simeq}\mathcal{A}}} L^{\simeq}\mathbb{H}^{\simeq}\mathcal{A} \xrightarrow{\epsilon_{\mathcal{A}}} \mathcal{A}$$

is a trivial fibration in 2Cat. Indeed, the 2-category $L^{\simeq}(\mathbb{H}^{\simeq}\mathcal{A})^c$ is obtained from $\mathcal{A}$ as follows. It has the same objects as $\mathcal{A}$; there is a copy $\overline{a}$ for each morphism a in $\mathcal{A}$ and a copy $\overline{u}$ for each adjoint equivalence u in $\mathcal{A}$, and morphisms in $L^{\simeq}(\mathbb{H}^{\simeq}\mathcal{A})^c$ are given by free composites of $\overline{a}$'s and $\overline{u}$'s; and 2-morphisms in $L^{\simeq}(\mathbb{H}^{\simeq}\mathcal{A})^c$ are given by 2-morphisms in $\mathcal{A}$ whose boundaries are the actual composites in $\mathcal{A}$ of the representatives of the free composites. With this description, we can see that the derived counit at $\mathcal{A}$ is a trivial fibration in 2Cat as it clearly satisfies the conditions of Proposition 6.1.11. In particular, it is a biequivalence. This shows that $L^{\simeq} \dashv \mathbb{H}^{\simeq}$ is a Quillen reflection. □

Remark 8.4.8. The components of the derived unit of the adjunction $L^{\simeq} \dashv \mathbb{H}^{\simeq}$ are not weak equivalences in DblCat in general. To see this, consider the double category $\mathbb{V}[1]$ free on a vertical morphism. Then $\mathbb{V}[1]$ is cofibrant in DblCat by Corollary 8.1.6, and we have that $\mathbb{H}^{\simeq}L^{\simeq}\mathbb{V}[1] = \mathbb{H}^{\simeq}E_{\mathrm{adj}}$. Since all objects are fibrant in 2Cat, the component of the derived unit at $\mathbb{V}[1]$ is given by the inclusion $\eta_{\mathbb{V}[1]}\colon \mathbb{V}[1] \to \mathbb{H}^{\simeq}E_{\mathrm{adj}}$ sending the non trivial vertical morphism of $\mathbb{V}[1]$ to the vertical morphism of $\mathbb{H}^{\simeq}E_{\mathrm{adj}}$ represented by the adjoint equivalence $0 \xrightarrow{\simeq} 1$ in E_{adj}. Furthermore, since $\mathbb{V}[1]$ is weakly horizontally invariant, it is enough to see that $\eta_{\mathbb{V}[1]}\colon \mathbb{V}[1] \to \mathbb{H}^{\simeq}E_{\mathrm{adj}}$ is not a double biequivalence, to show that it is not a weak equivalence in DblCat, by Proposition 8.3.4. Since $\mathbb{V}[1]$ and $\mathbb{H}^{\simeq}E_{\mathrm{adj}}$ have the same objects 0 and 1, but $\mathbb{H}^{\simeq}E_{\mathrm{adj}}$ has non trivial horizontal morphisms between 0 and 1, then $\eta_{\mathbb{V}[1]}$ can not satisfy (db2) of Definition 7.2.1 and therefore it is not a double biequivalence.

We now prove that the functor $\mathbb{H}^{\simeq}$ provides a level-wise fibrant replacement of $\mathbb{H}$ in the model structure on DblCat for weakly horizontally invariant double categories.

Theorem 8.4.9. *Let $\mathcal{A}$ be a 2-category. The inclusion double functor $J_{\mathcal{A}}\colon \mathbb{H}\mathcal{A} \to \mathbb{H}^{\simeq}\mathcal{A}$ is a double biequivalence. In particular, this exhibits $\mathbb{H}^{\simeq}\mathcal{A}$ as a fibrant replacement of $\mathbb{H}\mathcal{A}$ in the model structure on* DblCat *of Theorem 8.1.15.*

Proof. We show that the inclusion double functor $J_{\mathcal{A}}\colon \mathbb{H}\mathcal{A} \to \mathbb{H}^{\simeq}\mathcal{A}$ satisfies (db1-4) of Definition 7.2.1. Since $\mathbf{H}\mathbb{H}\mathcal{A} = \mathcal{A} = \mathbf{H}\mathbb{H}^{\simeq}\mathcal{A}$, the inclusion $J_{\mathcal{A}}$ is the identity on underlying horizontal 2-categories. Since $\mathbb{H}\mathcal{A}$ is completely determined by its underlying horizontal 2-category, it follows that $J_{\mathcal{A}}$ satisfies (db1-2) and (db4). It remains to show (db3). Let $u\colon A \twoheadrightarrow A'$ be a vertical morphism in $\mathbb{H}^{\simeq}\mathcal{A}$, i.e., an adjoint equivalence $u\colon A \xrightarrow{\simeq} A'$ in $\mathcal{A}$. Then the following square

$$\begin{tikzcd} A \arrow[r, equal] \arrow[d, equal, "J_{\mathcal{A}}(e_A)"'] \arrow[dr, phantom, "\overset{\mathrm{id}_u}{\underset{\cong}{\swarrow}}"] & A \arrow[d, "u", "\wr"'] \\ A \arrow[r, "\simeq", "u"'] & A' \end{tikzcd}$$

gives a weakly horizontally invertible square in $\mathbb{H}^{\simeq}\mathcal{A}$ as required. This shows that $J_{\mathcal{A}}$ is a double biequivalence and hence it is a weak equivalence in DblCat by Proposition 8.1.18. □

As a direct consequence of this result and the Quillen reflection between the two model structures on DblCat, we get the following result.

Corollary 8.4.10. *The adjunction*

$$\begin{tikzcd} 2\mathrm{Cat} \arrow[r, bend right, "\mathbb{H}^{\simeq}"'] \arrow[r, phantom, "\perp"] & \mathrm{DblCat} \arrow[l, bend right, "L^{\simeq}"'] \end{tikzcd}$$

is a Quillen reflection, where 2Cat *is endowed with the model structure of Theorem 6.1.8 and* DblCat *is endowed with the model structure of Theorem 7.1.3.*

Proof. This follows directly from Theorems 8.4.1 and 8.4.7. □

While Theorem 8.4.7 implies that $\mathbb{H}^{\simeq}\colon 2\mathrm{Cat} \to \mathrm{DblCat}$ preserves weak equivalences and fibrations, the following result implies that it further reflects weak equivalences and fibrations. Hence the model structure on 2Cat is completely determined from the model structure on DblCat for weakly horizontally invariant double categories through its image under $\mathbb{H}^{\simeq}$.

Theorem 8.4.11. *The model structure on* 2Cat *of Theorem 6.1.8 is right-induced along the adjunction*

$$\begin{tikzcd} 2\mathrm{Cat} \arrow[r, bend right, "\mathbb{H}^{\simeq}"'] \arrow[r, phantom, "\perp"] & \mathrm{DblCat} \arrow[l, bend right, "L^{\simeq}"'] \end{tikzcd}$$

from the model structure on DblCat *of Theorem 8.1.15.*

Proof. We need to show that a 2-functor F is a biequivalence (resp. Lack fibration) in 2Cat if and only if the double functor $\mathbb{H}^{\simeq}F$ is a weak equivalence (resp. fibration) in DblCat. Since the functor $\mathbb{H}^{\simeq}$ is right Quillen by Theorem 8.4.7, it preserves fibrations and, since all objects in 2Cat are fibrant, it preserves weak equivalences by Corollary 4.4.7. This shows that, if F is a biequivalence (resp. Lack fibration) in 2Cat, then $\mathbb{H}^{\simeq}F$ is a weak equivalence (resp. fibration) in DblCat. Conversely, if $\mathbb{H}^{\simeq}F$ is a weak equivalence in DblCat, then it is a double biequivalence by Proposition 8.3.4, since its source is weakly horizontally invariant by Proposition 8.4.6. Hence $\mathbf{H}\mathbb{H}^{\simeq}F = F$ is a biequivalence in 2Cat by Proposition 7.2.5. Finally, if $\mathbb{H}^{\simeq}F$ is a fibration in DblCat, then it is a $\mathcal{J}_w$-injective by Proposition 8.2.15, since its target is weakly horizontally invariant by Proposition 8.4.6. Hence $\mathbf{H}\mathbb{H}^{\simeq}F = F$ is a Lack fibration in 2Cat by Remark 7.2.7, since F satisfies (df1-2) of Definition 7.2.6 by Proposition 8.2.4. Therefore, the model structure on 2Cat is right-induced along $\mathbb{H}^{\simeq}$ from that on DblCat. ☐

Finally, by composing the Quillen reflection $P \dashv D$ between Cat and 2Cat with the Quillen reflection $L^{\simeq} \dashv \mathbb{H}^{\simeq}$ between 2Cat and DblCat, we get the following result.

Corollary 8.4.12. *The adjunction*

$$\begin{tikzcd} \mathrm{Cat} \arrow[r, bend right, "\mathbb{H}^{\simeq}D"'] \arrow[r, phantom, "\perp"] & \mathrm{DblCat} \arrow[l, bend right, "PL^{\simeq}"'] \end{tikzcd}$$

is a Quillen reflection, where Cat *is endowed with the model structure of Theorem 6.1.3 and* DblCat *is endowed with the model structure of Theorem 8.1.15.*

Proof. This follows directly from Theorems 8.4.7 and 6.1.14. ☐

As the canonical model structure on Cat is right-induced along D from Lack's model structure on 2Cat and this latter is right-induced along $\mathbb{H}^{\simeq}$ from the model structure on DblCat for weakly horizontally invariant double categories, we get the following result.

Corollary 8.4.13. *The model structure on* Cat *of Theorem 6.1.3 is right-induced along the adjunction*

$$\begin{tikzcd} \mathrm{Cat} \arrow[r, bend right, "\mathbb{H}^{\simeq}D"'] \arrow[r, phantom, "\perp"] & \mathrm{DblCat} \arrow[l, bend right, "PL^{\simeq}"'] \end{tikzcd}$$

from the model structure on DblCat *of Theorem 8.1.15.*

Proof. This follows directly from Theorems 8.4.11 and 6.1.16. ☐

8.5. **Monoidality.** We now turn our attention to the monoidality of the model structure on DblCat for weakly horizontally invariant double categories. By a similar argument to the one in Remark 7.5.1, we can also show that this new model structure on DblCat is not monoidal with respect to the cartesian product. However, by fixing the asymmetry between the horizontal and vertical directions, as we show in this section, it is monoidal with respect to the Gray tensor product $\otimes_{\mathrm{Gr}}$ on DblCat, as introduced in Proposition 3.3.5.

We first give a description of the Gray tensor product of two double categories.

Description 8.5.1. Let $\mathbb{A}$ and $\mathbb{X}$ be double categories. Their Gray tensor product $\mathbb{A}\otimes_{\mathrm{Gr}}\mathbb{X}$ is the double category described by the following data:

(i) an object in $\mathbb{A} \otimes_{\mathrm{Gr}} \mathbb{X}$ is a pair (A, X) of an object $A \in \mathbb{A}$ and an object $X \in \mathbb{X}$,
(ii) horizontal morphisms in $\mathbb{A} \otimes_{\mathrm{Gr}} \mathbb{X}$ are generated by the following horizontal morphisms:

- a horizontal morphism $(a,X)\colon (A,X)\to(C,X)$, for each pair (a,X) of a horizontal morphism $a\colon A\to C$ in $\mathbb{A}$ and an object $X\in\mathbb{X}$, and
- a horizontal morphism $(A,x)\colon (A,X)\to(A,Z)$, for each pair (A,x) of an object $A\in\mathbb{A}$ and a horizontal morphism $x\colon X\to Z$ in $\mathbb{X}$,

(iii) vertical morphisms in $\mathbb{A}\otimes_{\mathrm{Gr}}\mathbb{X}$ are generated by the following vertical morphisms:

- a vertical morphism $(u,X)\colon (A,X)\to\!\!\!\!\bullet\ (A',X')$, for each pair (u,X) of a vertical morphism $u\colon A\to\!\!\!\!\bullet\ A'$ in $\mathbb{A}$ and an object $X\in\mathbb{X}$, and
- a vertical morphism $(A,t)\colon (A,X)\to\!\!\!\!\bullet\ (A,Z)$, for each pair (A,t) of an object $A\in\mathbb{A}$ and a vertical morphism $t\colon X\to\!\!\!\!\bullet\ Z$ in $\mathbb{X}$,

(iv) squares in $\mathbb{A}\otimes_{\mathrm{Gr}}\mathbb{X}$ are generated by the following squares:

- a square (α,X) as below left, for each pair (α,X) of a square $\alpha\colon(u\ {}^{a}_{a'}\ w)$ in $\mathbb{A}$ and an object $X\in\mathbb{X}$, and a square (A,χ) as below right, for each pair (A,χ) of an object $A\in\mathbb{A}$ and a square $\chi\colon(t\ {}^{b}_{b'}\ v)$ in $\mathbb{X}$,

$$\begin{tikzcd}
(A,X) \arrow[r, "{(a,X)}"] \arrow[d, "{(u,X)}"', "\bullet" marking] \arrow[dr, phantom, "{(\alpha,X)}"] & (C,X) \arrow[d, "{(w,X)}", "\bullet" marking] \\
(A',X) \arrow[r, "{(a',X)}"'] & (C',X)
\end{tikzcd}
\qquad
\begin{tikzcd}
(A,X) \arrow[r, "{(A,x)}"] \arrow[d, "{(A,t)}"', "\bullet" marking] \arrow[dr, phantom, "{(A,\chi)}"] & (A,Z) \arrow[d, "{(A,v)}", "\bullet" marking] \\
(A,X') \arrow[r, "{(A,x')}"'] & (A,Z')
\end{tikzcd}$$

- a square (a,t) as below left, for each pair (a,t) of a horizontal morphism $a\colon A\to C$ in $\mathbb{A}$ and a vertical morphism $t\colon X\to\!\!\!\!\bullet\ X'$ in $\mathbb{X}$, and a square (u,x) as below right, for each pair (u,x) of a vertical morphism $u\colon A\to\!\!\!\!\bullet\ A'$ in $\mathbb{A}$ and a horizontal morphism $x\colon X\to Z$ in $\mathbb{X}$,

$$\begin{tikzcd}
(A,X) \arrow[r, "{(a,X)}"] \arrow[d, "{(A,t)}"', "\bullet" marking] \arrow[dr, phantom, "{(a,t)}"] & (C,X) \arrow[d, "{(C,t)}", "\bullet" marking] \\
(A,X') \arrow[r, "{(a,X')}"'] & (C,X')
\end{tikzcd}
\qquad
\begin{tikzcd}
(A,X) \arrow[r, "{(A,x)}"] \arrow[d, "{(u,X)}"', "\bullet" marking] \arrow[dr, phantom, "{(u,x)}"] & (A,Z) \arrow[d, "{(u,Z)}", "\bullet" marking] \\
(A',X) \arrow[r, "{(A',x)}"'] & (A',Z)
\end{tikzcd}$$

- a vertically invertible square (a,x) as below, for each pair (a,x) of a horizontal morphism $a\colon A\to C$ in $\mathbb{A}$ and a horizontal morphism $x\colon X\to Z$ in $\mathbb{X}$,

$$\begin{tikzcd}
(A,X) \arrow[r, "{(a,X)}"] \arrow[d, equal, "\bullet" marking] \arrow[drr, phantom, "{(a,x)\ \cong}"] & (C,X) \arrow[r, "{(C,x)}"] & (C,Z) \arrow[d, equal, "\bullet" marking] \\
(A,X) \arrow[r, "{(A,x)}"'] & (A,Z) \arrow[r, "{(a,Z)}"'] & (C,Z)
\end{tikzcd}$$

- a horizontally invertible square (u,t) as below, for each pair (u,t) of a vertical morphism $u\colon A\to\!\!\!\!\bullet\ A'$ in $\mathbb{A}$ and a vertical morphism $t\colon X\to\!\!\!\!\bullet\ X'$ in $\mathbb{X}$.

$$\begin{tikzcd}
(A,X) \arrow[r, equal] \arrow[d, "{(u,X)}"', "\bullet" marking] & (A,X) \arrow[d, "{(A,t)}", "\bullet" marking] \\
(A',X) \arrow[r, phantom, "\overset{\cong}{(u,t)}"] \arrow[d, "{(A',t)}"', "\bullet" marking] & (A,X') \arrow[d, "{(u,X')}", "\bullet" marking] \\
(A',X') \arrow[r, equal] & (A',X')
\end{tikzcd}$$

subject to conditions which are equivalent to requiring that the below double functor $\Pi_{\mathbb{A},\mathbb{X}}\colon \mathbb{A}\otimes_{\mathrm{Gr}}\mathbb{X}\to\mathbb{A}\times\mathbb{X}$ is fully faithful on squares.

There is a double functor $\Pi_{\mathbb{A},\mathbb{X}}\colon \mathbb{A}\otimes_{\mathrm{Gr}}\mathbb{X}\to\mathbb{A}\times\mathbb{X}$, which is the identity on objects, sends the generating horizontal morphisms (a,X) and (A,x) of $\mathbb{A}\otimes_{\mathrm{Gr}}\mathbb{X}$ to the horizontal morphisms (a,id_X) and (id_A,x) in $\mathbb{A}\times\mathbb{X}$, respectively, sends the generating vertical morphisms (u,X) and (A,t) of $\mathbb{A}\otimes_{\mathrm{Gr}}\mathbb{X}$ to the vertical morphisms (u,e_X) and (e_A,t) in $\mathbb{A}\times\mathbb{X}$, respectively, and sends the generating squares (α,X), (A,χ), (a,t), (u,x), (a,x), and (u,t) of $\mathbb{A}\otimes_{\mathrm{Gr}}\mathbb{X}$ to the squares $(\alpha,\square_X)$, $(\square_A,\chi)$, (e_a,id_t), (id_u,e_x), $e_{(a,x)}$, and $\mathrm{id}_{(u,t)}$ in $\mathbb{A}\times\mathbb{X}$, respectively.

As in Lemma 6.3.3 dealing with the case of the tensor product for 2-categories, we can show that this projection double functor $\Pi_{\mathbb{A},\mathbb{X}}\colon \mathbb{A}\otimes_{\mathrm{Gr}}\mathbb{X}\to\mathbb{A}\times\mathbb{X}$ is a trivial fibration in the model structure on DblCat for weakly horizontally invariant double categories.

Lemma 8.5.2. *Let $\mathbb{A}$ and $\mathbb{X}$ be double categories. Then the projection double functor $\Pi_{\mathbb{A},\mathbb{X}}\colon \mathbb{A}\otimes_{\mathrm{Gr}}\mathbb{X}\to\mathbb{A}\times\mathbb{X}$ is a trivial fibration in the model structure on* DblCat *of Theorem 8.1.15.*

Proof. To show that $\Pi_{\mathbb{A},\mathbb{X}}$ is a trivial fibration, we use the characterization in Proposition 8.1.2. Since $\Pi_{\mathbb{A},\mathbb{X}}$ is the identity on objects, it is clearly surjective on objects. Given a horizontal morphism $(a,x)\colon (A,X)\to(C,Z)$ in $\mathbb{A}\times\mathbb{X}$, the composite

$$(A,X)\xrightarrow{(a,X)}(C,X)\xrightarrow{(C,x)}(C,Z)$$

of horizontal morphisms in $\mathbb{A}\otimes_{\mathrm{Gr}}\mathbb{X}$ is sent by $\Pi_{\mathbb{A},\mathbb{X}}$ to (a,x), which shows that $\Pi_{\mathbb{A},\mathbb{X}}$ is full on horizontal morphisms. Now, given a vertical morphism $(u,t)\colon (A,X)\,{\to\!\!\!\!\bullet}\,(A',X')$ in $\mathbb{A}\times\mathbb{X}$, the composite

$$(A,X)\overset{(u,X)}{\,{\to\!\!\!\!\bullet}\,}(A',X)\overset{(A',t)}{\,{\to\!\!\!\!\bullet}\,}(A',X')$$

of vertical morphisms in $\mathbb{A}\otimes_{\mathrm{Gr}}\mathbb{X}$ is sent by $\Pi_{\mathbb{A},\mathbb{X}}$ to (u,t), which shows that $\Pi_{\mathbb{A},\mathbb{X}}$ is full on vertical morphisms. Fully faithfulness on squares holds by Description 8.5.1 (iv). $\square$

As a consequence, we get the following lemma.

Lemma 8.5.3. *Let $\mathbb{A}$ and $\mathbb{B}$ be double categories, and let $F\colon\mathbb{A}\to\mathbb{B}$ be a double biequivalence. Then, for every double category $\mathbb{X}$, the induced double functors*

$$F\times\mathrm{id}_{\mathbb{X}}\colon \mathbb{A}\times\mathbb{X}\to\mathbb{B}\times\mathbb{X}\quad\textit{and}\quad F\otimes_{\mathrm{Gr}}\mathrm{id}_{\mathbb{X}}\colon \mathbb{A}\otimes_{\mathrm{Gr}}\mathbb{X}\to\mathbb{B}\otimes_{\mathrm{Gr}}\mathbb{X}$$

are also double biequivalences.

Proof. Let $\mathbb{X}$ be a double category. It is straightforward to see that (db1-4) of Definition 7.2.1 hold for the double functor $F\times\mathrm{id}_{\mathbb{X}}\colon \mathbb{A}\times\mathbb{X}\to\mathbb{B}\times\mathbb{X}$ since they hold for F. Therefore $F\times\mathrm{id}_{\mathbb{X}}$ is a double biequivalence.

Then, the double functors $\Pi_{\mathbb{A},\mathbb{X}}\colon \mathbb{A}\otimes_{\mathrm{Gr}}\mathbb{X}\to\mathbb{A}\times\mathbb{X}$ and $\Pi_{\mathbb{B},\mathbb{X}}\colon \mathbb{B}\otimes_{\mathrm{Gr}}\mathbb{X}\to\mathbb{B}\times\mathbb{X}$ are trivial fibrations, by Lemma 8.5.2, and hence they are in particular double biequivalences by Proposition 8.2.7. Since the following diagram commutes

$$\begin{tikzcd} \mathbb{A}\otimes_{\mathrm{Gr}}\mathbb{X} \arrow[r, "F\otimes_{\mathrm{Gr}}\mathrm{id}_{\mathbb{X}}"] \arrow[d, "\Pi_{\mathbb{A},\mathbb{X}}"'] & \mathbb{B}\otimes_{\mathrm{Gr}}\mathbb{X} \arrow[d, "\Pi_{\mathbb{B},\mathbb{X}}"] \\ \mathbb{A}\times\mathbb{X} \arrow[r, "F\times\mathrm{id}_{\mathbb{X}}"'] & \mathbb{B}\times\mathbb{X} \end{tikzcd}$$

and $F\times\mathrm{id}_{\mathbb{X}}$ is a double biequivalence, it follows by 2-out-of-3 that $F\otimes_{\mathrm{Gr}}\mathrm{id}_{\mathbb{X}}$ is also a double biequivalence. $\square$

Using this result, we can show that the same results holds for a general weak equivalence in the model structure on DblCat for weakly horizontally invariant double categories. For this we first need to show that $\mathbb{A}^{\mathrm{whi}} \times \mathbb{X}^{\mathrm{whi}}$ is a weakly horizontally invariant replacement of the product of the double categories $\mathbb{A}$ and $\mathbb{X}$.

Lemma 8.5.4. *Let $\mathbb{A}$ and $\mathbb{X}$ be double categories. Then the double category $\mathbb{A}^{\mathrm{whi}} \times \mathbb{X}^{\mathrm{whi}}$ is weakly horizontally invariant and the double functor $j_{\mathbb{A}} \times j_{\mathbb{X}} \colon \mathbb{A} \times \mathbb{X} \to \mathbb{A}^{\mathrm{whi}} \times \mathbb{X}^{\mathrm{whi}}$ is a weak equivalence in the model structure on* DblCat *of Theorem 8.1.15, where $j_{\mathbb{A}}$ and $j_{\mathbb{X}}$ are the double functors introduced in Construction 8.1.9. In particular, this gives a fibrant replacement of $\mathbb{A} \times \mathbb{X}$.*

Proof. First note that, since $\mathbb{A}^{\mathrm{whi}}$ and $\mathbb{X}^{\mathrm{whi}}$ are weakly horizontally invariant, they are fibrant, and hence their product $\mathbb{A}^{\mathrm{whi}} \times \mathbb{X}^{\mathrm{whi}}$ is also fibrant, i.e., it is weakly horizontally invariant, by Theorem 8.3.1.

By Construction 8.1.9, the projections $\pi_{\mathbb{A}} \colon \mathbb{A} \times \mathbb{X} \to \mathbb{A}$ and $\pi_{\mathbb{X}} \colon \mathbb{A} \times \mathbb{X} \to \mathbb{X}$ induce double functors $\pi_{\mathbb{A}}^{\mathrm{whi}} \colon (\mathbb{A} \times \mathbb{X})^{\mathrm{whi}} \to \mathbb{A}^{\mathrm{whi}}$ and $\pi_{\mathbb{X}}^{\mathrm{whi}} \colon (\mathbb{A} \times \mathbb{X})^{\mathrm{whi}} \to \mathbb{X}^{\mathrm{whi}}$ between the weakly horizontally invariant replacements. Hence this yields a commutative triangle

$$\begin{tikzcd}
& \mathbb{A} \times \mathbb{X} \arrow[dl, "j_{\mathbb{A}\times\mathbb{X}}"'] \arrow[dr, "j_{\mathbb{A}} \times j_{\mathbb{X}}"] & \\
(\mathbb{A} \times \mathbb{X})^{\mathrm{whi}} \arrow[rr, "(\pi_{\mathbb{A}}^{\mathrm{whi}}, \pi_{\mathbb{X}}^{\mathrm{whi}})"'] & & \mathbb{A}^{\mathrm{whi}} \times \mathbb{X}^{\mathrm{whi}} .
\end{tikzcd}$$

Since $j_{\mathbb{A}\times\mathbb{X}}$ is a weak equivalence by Corollary 8.3.3, to show that $j_{\mathbb{A}} \times j_{\mathbb{X}}$ is a weak equivalence, it is enough to show that $(\pi_{\mathbb{A}}^{\mathrm{whi}}, \pi_{\mathbb{X}}^{\mathrm{whi}})$ is a weak equivalence by 2-out-of-3. For this, we show that $(\pi_{\mathbb{A}}^{\mathrm{whi}}, \pi_{\mathbb{X}}^{\mathrm{whi}})$ is a trivial fibration using Proposition 8.1.2. First note that $(\pi_{\mathbb{A}}^{\mathrm{whi}}, \pi_{\mathbb{X}}^{\mathrm{whi}})$ is the identity on underlying horizontal categories, and it is fully faithful on squares since $j_{\mathbb{A}\times\mathbb{X}}$ and $j_{\mathbb{A}} \times j_{\mathbb{X}}$ are so. Hence $(\pi_{\mathbb{A}}^{\mathrm{whi}}, \pi_{\mathbb{X}}^{\mathrm{whi}})$ is clearly surjective on objects, full on horizontal morphisms, and fully faithful on squares. It remains to show that it is full on vertical morphisms. However, by studying the weakly horizontally invariant replacements, we can see that all the vertical morphisms that were freely added to $\mathbb{A}^{\mathrm{whi}} \times \mathbb{X}^{\mathrm{whi}}$ from the image of $\mathbb{A} \times \mathbb{X}$ were also freely added to $(\mathbb{A} \times \mathbb{X})^{\mathrm{whi}}$ from the image of $\mathbb{A} \times \mathbb{X}$. Hence it is clear that $(\pi_{\mathbb{A}}^{\mathrm{whi}}, \pi_{\mathbb{X}}^{\mathrm{whi}})$ is full on vertical morphisms. In particular, this shows that $(\pi_{\mathbb{A}}^{\mathrm{whi}}, \pi_{\mathbb{X}}^{\mathrm{whi}})$ is a weak equivalence, and hence that so is $j_{\mathbb{A}} \times j_{\mathbb{X}}$. □

Corollary 8.5.5. *Let $\mathbb{A}$ and $\mathbb{B}$ be double categories, and let $F \colon \mathbb{A} \to \mathbb{B}$ be a weak equivalence in the model structure on* DblCat *of Theorem 8.1.15. Then, for every double category $\mathbb{X}$, the induced double functors*

$$F \times \mathrm{id}_{\mathbb{X}} \colon \mathbb{A} \times \mathbb{X} \to \mathbb{B} \times \mathbb{X} \quad \textit{and} \quad F \otimes_{\mathrm{Gr}} \mathrm{id}_{\mathbb{X}} \colon \mathbb{A} \otimes_{\mathrm{Gr}} \mathbb{X} \to \mathbb{B} \otimes_{\mathrm{Gr}} \mathbb{X}$$

are also weak equivalences in DblCat*.*

Proof. Let $F \colon \mathbb{A} \to \mathbb{B}$ be a weak equivalence in DblCat. By definition, the weakly horizontally invariant replacement $F^{\mathrm{whi}} \colon \mathbb{A}^{\mathrm{whi}} \to \mathbb{B}^{\mathrm{whi}}$ is a double biequivalence. We show that $F \times \mathrm{id}_{\mathbb{X}} \colon \mathbb{A} \times \mathbb{X} \to \mathbb{B} \times \mathbb{X}$ is a weak equivalence by showing that it induces a double biequivalence between the fibrant replacements of $\mathbb{A} \times \mathbb{X}$ and $\mathbb{B} \times \mathbb{X}$ constructed in Lemma 8.5.4. In other words, we need to show that $F^{\mathrm{whi}} \times \mathrm{id}_{\mathbb{X}^{\mathrm{whi}}} \colon \mathbb{A}^{\mathrm{whi}} \times \mathbb{X}^{\mathrm{whi}} \to \mathbb{B}^{\mathrm{whi}} \times \mathbb{X}^{\mathrm{whi}}$ is a double biequivalence, which is the case by Lemma 8.5.3 since F^{whi} is a double biequivalence. This shows that $F \times \mathrm{id}_{\mathbb{X}}$ is a weak equivalence, by 2-out-of-3, since $j_{\mathbb{A}} \times j_{\mathbb{X}}$, $j_{\mathbb{B}} \times j_{\mathbb{X}}$, and $F^{\mathrm{whi}} \times \mathrm{id}_{\mathbb{X}^{\mathrm{whi}}}$ are weak equivalences. We can then deduce that $F \otimes_{\mathrm{Gr}} \mathrm{id}_{\mathbb{X}}$ is also a weak equivalence as in Lemma 8.5.3 since the trivial fibrations $\Pi_{\mathbb{A},\mathbb{X}}$ and $\Pi_{\mathbb{B},\mathbb{X}}$ are in particular weak equivalences. □

This allows us to prove that the model structure on DblCat for weakly horizontally invariant double categories is monoidal with respect to the Gray tensor product, inspired by the proof of Theorem 6.3.5 showing the monoidality of Lack's model structure on 2Cat.

Theorem 8.5.6. *The model structure on* DblCat *of Theorem 8.1.15 is monoidal with respect to the Gray tensor product* $\otimes_{\mathrm{Gr}}$.

Proof. We first show that the pushout-product $I \,\square_{\otimes_{\mathrm{Gr}}} J$ is a cofibration in DblCat, whenever I and J are cofibrations in DblCat. By Remark 4.5.7, it is enough to show the result when I and J are in the set of generating cofibrations $\mathcal{I}_w = \{I_1, I_2, I_3', I_4, I_5\}$ described in Notation 8.1.1. Furthermore, note that $I \,\square_{\otimes_{\mathrm{Gr}}} J \cong J \,\square_{\otimes_{\mathrm{Gr}}} I$ since the Gray tensor product is symmetric, and therefore it is enough to show the result for one of the two pushout-products.

Suppose that $I = I_1 \colon \emptyset \to [0]$. Since $\emptyset \otimes_{\mathrm{Gr}} \mathbb{A} \cong \emptyset$ and $[0] \otimes_{\mathrm{Gr}} \mathbb{A} \cong \mathbb{A}$ for every double category $\mathbb{A}$, then $I_1 \,\square_{\otimes_{\mathrm{Gr}}} J \cong J$ and it is a cofibration, for every $J \in \mathcal{I}_w$. Now suppose that I is one of the generating cofibrations $I_4 \colon \delta\mathbb{S} \to \mathbb{S}$ or $I_5 \colon \mathbb{S}_2 \to \mathbb{S}$. Then I is an isomorphism on underlying horizontal and vertical categories. Since $U\mathbf{H}$ and $U\mathbf{V}$ preserve pushouts and the underlying horizontal and vertical categories of the Gray tensor product $\mathbb{A} \otimes_{\mathrm{Gr}} \mathbb{B}$ only depends on the underlying horizontal and vertical categories of $\mathbb{A}$ and $\mathbb{B}$, for every pair of double categories $\mathbb{A}$ and $\mathbb{B}$, it follows that the functors

$$U\mathbf{H}(I \,\square_{\otimes_{\mathrm{Gr}}} J) \colon U\mathbf{H}(\mathbb{D} \otimes_{\mathrm{Gr}} \mathbb{B}) \coprod_{U\mathbf{H}(\mathbb{D} \otimes_{\mathrm{Gr}} \mathbb{A})} U\mathbf{H}(\mathbb{S} \otimes_{\mathrm{Gr}} \mathbb{A}) \to U\mathbf{H}(\mathbb{S} \otimes_{\mathrm{Gr}} \mathbb{B}),$$
$$U\mathbf{V}(I \,\square_{\otimes_{\mathrm{Gr}}} J) \colon U\mathbf{V}(\mathbb{D} \otimes_{\mathrm{Gr}} \mathbb{B}) \coprod_{U\mathbf{V}(\mathbb{D} \otimes_{\mathrm{Gr}} \mathbb{A})} U\mathbf{V}(\mathbb{S} \otimes_{\mathrm{Gr}} \mathbb{A}) \to U\mathbf{V}(\mathbb{S} \otimes_{\mathrm{Gr}} \mathbb{B})$$

are isomorphisms of categories, where $\mathbb{D}$ is either $\delta\mathbb{S}$ or $\mathbb{S}_2$, and $J \colon \mathbb{A} \to \mathbb{B}$ is in $\{I_2, I_3', I_4, I_5\}$. It follows from Corollary 8.1.5 that $I \,\square_{\otimes_{\mathrm{Gr}}} J$ is a cofibration. It remains to show that $I \,\square_{\otimes_{\mathrm{Gr}}} J$ is a cofibration for $I, J \in \{I_2, I_3'\}$. The pushout-product $I_2 \,\square_{\otimes_{\mathrm{Gr}}} I_2$ is given by the boundary inclusion $\delta(\mathbb{H}[1] \otimes_{\mathrm{Gr}} \mathbb{H}[1]) \to \mathbb{H}[1] \otimes_{\mathrm{Gr}} \mathbb{H}[1]$, where $\delta(\mathbb{H}[1] \otimes_{\mathrm{Gr}} \mathbb{H}[1])$ is the double subcategory of $\mathbb{H}[1] \otimes_{\mathrm{Gr}} \mathbb{H}[1]$ where we removed the non trivial squares. Similarly, the pushout-products $I_2 \,\square_{\otimes_{\mathrm{Gr}}} I_3'$ and $I_3' \,\square_{\otimes_{\mathrm{Gr}}} I_3'$ are given by the boundary inclusions $\delta(\mathbb{H}[1] \otimes_{\mathrm{Gr}} \mathbb{V}[1]) \to \mathbb{H}[1] \otimes_{\mathrm{Gr}} \mathbb{V}[1]$ and $\delta(\mathbb{V}[1] \otimes_{\mathrm{Gr}} \mathbb{V}[1]) \to \mathbb{V}[1] \otimes_{\mathrm{Gr}} \mathbb{V}[1]$, respectively. Since these three pushout-products induce isomorphisms on underlying horizontal and vertical categories, they are cofibrations by Corollary 8.1.5. This shows that $I \,\square_{\otimes_{\mathrm{Gr}}} J$ is a cofibration in DblCat whenever I and J are cofibrations in DblCat.

We now show that the pushout-product $I \,\square_{\otimes_{\mathrm{Gr}}} J$ is a trivial cofibration in DblCat, whenever I is a cofibration in DblCat and $J \colon \mathbb{A} \to \mathbb{B}$ is a trivial cofibration in DblCat. Again, it is enough to show the result for $I \in \mathcal{I}_w$. Note that all domains of the generating cofibrations in $\mathcal{I}_w$ are cofibrant by Corollary 8.1.6, since they have free underlying horizontal and vertical categories. Therefore, the generating cofibration $I \in \mathcal{I}_w$ is of the form $I \colon \mathbb{D} \to \mathbb{E}$ with $\mathbb{D}$ cofibrant. By Corollary 8.5.5, the double functors $\mathrm{id}_{\mathbb{D}} \otimes_{\mathrm{Gr}} J$ and $\mathrm{id}_{\mathbb{E}} \otimes_{\mathrm{Gr}} J$ are weak equivalences, since J is a weak equivalence. Furthermore, since $\mathbb{D}$ is cofibrant, J is a cofibration, and $\mathrm{id}_{\mathbb{D}} \otimes_{\mathrm{Gr}} J = (\emptyset \to \mathbb{D}) \,\square_{\otimes_{\mathrm{Gr}}} J$, it follows by the first part of the proof that $\mathrm{id}_{\mathbb{D}} \otimes_{\mathrm{Gr}} J$ is a cofibration in DblCat. Consider the following diagram.

$$\begin{tikzcd}
\mathbb{D} \otimes_{\mathrm{Gr}} \mathbb{A} \arrow[r, hook, "\mathrm{id}_{\mathbb{D}} \otimes_{\mathrm{Gr}} J", "\sim"'] \arrow[d, "I \otimes_{\mathrm{Gr}} \mathrm{id}_{\mathbb{A}}"'] & \mathbb{D} \otimes_{\mathrm{Gr}} \mathbb{B} \arrow[d] \arrow[ddr, bend left, "I \otimes_{\mathrm{Gr}} \mathrm{id}_{\mathbb{B}}"] & \\
\mathbb{E} \otimes_{\mathrm{Gr}} \mathbb{A} \arrow[r, hook, "K", "\sim"'] \arrow[urr, phantom, "\ulcorner" very near end] \arrow[drr, bend right, "\sim", "\mathrm{id}_{\mathbb{E}} \otimes_{\mathrm{Gr}} J"'] & \mathbb{P} \arrow[dr, dashed, "I \,\square_{\otimes_{\mathrm{Gr}}} J"] & \\
 & & \mathbb{E} \otimes_{\mathrm{Gr}} \mathbb{B}
\end{tikzcd}$$

Since trivial cofibrations are closed under pushouts and $\mathrm{id}_{\mathbb{D}} \otimes_{\mathrm{Gr}} J$ is a trivial cofibration by the above discussion, then K is also a trivial cofibration. Then $I \square_{\otimes_{\mathrm{Gr}}} J$ is a weak equivalence by 2-out-of-3 applied to $\mathrm{id}_{\mathbb{E}} \otimes_{\mathrm{Gr}} J = (I \square_{\otimes_{\mathrm{Gr}}} J)K$. This shows that $I \square_{\otimes_{\mathrm{Gr}}} J$ is a trivial cofibration in DblCat whenever I and J are cofibrations in DblCat such that one of I and J is trivial. This concludes the proof. $\square$

Remark 8.5.7. Recall that, by restricting the Gray tensor product $\otimes_{\mathrm{Gr}}$ in one variable along $\mathbb{H}\colon 2\mathrm{Cat} \to \mathrm{DblCat}$, we get the tensoring functor $\otimes\colon \mathrm{DblCat} \times 2\mathrm{Cat} \to \mathrm{DblCat}$ which gives an enrichment of DblCat over 2Cat as in Proposition 3.5.2. Since the functor $\mathbb{H}$ is left Quillen by Theorem 8.4.3, as a corollary of Theorem 8.5.6, we get that the model structure on DblCat of Theorem 8.1.15 is 2Cat-enriched with respect to the enrichment given by $\mathbf{H}[-,-]_{\mathrm{ps}}$.

PART IV.

HOMOTOPY THEORY OF ∞-ANALOGUES OF 2-DIMENSIONAL CATEGORIES

Higher category theory aims to study more structured objects than categories. Recall that a category consists of objects and a set morphisms between every pair of objects such that these morphisms compose associatively. Higher structures on categories can be obtained by adding higher morphisms, and by weakening the associativity constraint of the different compositions. As we have seen in Part I., 2-categories are obtained this way by also including 2-morphisms between the morphisms, but without changing the strictness of the associativity constraint for compositions. If this latter is relaxed to a 2-isomorphism, we obtain a *bicategory.* Continuing this process up to n-morphisms between $(n-1)$-morphisms, we obtain notions of *n-categories* and *weak n-categories*, depending on whether the compositions of k-morphisms are associative on the nose or up to higher invertible morphisms, for $k < n$. To encode full coherence, it is often convenient to allow morphisms in all dimensions and, by requiring all k-morphisms to be invertible for $k > n$, we obtain the notion of an *(∞, n)-category.* This latter should be thought of as a homotopical version of a (weak) n-category.

Higher categories can be obtained using the categorical tools of *enrichment* and *internalization.* As introduced in Section 1.1, an enriched category is a generalization of a category where the sets of morphisms are replaced by more structured objects, e.g. categories, simplicial sets, or topological spaces. In particular, a 2-category is obtained this way as a category enriched over categories. We can then iterate this process and define an n-category as a category enriched in $(n-1)$-categories. Accordingly, an (∞, n)-category should be a category enriched in $(\infty, n-1)$-categories. On the other hand, we recall that an internal category is a diagram in a category consisting of an object of objects and an object of morphisms, together with a composition given internally to the ambient category. In particular, as described in Remark 3.1.2, a double category is obtained this way as an internal category to categories. Moreover, in Remark 3.1.8, we have seen that a 2-category can also be seen as an internal category to categories with discrete category of objects. Going up in dimensions, an n-category can also be seen as an internal category to $(n-1)$-categories whose $(n-1)$-category of objects is discrete. In this sense, an (∞, n)-category should then correspond to an internal category to $(\infty, n-1)$-categories whose $(\infty, n-1)$-category of objects is discrete.

While n-categories are well-defined via these methods, to make sense of a notion of (∞, n)-categories, we need models. The machinery used here is often that of model categories, presented in Part II.. Model categories were first introduced by Quillen [Qui67] to axiomatize the homotopy theory of spaces, which are equivalent to $(\infty, 0)$-categories – also called *∞-groupoids* – by Grothendieck's homotopy hypothesis. In particular, the Kan-Quillen model structure for simplicial sets given in Theorem 5.2.7 gives a model for ∞-groupoids. As for $(\infty, 1)$-categories, they were first modeled by quasi-categories, originally defined by Boardman and Vogt [BV73], and further developed by Joyal [Joy02] and Lurie [Lur09]. Following the idea that an $(\infty, 1)$-category is an enriched category in ∞-groupoids, another model for $(\infty, 1)$-categories is given by simplicial categories whose simplicial homs are Kan complexes, as constructed by Bergner [Ber07]. In this thesis, we adopt the other point of view, which sees an $(\infty, 1)$-category as an internal category to ∞-groupoids with "discrete" ∞-groupoid of objects. This is given by the model of *complete Segal spaces*, introduced by Rezk [Rez01], where the discreteness condition is given by a *completeness* condition implying that the space of objects is weakly equivalent to the underlying ∞-groupoid of the $(\infty,1)$-category. There are also other models of $(\infty,1)$-categories and these were all shown to be equivalent.

Going one dimension up, we now want to describe ∞-analogues of the 2-dimensional categories introduced in Part I.. For this, we adopt the point of view that a double category is an internal category to categories and that a 2-category is also such a structure with discrete category of objects. Hence, we define a double $(\infty,1)$-category as an internal category to $(\infty,1)$-categories modeled by complete Segal spaces. In particular, this corresponds to the notion of Segal objects in complete Segal spaces introduced by Haugseng in [Hau13] as a model for double $(\infty,1)$-categories. However, in this thesis, since we want to consider $(\infty,2)$-categories as horizontally embedded in double $(\infty,1)$-categories, we require the completeness condition to hold in the horizontal direction, rather than in the vertical direction as in Haugseng's definition. Then, an $(\infty,2)$-category can be defined as a complete Segal object in complete Segal spaces whose complete Segal space of objects is essentially constant. This model of $(\infty,2)$-categories was introduced by Barwick in [Bar05] under the name of *2-fold complete Segal spaces*. In particular, both models of double $(\infty,1)$-categories and 2-fold complete Segal spaces are obtained as a left Bousfield localization (see Section 5.2) of the Reedy model structure on bisimplicial spaces. Moreover, the $(\infty,2)$-categorical model is a localization of the double $(\infty,1)$-categorical model, which gives us the desired "horizontal embedding" of $(\infty,2)$-categories into double $(\infty,1)$-categories.

In Section 9, we first recall the main features of the Reedy model structure on simplicial spaces, and then localize this model structure in order to obtain a model structure for complete Segal spaces. Then, in Section 10, after introducing the Reedy model structure on bisimplicial spaces, we localize it to obtain the two desired models of double $(\infty,1)$-categories, and $(\infty,2)$-categories in the form of 2-fold complete Segal spaces.

9. Complete Segal spaces as a model of $(\infty,1)$-categories

We present in this section the model for $(\infty,1)$-categories given by Rezk's complete Segal spaces, introduced in [Rez01]. A *Segal space* is a simplicial space $X\colon \Delta^{\mathrm{op}} \to \mathrm{sSet}$, where X_0 is thought of as the space of objects, X_1 as the space of morphisms, and X_2 as the space of composable pairs of morphisms, given by a *Segal condition* which requires that the spaces X_2 and $X_1 \times_{X_0} X_1$ are the same up to weak equivalence. Going up in dimensions, the space X_n is also identified with the space of n composable morphisms, for $n \geq 3$. Hence, in such a *Segal space*, there is a notion of *homotopy equivalences*, which are defined as the vertices of X_1 which admit an inverse up to higher homotopies. In particular, this yields two different underlying ∞-groupoids for the Segal space X: the space X_0 of objects and the underlying space of X_1 consisting of the homotopy equivalences. However, an $(\infty,1)$-category should have only one underlying ∞-groupoid, and therefore, instead of requiring the space X_0 of objects to be discrete, we impose a *completeness condition*, which identifies the two ∞-groupoids mentioned above. A Segal space satisfying this completeness condition is then called a *complete Segal space* and models an $(\infty,1)$-category.

There is a model structure on the category of simplicial spaces whose fibrant objects are precisely the complete Segal spaces. It is obtained as a left Bousfield localization of the Reedy model structure on the category of simplicial spaces. In Section 9.1, we first recall the main features of this Reedy model structure, which coincides with the injective model structure. Then, in Section 9.2, we introduce complete Segal spaces, following the paper [Rez01] by Rezk, and give the model structure for complete Segal spaces.

9.1. **Reedy and injective model structures on simplicial spaces.** We first introduce the Reedy and injective model structures on the category $\mathrm{sSet}^{\Delta^{\mathrm{op}}}$ of simplicial objects in sSet, where sSet is the category of simplicial sets endowed with the Kan-Quillen model structure (see Theorem 5.2.7). The Reedy and injective model structures are both constructed in such a way that their weak equivalences are level-wise. Furthermore, these two model structures on $\mathrm{sSet}^{\Delta^{\mathrm{op}}}$ actually coincide. This allows us to describe both the (trivial) cofibrations as the level-wise (trivial) cofibrations by definition of the injective model structure, and the fibrations as the *Reedy* fibrations, i.e., maps such that their associated matching map is a fibration, by definition of the Reedy model structure.

Since the objects considered here are simplicial objects in sSet, they can also be seen as bisimplicial sets $\Delta^{\mathrm{op}} \times \Delta^{\mathrm{op}} \to \mathrm{Set}$. We now introduce notations for the representable functors in each of the copy of Δ^{op}, as well as their boundary and horn inclusions.

Notation 9.1.1. Consider the category $\mathrm{sSet}^{\Delta^{\mathrm{op}}} \cong \mathrm{Set}^{\Delta^{\mathrm{op}} \times \Delta^{\mathrm{op}}}$ of bisimplicial sets. We denote by $\Delta[n]$ the representable functor

$$\Delta[n] \colon \Delta^{\mathrm{op}} \times \Delta^{\mathrm{op}} \to \mathrm{Set}, ([k],[p]) \mapsto \Delta([p],[n])$$

constant in the first variable, for all $n \geq 0$, and by $F[k]$ the representable functor

$$F[k] \colon \Delta^{\mathrm{op}} \times \Delta^{\mathrm{op}} \to \mathrm{Set}, \ ([l],[n]) \mapsto \Delta([l],[k])$$

constant in the second variable, for all $k \geq 0$. As in Definition 5.2.4, we write $\delta\Delta[n]$ and $\delta F[k]$ for the boundaries of $\Delta[n]$ and $F[k]$, for all $n, k \geq 0$, and $\Lambda^t[n]$ for the (n,t)-horn of $\Delta[n]$, for all $n \geq 1$ and $0 \leq t \leq n$. These come with inclusion maps $\iota_n^\Delta \colon \delta\Delta[n] \to \Delta[n]$, $\iota_k^F \colon \delta F[k] \to F[k]$, and $\ell_{n,t}^\Delta \colon \Lambda^t[n] \to \Delta[n]$.

Notation 9.1.2. Given a bisimplicial set $X \colon \Delta^{\mathrm{op}} \to \mathrm{sSet}$, we denote by $X_k \coloneqq X([k])$ its kth simplicial set, and by $X_{k,n} \coloneqq X_k([n])$ the set of n-simplices of X_k, for $k, n \geq 0$. Note that, by the Yoneda Lemma, there is an isomorphism $X_{k,n} \cong \mathrm{sSet}^{\Delta^{\mathrm{op}}}(F[k] \times \Delta[n], X)$, for all $k, n \geq 0$.

We refer to the direction given by the $\Delta[n]$'s as the *space* direction, and the one given by the $F[k]$'s as the *categorical* direction. This terminology comes from the fact that, given a bisimplicial set $X \colon \Delta^{\mathrm{op}} \to \mathrm{sSet}$ which is an "$(\infty,1)$-category", we think of X_0 as the space of objects, X_1 as the space of morphisms, X_2 as the space of pair of composable morphisms, etc. Hence $X_{k,n} \cong \mathrm{sSet}^{\Delta^{\mathrm{op}}}(F[k] \times \Delta[n], X)$ can be thought of as the set of n-simplices in the space of k composable morphisms, so that $F[k]$ represents the categorical part of X and $\Delta[n]$ the topological part of X.

There is an inclusion of sSet into $\mathrm{sSet}^{\Delta^{\mathrm{op}}}$ which looks at a simplicial set as a bisimplicial set constant in the categorical direction. Since sSet models spaces, and hence ∞-groupoids, this inclusion mirrors the inclusion of sets into categories in the ∞-setting.

Notation 9.1.3. We define the inclusion functor $c \colon \mathrm{sSet} \to \mathrm{sSet}^{\Delta^{\mathrm{op}}}$, which sends a simplicial set K to the bisimplicial set $c(K)$ constant at K in the categorical direction, i.e., we have that $c(K)_k = K$, for all $k \geq 0$, and a simplicial map $f \colon K \to L$ to the map $c(f) \colon c(K) \to c(L)$ constant at f in the categorical direction, i.e., we have that $c(f)_k = f$, for all $k \geq 0$. This functor clearly preserves colimits.

Remark 9.1.4. Note that $\Delta[n]$ is actually $c(\Delta[n])$, where $\Delta[n]$ is the standard n-simplex in sSet.

With this inclusion functor, we define a simplicial enrichment on $\mathrm{sSet}^{\Delta^{\mathrm{op}}}$, which is both tensored and cotensored. In particular, we will see that the Reedy/injective model structure on $\mathrm{sSet}^{\Delta^{\mathrm{op}}}$ is simplicial with respect to this enrichment.

Proposition 9.1.5. *The category* $\mathrm{sSet}^{\Delta^{\mathrm{op}}}$ *is a tensored and cotensored simplicial category with*

(i) *simplicial homs* $\mathrm{Map}(X,Y)$ *given by*

$$\mathrm{Map}(X,Y)_n = \mathrm{sSet}^{\Delta^{\mathrm{op}}}(X \times \Delta[n], Y),$$

for all $n \geq 0$ *and* $X, Y \in \mathrm{sSet}^{\Delta^{\mathrm{op}}}$,

(ii) *tensors given by* $X \otimes K \coloneqq X \times c(K)$, *for all* $X \in \mathrm{sSet}^{\Delta^{\mathrm{op}}}$ *and* $K \in \mathrm{sSet}$,

(iii) *cotensors* Y^K *given by*

$$(Y^K)_k = \mathrm{Map}(F[k] \times c(K), Y),$$

for all $k \geq 0$, $Y \in \mathrm{sSet}^{\Delta^{\mathrm{op}}}$, *and* $K \in \mathrm{sSet}$.

Proof. First note that, by the Yoneda Lemma, we have

$$\mathrm{sSet}(\Delta[n], \mathrm{Map}(X,Y)) \cong \mathrm{Map}(X,Y)_n = \mathrm{sSet}^{\Delta^{\mathrm{op}}}(X \times \Delta[n], Y),$$

for all $n \geq 0$. Now, since every simplicial set K can be obtained as a colimit of $\Delta[n]$'s of the form $K \cong \mathrm{colim}_{\sigma \in K_n, n \geq 0} \Delta[n]$, it follows that

$$\begin{aligned}\mathrm{sSet}(K, \mathrm{Map}(X,Y)) &\cong \mathrm{sSet}(\mathrm{colim}_{\sigma,n} \Delta[n], \mathrm{Map}(X,Y)) \cong \lim_{\sigma,n} \mathrm{sSet}(\Delta[n], \mathrm{Map}(X,Y)) \\ &\cong \lim_{\sigma,n} \mathrm{sSet}^{\Delta^{\mathrm{op}}}(X \times \Delta[n], Y) \cong \mathrm{sSet}^{\Delta^{\mathrm{op}}}(\mathrm{colim}_{\sigma,n}(X \times \Delta[n]), Y) \\ &\cong \mathrm{sSet}^{\Delta^{\mathrm{op}}}(X \times (\mathrm{colim}_{\sigma,n} \Delta[n]), Y) \cong \mathrm{sSet}^{\Delta^{\mathrm{op}}}(X \times c(K), Y),\end{aligned}$$

where we used that the hom-functors send colimits in the first variable to limits, that products commute with colimits in $\mathrm{sSet}^{\Delta^{\mathrm{op}}}$, and that the isomorphism for $\Delta[n]$ holds by the above argument. Now, again by the Yoneda Lemma, we can see that

$$\mathrm{sSet}^{\Delta^{\mathrm{op}}}(F[k] \times \Delta[n], Y^K) \cong (Y^K)_{k,n} = \mathrm{sSet}^{\Delta^{\mathrm{op}}}(F[k] \times \Delta[n] \times c(K), Y),$$

for all $k, n \geq 0$, by definition of Y^K and $\mathrm{Map}(-,-)$. Since every bisimplicial set X can be obtained as a colimit of $F[k] \times \Delta[n]$'s of the form $X \cong \mathrm{colim}_{\sigma \in X_{k,n}, k,n \geq 0} F[k] \times \Delta[n]$, using similar arguments to the ones above, we can show that

$$\mathrm{sSet}^{\Delta^{\mathrm{op}}}(X, Y^K) \cong \mathrm{sSet}^{\Delta^{\mathrm{op}}}(X \times c(K), Y).$$

Furthermore, since the inclusion functor $c\colon \mathrm{sSet} \to \mathrm{sSet}^{\Delta^{\mathrm{op}}}$ preserves products, we have isomorphisms

$$(X \otimes K) \otimes L \cong X \times c(K) \times c(L) \cong X \times c(K \times L) \cong X \otimes (K \times L)$$

natural in X, K, and L, for all $X \in \mathrm{sSet}^{\Delta^{\mathrm{op}}}$ and $K, L \in \mathrm{sSet}$. Hence, it follows from Corollary 1.1.16 that $\mathrm{sSet}^{\Delta^{\mathrm{op}}}$ is tensored and cotensored over sSet. $\square$

We now describe the Reedy/injective model structure on $\mathrm{sSet}^{\Delta^{\mathrm{op}}}$.

Theorem 9.1.6. *Let* sSet *be the category of simplicial sets endowed with the Kan-Quillen model structure of Theorem 5.2.7. The* ***Reedy/injective model structure*** *on* $\mathrm{sSet}^{\Delta^{\mathrm{op}}}$ *exists and is such that*

(i) *the cofibrations are the monomorphisms; in particular, every object is cofibrant,*

(ii) *the weak equivalences are the level-wise weak equivalences,*

(iii) *the fibrations are the* ***Reedy fibrations****, i.e., the maps* $f\colon X \to Y$ *in* $\mathrm{sSet}^{\Delta^{\mathrm{op}}}$ *such that the pullback corner map*

$$((\iota_k^F)^*, f_*)\colon \mathrm{Map}(F[k], X) \to \mathrm{Map}(\delta F[k], X) \underset{\mathrm{Map}(\delta F[k], Y)}{\times} \mathrm{Map}(F[k], Y)$$

is a Kan fibration in sSet, *for all* $k \geq 0$,

(iv) *the fibrant objects are the* ***Reedy fibrant*** *ones, i.e., the objects* $X \in \mathrm{sSet}^{\Delta^{\mathrm{op}}}$ *such that the induced map*

$$(\iota_k^F)^*\colon X_k \cong \mathrm{Map}(F[k], X) \to \mathrm{Map}(\delta F[k], X)$$

is a Kan fibration in sSet, *for all* $k \geq 0$.

Moreover, it is combinatorial, and simplicial for the enrichment of Proposition 9.1.5.

Proof. The existence of the injective model structure, although classical, can be seen as a special instance of [HKRS17, Theorem 3.4.1], where it is constructed as a left-induced model structure along the left Kan extension $i_!\colon \mathrm{sSet}^{\mathrm{Ob}(\Delta^{\mathrm{op}})} \to \mathrm{sSet}^{\Delta^{\mathrm{op}}}$ of the inclusion functor $i\colon \mathrm{Ob}(\Delta^{\mathrm{op}}) \to \Delta^{\mathrm{op}}$, where $\mathrm{Ob}(\Delta^{\mathrm{op}})$ is the discrete category of objects of Δ^{op}. A proof for the existence of the Reedy model structure can be found, for example, in [Hir03, Theorem 15.3.4] or in [RV14, Theorem 4.18]. The fact that they coincide follows, for example, from [BR13, Proposition 3.15 and Corollary 4.5].

A proof of the fact that the Reedy model structure is cofibrantly generated can be found in [Hir03, Theorem 15.6.27] or [RV14, Proposition 7.7]. It also follows from Remark 9.1.7 below, where we give sets of generating cofibrations and trivial cofibrations. Finally, it is simplicial since the model structure on sSet is simplicial by Theorem 5.2.7, and by the fact that the (trivial) cofibrations, tensors, and colimits in $\mathrm{sSet}^{\Delta^{\mathrm{op}}}$ are defined level-wise. Indeed, if $i\colon A \hookrightarrow B$ is a cofibration in $\mathrm{sSet}^{\Delta^{\mathrm{op}}}$ and $j\colon K \hookrightarrow L$ is a cofibration in sSet, then their pushout-product

$$i \,\square_\times\, c(j)\colon A \times c(L) \coprod_{A\times c(K)} B \times c(K) \to B \times c(L)$$

is given level-wise in sSet by $i_k \,\square_\times\, j$, where i_k and j are cofibrations in sSet, for all $k \geq 0$. Hence it is a cofibration in sSet by (emc3') of Proposition 4.5.6, since sSet is simplicial. Moreover, if one of i or j is trivial, then so is $i_k \,\square_\times\, j$, for all $k \geq 0$. This shows (emc3') for $\mathrm{sSet}^{\Delta^{\mathrm{op}}}$. $\square$

Remark 9.1.7. A set of generating cofibrations for the Reedy/injective model structure on $\mathrm{sSet}^{\Delta^{\mathrm{op}}}$ is given by

$$\{\iota^F_k \,\square_\times\, \iota^\Delta_n\colon \delta F[k] \times \Delta[n] \coprod_{\delta F[k]\times \delta\Delta[n]} F[k] \times \delta\Delta[n] \to F[k] \times \Delta[n] \mid k, n \geq 0\}.$$

Indeed, these maps generate all monomorphisms in $\mathrm{sSet}^{\Delta^{\mathrm{op}}}$ under transfinite compositions, pushouts, and retracts. Moreover, a set of generating trivial cofibrations is given by

$$\{\iota^F_k \,\square_\times\, \ell^\Delta_{n,t}\colon \delta F[k] \times \Delta[n] \coprod_{\delta F[k]\times \Lambda^t[n]} F[k] \times \Lambda^t[n] \to F[k] \times \Delta[n] \mid k \geq 0,\, n \geq 1, 0 \leq t \leq n\}.$$

Let $f\colon X \twoheadrightarrow Y$ be a Reedy fibration in $\mathrm{sSet}^{\Delta^{\mathrm{op}}}$. Then there is a lift in every commutative diagram in $\mathrm{sSet}^{\Delta^{\mathrm{op}}}$ of the form

$$\begin{tikzcd}
\delta F[k] \times \Delta[n] \coprod_{\delta F[k]\times \Lambda^t[n]} F[k] \times \Lambda^t[n] \arrow[r] \arrow[d, hook, "\iota^F_k \,\square_\times\, \ell^\Delta_{n,t}"', "\wr"] & X \arrow[d, two heads, "f"] \\
F[k] \times \Delta[n] \arrow[r] \arrow[ur, dashed] & Y
\end{tikzcd}$$

if and only if there is a lift in the every commutative diagram in sSet of the form

$$\begin{tikzcd}
\Lambda^t[n] \arrow[r] \arrow[d, hook, "\ell^\Delta_{n,t}"', "\wr"] & \mathrm{Map}(F[k], X) \arrow[d, "((\iota^F_k)^*, f_*)"] \\
\Delta[n] \arrow[r] \arrow[ur, dashed] & \mathrm{Map}(\delta F[k], X) \times_{\mathrm{Map}(\delta F[k], Y)} \mathrm{Map}(F[k], Y)
\end{tikzcd},$$

for all $k \geq 0$, $n \geq 1$, and $0 \leq t \leq n$. But, by definition, the map f is a Reedy fibration if and only if there is a lift in the second diagrams, for all $k \geq 0$, $n \geq 1$, and $0 \leq t \leq n$. Hence this holds if and only if there is a lift in the first diagrams, for all $k \geq 0$, $n \geq 1$, and $0 \leq t \leq n$. This shows that the maps $\iota_k^F \square_\times \ell_{n,t}^\Delta$ form a set of generating trivial cofibrations.

Remark 9.1.8. Since the Reedy model structure on $\mathrm{sSet}^{\Delta^{\mathrm{op}}}$ is simplicial and every object is cofibrant, the map

$$\mathrm{id}_{F[k]} \times \ell_{n,t}^\Delta = (\emptyset \to F[k]) \square_\times \ell_{n,t}^\Delta \colon F[k] \times \Lambda^t[n] \to F[k] \times \Delta[n]$$

is a trivial cofibration in $\mathrm{sSet}^{\Delta^{\mathrm{op}}}$. Hence, if $f\colon X \twoheadrightarrow Y$ is a Reedy fibration in $\mathrm{sSet}^{\Delta^{\mathrm{op}}}$, then it is level-wise a Kan fibration, i.e., the map $f_k\colon X_k \to Y_k$ is a Kan fibration in sSet, for all $k \geq 0$. To see this, note that there is a lift in the below left diagram in sSet if and only if there is a lift in the below right diagram in $\mathrm{sSet}^{\Delta^{\mathrm{op}}}$.

$$\begin{tikzcd}
\Lambda^t[n] \arrow[r] \arrow[d, hook, "\ell_{n,t}^\Delta"', "\wr"] & X_k \cong \mathrm{Map}(F[k], X) \arrow[d, "f_k"] \\
\Delta[n] \arrow[r] \arrow[ur, dashed] & Y_k \cong \mathrm{Map}(F[k], Y)
\end{tikzcd}
\qquad
\begin{tikzcd}
F[k] \times \Lambda^t[n] \arrow[r] \arrow[d, hook, "\mathrm{id}_{F[k]} \times \ell_{n,t}^\Delta"', "\wr"] & X \arrow[d, two heads, "f"] \\
F[k] \times \Delta[n] \arrow[r] \arrow[ur, dashed] & Y
\end{tikzcd}$$

Since f is a Reedy fibration and $\mathrm{id}_{F[k]} \times \ell_{n,t}^\Delta$ is a trivial cofibration by the above argument, there is a lift in the above right diagram, and this shows that f_k is a Kan fibration, for all $k \geq 0$. As a direct consequence, we have that, if $X \in \mathrm{sSet}^{\Delta^{\mathrm{op}}}$ is Reedy fibrant, then the simplicial set X_k is a Kan complex, for all $k \geq 0$.

9.2. **Complete Segal spaces.** As mentioned above, an $(\infty, 1)$-category can be modeled by a bisimplicial set $X\colon \Delta^{\mathrm{op}} \to \mathrm{sSet}$, by looking at X_0 as its space of objects, X_1 as its space of morphisms, X_2 as its space of pair of composable morphisms, etc. In particular, to have that X_2 models pair of composable morphisms, we need to require that a certain map, called the *Segal map*, between X_2 and the fibered product $X_1 \times_{X_0} X_1$ is a weak equivalence. This gives rise to the notion of a *Segal space*. These do not quite model $(\infty, 1)$-categories since we have a space of objects rather than a set, and we need to impose a further condition, called the *completeness* condition, which yields the notion of a *complete Segal space*, introduced by Rezk in [Rez01]. In particular, we also recall Rezk's construction of a model structure on $\mathrm{sSet}^{\Delta^{\mathrm{op}}}$ in which the fibrant objects are precisely the complete Segal spaces, which is obtained as a localization of the Reedy/injective model structure on $\mathrm{sSet}^{\Delta^{\mathrm{op}} \times \Delta^{\mathrm{op}}}$.

Let us first introduce the Segal maps.

Notation 9.2.1. Let $k \geq 1$. We write $G[k] \coloneqq F[1] \sqcup_{F[0]} \ldots \sqcup_{F[0]} F[1]$ for the colimit of the following diagram in $\mathrm{sSet}^{\Delta^{\mathrm{op}}}$

$$\begin{tikzcd}[column sep=small]
 & F[0] \arrow[dl, "d^0"'] \arrow[dr, "d^1"] & & F[0] \arrow[dl, "d^0"'] \arrow[dr] & & F[0] \arrow[dl] \arrow[dr, "d^1"] & \\
F[1] & & F[1] & & \cdots & & F[1]\,,
\end{tikzcd}$$

where k copies of $F[1]$ appear. We set $G[0] \coloneqq F[0]$. In particular, the maps $\rho_i\colon [1] \to [k]$ of Δ with image $\{i-1, i\}$, for all $1 \leq i \leq k$, induce an inclusion map

$$g_k^F \coloneqq \rho_1 \sqcup \ldots \sqcup \rho_k \colon G[k] \to F[k].$$

Definition 9.2.2. Let $X\colon \Delta^{\mathrm{op}} \to \mathrm{sSet}$ be a bisimplicial set and $k \geq 0$. Then the map $g_k^F\colon G[k] \to F[k]$ induces a map

$$(g_k^F)^*\colon X_k \cong \mathrm{Map}(F[k], X) \to \mathrm{Map}(G[k], X) \cong X_1 \times_{X_0} \ldots \times_{X_0} X_1,$$

called the **Segal map**.

A Segal space can then be defined by requiring that the Segal maps are weak equivalences of simplicial sets. We further impose a Reedy fibrancy condition, since we want these to be the fibrant objects of a localization of the Reedy model structure.

Definition 9.2.3. A bisimplicial set $X\colon \Delta^{\mathrm{op}} \to \mathrm{sSet}$ is a **Segal space** if

(i) it is Reedy fibrant,

(ii) the Segal map $X_k \xrightarrow{\sim} X_1 \times_{X_0} \ldots \times_{X_0} X_1$ is a weak equivalence in sSet, for all $k \geq 0$.

Since $(\infty, 1)$-categories should be thought of as categories enriched in ∞-groupoids – modeled here by Kan complexes – there should be a Kan complex of morphisms between any two *vertices* in X_0, i.e., 0-simplices of the simplicial set X_0, for a Segal space X. We now construct such enriched homs.

Remark 9.2.4. If X is Reedy fibrant, note that the map

$$(\iota_1^F)^* \cong (d_1, d_0)\colon X_1 \cong \mathrm{Map}(F[1], X) \twoheadrightarrow \mathrm{Map}(\delta F[1], X) \cong X_0 \times X_0$$

is a Kan fibration in sSet.

Definition 9.2.5. Let X be a Segal space and let $x, y \in X_0$ be vertices. We define the **mapping space** $X(x, y)$ to be the following pullback in sSet.

$$\begin{tikzcd} X(x,y) \arrow[r] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner", very near start] & X_1 \arrow[d, two heads, "{(d_1, d_0)}"] \\ \Delta[0] \arrow[r, "{(x,y)}"'] & X_0 \times X_0 \end{tikzcd}$$

Since the map $(d_1, d_0)\colon X_1 \twoheadrightarrow X_0 \times X_0$ is a Kan fibration, its pullback $X(x, y) \twoheadrightarrow \Delta[0]$ is also a Kan fibration, and hence $X(x, y)$ is a Kan complex.

In particular, in order to model an $(\infty, 1)$-category, we want to define a composition "up to homotopy" on the mapping spaces of a Segal space. This can be achieved as follows.

Construction 9.2.6. Let X be a Segal space and x, y, z be vertices in X_0. Similarly, we can define a Kan complex $X(x, y, z)$ as the fiber of the fibration $X_2 \twoheadrightarrow X_0 \times X_0 \times X_0$ at (x, y, z). This yields the following diagram in sSet, where both squares are pullbacks.

$$\begin{tikzcd} X(x,y,z) \arrow[r] \arrow[d, "\wr"'] \arrow[dr, phantom, "\lrcorner", very near start] & X_2 \arrow[d, "\wr"', "{(d_2, d_0)}"] \\ X(x,y) \times X(y,z) \arrow[r] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner", very near start] & X_1 \times_{X_0} X_1 \arrow[d, two heads, "{(d_1, d_0)}"] \\ \Delta[0] \arrow[r, "{(x,y)}"'] & X_0 \times X_0 \times X_0 \end{tikzcd}$$

Since all objects are Kan complexes in this diagram, i.e., they are all fibrant, the pullback $X(x, y, z) \xrightarrow{\sim} X(x, y) \times X(y, z)$ of the weak equivalence $X_2 \xrightarrow{\sim} X_1 \times_{X_0} X_1$ is also a weak equivalence in sSet. We then define a map $d_1\colon X(x, y, z) \to X(x, z)$ to be the unique map given by the universal property of pullbacks which makes the following diagram commute.

$$\begin{tikzcd}
X(x,y,z) \arrow[r] \arrow[dr, dashed, "d_1"] \arrow[ddr, bend right] & X_2 \arrow[dr, "d_1"] & \\
& X(x,z) \arrow[r] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner", very near start] & X_1 \arrow[d, two heads, "{(d_1,d_0)}"] \\
& \Delta[0] \arrow[r, "{(x,z)}"'] & X_0 \times X_0
\end{tikzcd}$$

Then the diagram

$$X(x,y) \times X(y,z) \xleftarrow{\sim} X(x,y,z) \xrightarrow{d_1} X(x,z)$$

gives the desired composition up to homotopy, since the map going in the wrong direction is a weak equivalence.

Using these constructions, we can define the *homotopy category* of a Segal space X, by taking $X_{0,0}$ as its set of objects, and the sets of path components of the mapping spaces as its hom-sets. Recall that, given a simplicial set K, its *set of path components* $\pi_0 K$ is defined to be the following co-equalizer in Set

$$K_1 \underset{d_0}{\overset{d_1}{\rightrightarrows}} K_0 \longrightarrow \pi_0 K .$$

Moreover, recall that a weak equivalence $f\colon K \xrightarrow{\sim} L$ in the model structure on sSet of Theorem 5.2.7 induces an isomorphism $\pi_0 f\colon \pi_0 K \xrightarrow{\cong} \pi_0 L$ between the sets of path components. Hence this gives the following.

Definition 9.2.7. Let X be a Segal space. We define its **homotopy category** $\mathrm{ho}(X)$ to be the category whose

(i) objects are the vertices of X_0,
(ii) hom-sets are given by $\mathrm{ho}(X)(x,y) = \pi_0 X(x,y)$, for all $x, y \in X_0$,
(iii) composition is given by

$$\pi_0 X(x,y) \times \pi_0 X(y,z) \cong \pi_0 X(x,y,z) \xrightarrow{\pi_0 d_1} \pi_0 X(x,z),$$

for all $x, y, z \in X_0$,
(iv) identities are given by $[s_0 x] \in \pi_0 X(x,x)$, for all $x \in X_0$.

By looking at the maps that are invertible in the homotopy category, we get a notion of *homotopy equivalences* in a Segal space, which correspond to the homotopically invertible morphisms of the $(\infty,1)$-category.

Definition 9.2.8. Let X be a Segal space. A **homotopy equivalence** in X is a vertex $f \in X_1$ such that its homotopy class $[f] \in \mathrm{ho}(X)$ is an isomorphism.

By [Rez01, Lemma 5.8], if two vertices $f, g \in X_1$ are in the same path component and one of them is a homotopy equivalence, then so is the other. Hence we can define the following.

Definition 9.2.9. Let X be a Segal space. We define its **space of homotopy equivalences** X_1^{heq} to be the subspace of X_1 consisting of those components of X_1 whose vertices are homotopy equivalences. In particular, every degenerate simplex $s_0 x$ is a homotopy equivalence, for every vertex $x \in X_0$, and the map $s_0\colon X_0 \to X_1$ factors through $s_0\colon X_0 \to X_1^{\mathrm{heq}}$.

Given a Segal space, we now have two ∞-groupoids of objects; one given by the space X_0 of objects, and the other one given by the underlying ∞-groupoid X_1^{heq}, as defined above.

In order to model $(\infty, 1)$-categories, we need to require that these two ∞-groupoids are weakly equivalent.

Definition 9.2.10. A simplicial space $X\colon \Delta^{\mathrm{op}} \to \mathrm{sSet}$ is a **complete Segal space** if

(i) it is Reedy fibrant,
(ii) the Segal map $X_k \xrightarrow{\simeq} X_1 \times_{X_0} \ldots \times_{X_0} X_1$ is a weak equivalence in sSet, for all $k \geq 0$,
(iii) the map $s^0\colon X_0 \xrightarrow{\simeq} X_1^{\mathrm{heq}}$ is a weak equivalence in sSet.

The space of homotopy equivalences can be modeled by the simplicial mapping space out of the discrete nerve of the "free-living isomorphism". We can then re-express the last condition of a complete Segal space by requiring that some inclusion induces a weak equivalence between simplicial homs, which is useful to describe the set of cofibrations at which we want to localize.

Notation 9.2.11. We denote by $N\colon \mathrm{Cat} \to \mathrm{Set}^{\Delta^{\mathrm{op}} \times \Delta^{\mathrm{op}}}$ the discrete nerve constant in the space direction. It is given by $(N\mathcal{C})_{k,n} = \mathrm{Cat}([k], \mathcal{C})$, for every category $\mathcal{C}$.

Example 9.2.12. We define the category $I = \{x \cong y\}$ to be the "free-living isomorphism". Its discrete nerve is given by $(NI)_{k,n} = \mathrm{Cat}([k], I)$. In particular, a functor $[k] \to I$ can be described as a word of k letters in $\{x, y\}$. For example, when $k = 0$, we have that $(NI)_{0,n} = \{x, y\}$; and, when $k = 1$, $(NI)_{1,n} = \{xx, xy, yx, yy\}$ where xx and yy are degenerate and represent the identities at x and y, and xy and yx represent the two inverse morphisms between x and y. In particular, it comes with an inclusion map $e\colon F[0] \to NI$, where the unique 0-simplex of $F[0]$ is sent to x.

Proposition 9.2.13. *Let X be a Segal space. There is an isomorphism of simplicial sets $X_1^{\mathrm{heq}} \cong \mathrm{Map}(NI, X)$. Moreover, the map $s_0\colon X_0 \to X_1^{\mathrm{heq}}$ is a weak equivalence in* sSet *if and only if the map*

$$e^*\colon X_1^{\mathrm{heq}} \cong \mathrm{Map}(NI, X) \to \mathrm{Map}(F[0], X) \cong X_0$$

is a weak equivalence in sSet*.*

Proof. The first isomorphism follows from [Rez01, Theorem 6.2]. For the second one, note that the composite $F[0] \to NI \to F[0]$ is the identity, and hence the composite

$$X_0 \xrightarrow{s_0} X_1^{\mathrm{heq}} \cong \mathrm{Map}(NI, X) \xrightarrow{e^*} X_0 \cong \mathrm{Map}(F[0], X)$$

is also the identity. We conclude, by 2-out-of-3, that s_0 is a weak equivalence if and only if e^* is. □

We are now ready to state the main theorem, which constructs the model structure for complete Segal spaces as a localization of the Reedy/injective model structure.

Theorem 9.2.14. *There is a model structure on* $\mathrm{sSet}^{\Delta^{\mathrm{op}}}$*, denoted by* CSS*, such that*

(i) the cofibrations in CSS *are the monomorphisms; in particular, every object is cofibrant,*
(ii) the fibrant objects in CSS *are the complete Segal spaces.*

In particular, it is obtained as a localization of the Reedy/injective model structure on $\mathrm{sSet}^{\Delta^{\mathrm{op}}}$ *of Theorem 9.1.6 at the set of monomorphisms*

$$\{g_k^F\colon G[k] \to F[k] \mid k \geq 0\} \bigcup \{e\colon F[0] \to NI\}.$$

Moreover, this model structure is combinatorial, and simplicial for the enrichment of Proposition 9.1.5.

Proof. Since the Reedy model structure on $\mathrm{sSet}^{\Delta^{\mathrm{op}}}$ is a combinatorial, simplicial model structure such that all objects are cofibrant by Theorem 9.1.6, its left Bousfield localization CSS at the set of monomorphisms

$$\mathcal{S} = \{g_k^F : G[k] \to F[k] \mid k \geq 0\} \bigcup \{e \colon F[0] \to NI\}$$

exists and is again simplicial and combinatorial, by Theorem 5.2.15. Moreover, the cofibrations in CSS are the monomorphisms, since they are the cofibrations of the Reedy model structure. Finally, the fibrant objects are the $\mathcal{S}$-local objects by Proposition 5.2.18, i.e., an object $X \in \mathrm{CSS}$ is fibrant if and only if it is Reedy fibrant and the maps

$$(g_k^F)^* \colon X_k \xrightarrow{\sim} X_1 \times_{X_0} \ldots \times_{X_0} X_1 \quad \text{and} \quad e^* \colon X_1^{\mathrm{heq}} \xrightarrow{\sim} X_0$$

are weak equivalences in sSet, for all $k \geq 0$. This shows that the fibrant objects are precisely the complete Segal spaces, by Proposition 9.2.13. $\square$

Finally, we show that every category gives rise to a complete Segal space.

Example 9.2.15. In [Rez01, §3.5], Rezk constructs a nerve $N_{\mathrm{Rezk}} \colon \mathrm{Cat} \to \mathrm{sSet}^{\Delta^{\mathrm{op}}}$ which associates to a category $\mathcal{C}$ the bisimplicial set

$$N_{\mathrm{Rezk}}\mathcal{C} \colon \Delta^{\mathrm{op}} \times \Delta^{\mathrm{op}} \to \mathrm{Set}, \quad ([k],[n]) \mapsto \mathrm{Cat}([k] \times I[n], \mathcal{C}),$$

where $I[n]$ is the category with object set $\{0, \ldots, n\}$ and a unique isomorphism between any two objects. In particular, by [Rez01, Proposition 6.1], the nerve $N_{\mathrm{Rezk}}\mathcal{C}$ is a complete Segal space, for every category $\mathcal{C}$.

10. Models of double $(\infty,1)$-categories and $(\infty,2)$-categories

Given the model of $(\infty,1)$-categories presented above, we want to use it to define ∞-analogues of double categories and 2-categories. For this, we consider bisimplicial spaces $X \colon \Delta^{\mathrm{op}} \times \Delta^{\mathrm{op}} \to \mathrm{sSet}$, where the additional copy of Δ^{op} allows us to have one more dimension. In particular, a double $(\infty,1)$-category $X \colon \Delta^{\mathrm{op}} \times \Delta^{\mathrm{op}} \to \mathrm{sSet}$ should be such that $X_{0,0}$ forms its space of objects, $X_{1,0}$ its space of horizontal morphisms, $X_{0,1}$ its space of vertical morphisms, and $X_{1,1}$ its space of squares. The higher simplicial sets should model horizontal and vertical composites of such. For this to be the case, we require that a double $(\infty,1)$-category satisfies the Segal condition in both directions. We can further impose a completeness condition in one direction, and our convention here is to put it in the horizontal direction. Note that we cannot have the completeness condition in both directions since this would imply that $X_{1,0}^{\mathrm{heq}} \simeq X_{0,0} \simeq X_{0,1}^{\mathrm{heq}}$ and hence that two objects in $X_{0,0}$ are horizontally equivalent if and only if they are vertically equivalent. However, even in a strict double category, it is not true that two objects are horizontally isomorphic if and only they are vertically isomorphic, so this is not a desirable condition. By choosing the completeness condition to be in the vertical direction instead, we recover the model of double $(\infty,1)$-categories defined by Haugseng [Hau13]; this gives an equivalent model to the horizontally complete one via a transpose functor which interchanges the horizontal and vertical directions.

Using the fact that an $(\infty,2)$-category should be a horizontal double $(\infty,1)$-category, we define an $(\infty,2)$-category as a double $(\infty,1)$-category $X \colon \Delta^{\mathrm{op}} \times \Delta^{\mathrm{op}} \to \mathrm{sSet}$ such that its space of vertical morphisms $X_{0,1}$ is discrete. For this, we identify all the spaces $X_{0,k}$ of k composable vertical morphisms with the space of objects $X_{0,0}$ up to weak equivalence. In this case, there is no problem about also requiring the completeness condition in the vertical direction, and this yields the notion of 2-fold complete Segal spaces introduced by Barwick [Bar05].

As in the case of complete Segal spaces, there are model structures on the category of bisimplicial spaces whose fibrant objects are the double $(\infty,1)$-categories and 2-fold complete Segal spaces, respectively. They are obtained as left Bousfield localizations of the Reedy/injective model structure on the category of bisimplicial spaces, by similar methods to the ones presented in Section 9. In Section 10.1, we first recall the main features of the Reedy/injective model structure for bisimplicial spaces, and then localize it in Section 10.2 to obtain a model structure for double $(\infty,1)$-categories. Finally, in Section 10.3, we further localize the model structure for double $(\infty,1)$-categories to obtain a model structure for 2-fold complete Segal spaces. In particular, the identity on bisimplicial spaces is a Quillen reflection embedding the homotopy theory of 2-fold complete Segal spaces into that of double $(\infty,1)$-categories. This identity functor can therefore be interpreted as the ∞-analogue of the horizontal embedding of 2-categories into double categories.

10.1. **Reedy and injective model structures on bisimplicial spaces.** We now consider the category $\mathrm{sSet}^{\Delta^{\mathrm{op}}\times\Delta^{\mathrm{op}}}$ of simplicial objects in $\mathrm{sSet}^{\Delta^{\mathrm{op}}}$ or bisimplicial objects in sSet. Similarly to Section 9.1, we describe the Reedy and injective model categories on this category, where sSet is endowed with the Kan-Quillen model structure of Theorem 5.2.7. As it was the case of $\mathrm{sSet}^{\Delta^{\mathrm{op}}}$, the Reedy and injective model structures on $\mathrm{sSet}^{\Delta^{\mathrm{op}}\times\Delta^{\mathrm{op}}}$ coincide, and hence we can describe both the cofibrations and fibrations in this model structure.

Since bisimplicial objects in sSet can equivalently be described as trisimplicial sets $(\Delta^{\mathrm{op}})^{\times 3}\to \mathrm{Set}$, there is one more simplicial direction. We first introduce our notations for the representable in each simplicial direction and for their respective boundary and horn inclusions.

Notation 10.1.1. Consider the category $\mathrm{sSet}^{\Delta^{\mathrm{op}}\times\Delta^{\mathrm{op}}}\cong \mathrm{Set}^{(\Delta^{\mathrm{op}})^{\times 3}}$ of trisimplicial sets. We denote by $\Delta[n]$ the representable functor

$$\Delta[n]\colon \Delta^{\mathrm{op}}\times\Delta^{\mathrm{op}}\to \mathrm{Set}, ([m],[k],[p])\mapsto \Delta([p],[n])$$

constant in the first two variables, for all $n\geq 0$, by $F[k]$ the representable functor

$$F[k]\colon \Delta^{\mathrm{op}}\times\Delta^{\mathrm{op}}\to \mathrm{Set},\ ([m],[l],[n])\mapsto \Delta([l],[k])$$

constant in the first and third variables, for all $k\geq 0$, and by $R[m]$ the representable functor

$$R[m]\colon \Delta^{\mathrm{op}}\times\Delta^{\mathrm{op}}\to \mathrm{Set},\ ([q],[k],[n])\mapsto \Delta([q],[m])$$

constant in the last two variables, for all $m\geq 0$. As in Definition 5.2.4, we write $\delta\Delta[n]$, $\delta F[k]$, and $\delta R[m]$ for the boundaries of $\Delta[n]$, $F[k]$, and $R[m]$, for all $n,k,m\geq 0$, and $\Lambda^t[n]$ for the (n,t)-horn of $\Delta[n]$, for all $n\geq 1$ and $0\leq t\leq n$. These come with inclusion maps $\iota^\Delta_n\colon \delta\Delta[n]\to\Delta[n]$, $\iota^F_k\colon \delta F[k]\to F[k]$, $\iota^R_m\colon \delta R[m]\to R[m]$, and $\ell^\Delta_{n,t}\colon \Lambda^t[n]\to\Delta[n]$.

Notation 10.1.2. Given a trisimplicial set $X\colon \Delta^{\mathrm{op}}\times\Delta^{\mathrm{op}}\to \mathrm{sSet}$, we denote its (m,k)th simplicial set by $X_{m,k}\coloneqq X([m],[k])$, and by $X_{m,k,n}\coloneqq X_{m,k}([n])$ the set of n-simplices of $X_{m,k}$, for $m,k,n\geq 0$. Note that, by the Yoneda Lemma, there is an isomorphism $X_{m,k,n}\cong \mathrm{sSet}^{\Delta^{\mathrm{op}}\times\Delta^{\mathrm{op}}}(R[m]\times F[k]\times\Delta[n],X)$, for all $m,k,n\geq 0$. We also write

$$X_{m,-}\coloneqq X([m],-)\colon \Delta^{\mathrm{op}}\to \mathrm{sSet}\quad\text{and}\quad X_{-,k}\coloneqq X(-,[k])\colon \Delta^{\mathrm{op}}\to \mathrm{sSet}$$

for the induced bisimplicial sets, for $m,k\geq 0$.

We refer to the direction given by the $\Delta[n]$'s as the *space* direction, the one given by the $F[k]$'s as the *vertical* direction, and the one given by the $R[m]$'s as the *horizontal* direction. The idea this time is that a double $(\infty,1)$-category can be modeled by a trisimplicial set $X\colon \Delta^{\mathrm{op}}\times\Delta^{\mathrm{op}}\to \mathrm{sSet}$, by looking at $X_{0,0}$ as its space of object, $X_{0,1}$ as its space of horizontal morphisms, $X_{1,0}$ as its space of vertical morphisms, and $X_{1,1}$ as its space of squares. Since $\mathrm{sSet}^{\Delta^{\mathrm{op}}\times\Delta^{\mathrm{op}}}(R[1],X)\cong X_{1,0,0}$ and $\mathrm{sSet}^{\Delta^{\mathrm{op}}\times\Delta^{\mathrm{op}}}(F[1],X)\cong X_{0,1,0}$

represent the sets of horizontal morphisms and vertical morphisms of the "double $(\infty,1)$-category" X, respectively, which justifies the fact that $R[m]$ represents the horizontal direction of X, while $F[k]$ represents the vertical one.

There is also an inclusion of sSet into $\mathrm{sSet}^{\Delta^{\mathrm{op}}\times\Delta^{\mathrm{op}}}$ which looks at a simplicial set as a constant trisimplicial set in the horizontal and vertical directions. This can be thought of as the ∞-version of the fact that any set can be seen as a discrete double category with only trivial morphisms.

Notation 10.1.3. There is an inclusion functor $c\colon \mathrm{sSet}\to\mathrm{sSet}^{\Delta^{\mathrm{op}}\times\Delta^{\mathrm{op}}}$, which sends a simplicial set K to the trisimplicial set $c(K)$ constant in the horizontal and vertical directions, i.e., we have that $c(K)_{m,k}=K$, for all $m,k\geq 0$, and a simplicial map $f\colon K\to L$ to the map $c(f)\colon c(K)\to c(L)$ constant at f in the horizontal and vertical directions, i.e., we have that $c(f)_{m,k}=f$, for all $m,k\geq 0$. This functor clearly preserves colimits.

Remark 10.1.4. Note that $\Delta[n]$ is actually $c(\Delta[n])$, where $\Delta[n]$ is the standard n-simplex in sSet.

The category $\mathrm{sSet}^{\Delta^{\mathrm{op}}\times\Delta^{\mathrm{op}}}$ is also simplicially enriched, tensored, and cotensored, and the Reedy/injective model structure is simplicial with respect to this enrichment, as we will see below.

Proposition 10.1.5. *The category* $\mathrm{sSet}^{\Delta^{\mathrm{op}}\times\Delta^{\mathrm{op}}}$ *is a tensored and cotensored simplicial category with*

(i) simplicial homs $\mathrm{Map}(X,Y)$ *given by*
$$\mathrm{Map}(X,Y)_n=\mathrm{sSet}^{\Delta^{\mathrm{op}}\times\Delta^{\mathrm{op}}}(X\times\Delta[n],Y),$$
for all $n\geq 0$ *and* $X,Y\in\mathrm{sSet}^{\Delta^{\mathrm{op}}\times\Delta^{\mathrm{op}}}$*,*

(ii) tensors given by $X\otimes K\coloneqq X\times c(K)$*, for all* $X\in\mathrm{sSet}^{\Delta^{\mathrm{op}}\times\Delta^{\mathrm{op}}}$ *and* $K\in\mathrm{sSet}$*,*

(iii) cotensors Y^K *given by*
$$(Y^K)_{m,k}=\mathrm{Map}(R[m]\times F[k]\times c(K),Y),$$
for all $m,k\geq 0$*,* $Y\in\mathrm{sSet}^{\Delta^{\mathrm{op}}\times\Delta^{\mathrm{op}}}$*, and* $K\in\mathrm{sSet}$*.*

Proof. The proof is similar to the one of Proposition 9.1.5. □

We introduce the following notation, which represents the boundary of the product $R[m]\times F[k]$.

Notation 10.1.6. Let $m,k\geq 0$. We write
$$\delta(R[m]\times F[k])\coloneqq \delta R[m]\times F[k]\coprod_{\delta R[m]\times\delta F[k]} R[m]\times\delta F[k].$$
Hence the pushout-product $\iota^R_m\,\square_\times\,\iota^F_k$ is the inclusion $\delta(R[m]\times F[k])\to R[m]\times F[k]$.

Using this notation, we now describe the Reedy/injective model structure on $\mathrm{sSet}^{\Delta^{\mathrm{op}}\times\Delta^{\mathrm{op}}}$.

Theorem 10.1.7. *Let* sSet *be the category of simplicial sets endowed with the Kan-Quillen model structure of Theorem 5.2.7. The* ***Reedy/injective model structure*** *on* $\mathrm{sSet}^{\Delta^{\mathrm{op}}\times\Delta^{\mathrm{op}}}$ *exists and is such that*

(i) the cofibrations are the monomorphisms; in particular, every object is cofibrant,

(ii) the weak equivalence are the level-wise weak equivalences,

(iii) the fibrations are the ***Reedy fibrations****, i.e., the maps* $f\colon X\to Y$ *in* $\mathrm{sSet}^{\Delta^{\mathrm{op}}\times\Delta^{\mathrm{op}}}$ *such that the pullback corner map induced by* f *and* $\iota^R_m\,\square_\times\,\iota^F_k$
$$\mathrm{Map}(R[m]\times F[k],X)\to\mathrm{Map}(\delta(R[m]\times F[k]),X)\underset{\mathrm{Map}(\delta(R[m]\times F[k]),Y)}{\times}\mathrm{Map}(R[m]\times F[k],Y)$$
is a Kan fibration in sSet*, for all* $m,k\geq 0$*,*

(iv) the fibrant objects are the ***Reedy fibrant*** *ones, i.e., the objects* $X \in \mathrm{sSet}^{\Delta^{\mathrm{op}}}$ *such that the induced map*

$$(\iota^R_m \,\square_\times\, \iota^F_k)^* \colon X_{m,k} \cong \mathrm{Map}(R[m] \times F[k], X) \to \mathrm{Map}(\delta(R[m] \times F[k]), X)$$

is a Kan fibration in sSet*, for all* $m, k \geq 0$*.*

Moreover, it is combinatorial, and simplicial for the enrichment of Proposition 10.1.5.

Proof. The proof of this result is similar to the one of Theorem 9.1.6. Sets of generating cofibrations and trivial cofibrations are given just below in Remark 10.1.8. □

Remark 10.1.8. A set of generating cofibrations for the Reedy/injective model structure on $\mathrm{sSet}^{\Delta^{\mathrm{op}} \times \Delta^{\mathrm{op}}}$ is given by

$$\{(\iota^R_m \,\square_\times\, \iota^F_k) \,\square_\times\, \iota^\Delta_n \mid m, k, n \geq 0\},$$

where $(\iota^R_m \,\square_\times\, \iota^F_k) \,\square_\times\, \iota^\Delta_n$ is the inclusion

$$\delta(R[m] \times F[k]) \times \Delta[n] \coprod_{\delta(R[m] \times F[k]) \times \delta\Delta[n]} R[m] \times F[k] \times \delta\Delta[n] \to R[m] \times F[k] \times \Delta[n],$$

and a set of generating trivial cofibrations is given by

$$\{(\iota^R_m \,\square_\times\, \iota^F_k) \,\square_\times\, \ell^\Delta_{n,t} \mid m, k \geq 0,\ n \geq 1, 0 \leq t \leq n\},$$

where $(\iota^R_m \,\square_\times\, \iota^F_k) \,\square_\times\, \ell^\Delta_{n,t}$ is the inclusion

$$\delta(R[m] \times F[k]) \times \Delta[n] \coprod_{\delta(R[m] \times F[k]) \times \Lambda^t[n]} R[m] \times F[k] \times \Lambda^t[n] \to R[m] \times F[k] \times \Delta[n].$$

One can check that these indeed give generating sets for the Reedy/injective model structure on $\mathrm{sSet}^{\Delta^{\mathrm{op}} \times \Delta^{\mathrm{op}}}$ using similar arguments to the ones presented in Remark 9.1.7.

Remark 10.1.9. As in Remark 9.1.8, one can show that, if $f \colon X \twoheadrightarrow Y$ is a Reedy fibration in $\mathrm{sSet}^{\Delta^{\mathrm{op}} \times \Delta^{\mathrm{op}}}$, then it is level-wise a Kan fibration, i.e., the map $f_{m,k} \colon X_{m,k} \to Y_{m,k}$ is a Kan fibration in sSet, for all $m, k \geq 0$. Hence, if $X \in \mathrm{sSet}^{\Delta^{\mathrm{op}} \times \Delta^{\mathrm{op}}}$ is Reedy fibrant, then the simplicial set $X_{m,k}$ is a Kan complex, for all $m, k \geq 0$.

10.2. **Double $(\infty, 1)$-categories.** In analogy to the strict case where a double category is defined as an internal category to categories (see Remark 3.1.2), a double $(\infty, 1)$-category should be a "homotopical" internal category to $(\infty, 1)$-categories. As we have seen in Section 9.2, a complete Segal space is an example of a "homotopical" internal category to ∞-groupoids. By iterating this process, we could think of a double $(\infty, 1)$ category as a double complete Segal space, i.e., as a trisimplicial set $X \colon \Delta^{\mathrm{op}} \times \Delta^{\mathrm{op}} \to \mathrm{sSet}$ such that all rows $X_{-,k} \colon \Delta^{\mathrm{op}} \to \mathrm{sSet}$ and all columns $X_{m,-} \colon \Delta^{\mathrm{op}} \to \mathrm{sSet}$ are complete Segal spaces. However, by setting this condition, we get that $X^{\mathrm{heq}}_{1,0} \simeq X_{0,0} \simeq X^{\mathrm{heq}}_{0,1}$, which says that the space of horizontal homotopy equivalences is weakly equivalent to the space of vertical homotopy equivalences. However, as mentioned in the introduction, this is not a desirable condition, since, even in the strict case, two objects in a double category are not horizontally isomorphic (or equivalent) if and only if they are vertically isomorphic (or equivalent). Instead, we choose to put the completeness condition in only one direction and, since we want to see an $(\infty, 2)$-category as a horizontal double $(\infty, 1)$-category, we require the double $(\infty, 1)$-categories to be *horizontally complete*.

Definition 10.2.1. A trisimplicial set $X \colon \Delta^{\mathrm{op}} \times \Delta^{\mathrm{op}} \to \mathrm{sSet}$ is a **horizontally complete double $(\infty, 1)$-category** if

(i) it is Reedy fibrant,
(ii) the bisimplicial set $X_{m,-} \colon \Delta^{\mathrm{op}} \to \mathrm{sSet}$ is a Segal space, for all $m \geq 0$,
(iii) the bisimplicial set $X_{-,k} \colon \Delta^{\mathrm{op}} \to \mathrm{sSet}$ is a complete Segal space, for all $k \geq 0$.

We introduce the following inclusion maps, which induce the Segal maps in the horizontal and vertical directions on simplicial homs.

Notation 10.2.2. As in Notation 9.2.1, we define $G[k] \coloneqq F[1] \sqcup_{F[0]} \ldots \sqcup_{F[0]} F[1]$ to be the colimit of k copies of $F[1]$ under $F[0]$, for $k \geq 1$, and set $G[0] \coloneqq F[0]$. It comes with an inclusion map $g_k^F \colon G[k] \to F[k]$. Similarly, we define $Q[m] \coloneqq R[1] \sqcup_{R[0]} \ldots \sqcup_{R[0]} R[1]$ to be the colimit of m copies of $R[1]$ under $R[0]$, for $m \geq 1$, and set $Q[0] \coloneqq R[0]$. It comes with an inclusion map $q_m^R \colon Q[m] \to R[m]$.

To be able to express the completeness condition in both directions, we introduce the following discrete nerves functors, as well as notations for the discrete nerves of the "free-living isomorphism".

Notation 10.2.3. We denote by $N^R \colon \mathrm{Cat} \to \mathrm{Set}^{(\Delta^{\mathrm{op}})^{\times 3}}$ the discrete nerve constant in the vertical and space directions. It is given by $(N^R\mathcal{C})_{m,k,n} = \mathrm{Cat}([m], \mathcal{C})$, for every category $\mathcal{C}$. Similarly, we denote by $N^F \colon \mathrm{Cat} \to \mathrm{Set}^{(\Delta^{\mathrm{op}})^{\times 3}}$ the discrete nerve constant in the horizontal and space directions. It is given by $(N^F\mathcal{C})_{m,k,n} = \mathrm{Cat}([k], \mathcal{C})$, for every category $\mathcal{C}$.

Below, we consider the discrete nerves $N^R I$ and $N^F I$ of the "free-living isomorphism", which can be described as in Example 9.2.12, and these come with inclusions maps $e^R \colon R[0] \to N^R I$ and $e^F \colon F[0] \to N^F I$.

We can now obtain the model structure for double $(\infty, 1)$-categories by localizing the Reedy/injective model structure on $\mathrm{sSet}^{\Delta^{\mathrm{op}} \times \Delta^{\mathrm{op}}}$ at the monomorphisms introduced above.

Theorem 10.2.4. *There is a model structure on* $\mathrm{sSet}^{\Delta^{\mathrm{op}} \times \Delta^{\mathrm{op}}}$, *denoted by* DblCat_∞^h, *such that*

(i) the cofibrations in DblCat_∞^h *are the monomorphisms; in particular, every object is cofibrant,*

(ii) the fibrant objects in DblCat_∞^h *are the horizontally complete double* $(\infty, 1)$*-categories.*

In particular, it is obtained as a localization of the Reedy model structure on $\mathrm{sSet}^{\Delta^{\mathrm{op}} \times \Delta^{\mathrm{op}}}$ *of Theorem 10.1.7 at the set of monomorphisms*

$$\{\mathrm{id}_{R[m]} \times g_k^F \colon R[m] \times G[k] \to R[m] \times F[k],\ q_m^R \times \mathrm{id}_{F[k]} \colon Q[m] \times F[k] \to R[m] \times F[k] \mid m, k \geq 0\}$$
$$\bigcup \{e^R \times \mathrm{id}_{F[k]} \colon F[k] \cong R[0] \times F[k] \to N^R I \times F[k] \mid k \geq 0\}.$$

Moreover, this model structure is combinatorial, and simplicial for the enrichment of Proposition 10.1.5.

Proof. Since the Reedy model structure on $\mathrm{sSet}^{\Delta^{\mathrm{op}} \times \Delta^{\mathrm{op}}}$ is a combinatorial, simplicial model structure such that all objects are cofibrant, its left Bousfield localization DblCat_∞^h at the set $\mathcal{S}$ of monomorphisms

$$\{\mathrm{id}_{R[m]} \times g_k^F \colon R[m] \times G[k] \to R[m] \times F[k],\ q_m^R \times \mathrm{id}_{F[k]} \colon Q[m] \times F[k] \to R[m] \times F[k] \mid m, k \geq 0\}$$
$$\bigcup \{e^R \times \mathrm{id}_{F[k]} \colon F[k] \cong R[0] \times F[k] \to N^R I \times F[k] \mid k \geq 0\}.$$

exists and is again simplicial and combinatorial, by Theorem 5.2.15. Moreover, the cofibrations in DblCat_∞^h are the monomorphisms, since they are the cofibrations of the Reedy model structure on $\mathrm{sSet}^{\Delta^{\mathrm{op}} \times \Delta^{\mathrm{op}}}$. Finally, the fibrant objects are the $\mathcal{S}$-local objects by Proposition 5.2.18, i.e., an object $X \in \mathrm{DblCat}_\infty^h$ is fibrant if and only if it is Reedy fibrant and the maps

$$(\mathrm{id}_{R[m]} \times g_k^F)^* \colon X_{m,k} \xrightarrow{\sim} X_{m,1} \times_{X_{m,0}} \ldots \times_{X_{m,0}} X_{m,1} \cong \mathrm{Map}(R[m] \times G[k], X),$$

$$(q_m^R \times \mathrm{id}_{F[k]})^* \colon X_{m,k} \xrightarrow{\sim} X_{1,k} \times_{X_{0,k}} \ldots \times_{X_{0,k}} X_{1,k} \cong \mathrm{Map}(Q[m] \times F[k], X),$$

$$(e^R \times \mathrm{id}_{F[k]})^*\colon X^{\mathrm{heq}}_{1,k} \cong \mathrm{Map}(N^R I \times F[k], X) \xrightarrow{\sim} X_{0,k} \cong \mathrm{Map}(R[0] \times F[k], X)$$

are weak equivalences in sSet, where we identified $X_{m,k} \cong \mathrm{Map}(R[m] \times F[k], X)$, for all $m, k \geq 0$. The first weak equivalences tell us that the bisimplicial set $X_{m,-}$ is a Segal space, for all $m \geq 0$, while the two other weak equivalences tell us that the bisimplicial set $X_{-,k}$ is a complete Segal space, for all $k \geq 0$. This shows that the fibrant objects are precisely the horizontally complete double $(\infty, 1)$-categories. □

While we chose the completeness in the horizontal direction, one could have chosen to put it in the vertical direction instead. In this case, the double $(\infty, 1)$-categories correspond to Segal objects in complete Segal spaces, and hence correspond to the model of double $(\infty, 1)$-categories introduced by Haugseng in [Hau13, Definition 2.2.2.1]. However, as we show below, these two models are Quillen equivalent.

Remark 10.2.5. By requiring in the definition of a double $(\infty, 1)$-category that we instead have $X_{m,-}$ a complete Segal space and $X_{k,-}$ a Segal space, for all $m, k \geq 0$, we obtain the transposed notion of **vertically complete double** $(\infty, 1)$**-categories**. In particular, we can also define a model structure DblCat^v_∞ on $\mathrm{sSet}^{\Delta^{\mathrm{op}} \times \Delta^{\mathrm{op}}}$, whose fibrant objects are the vertically complete double $(\infty, 1)$-categories, by localizing the Reedy model structure on $\mathrm{sSet}^{\Delta^{\mathrm{op}} \times \Delta^{\mathrm{op}}}$ at the same monomorphisms inducing the Segal maps and the monomorphisms $\mathrm{id}_{R[m]} \times e^F\colon R[m] \cong R[m] \times F[0] \to R[m] \times N^F I$, for all $m \geq 0$, in place of the monomorphisms $e^R \times \mathrm{id}_{F[k]}$.

These two models of double $(\infty, 1)$-categories are then Quillen equivalent. To see this, consider the functor $t\colon \Delta \times \Delta \to \Delta \times \Delta$ which switches the two copies of Δ, i.e., we have $t([m], [k]) = t([k], [m])$, for all $m, k \geq 0$. Pre-composing with t induces a functor $t^*\colon \mathrm{sSet}^{\Delta^{\mathrm{op}} \times \Delta^{\mathrm{op}}} \to \mathrm{sSet}^{\Delta^{\mathrm{op}} \times \Delta^{\mathrm{op}}}$, which can be thought of as the "transpose" functor. Since $t \circ t = \mathrm{id}_{\Delta \times \Delta}$, we get an adjunction

$$\begin{tikzcd} \mathrm{DblCat}^h_\infty \arrow[r, bend right, "t^*"'] \arrow[r, phantom, "\perp"] & \mathrm{DblCat}^v_\infty \arrow[l, bend right, "t^*"'] \end{tikzcd},$$

which gives a Quillen equivalence between the two models of double $(\infty, 1)$-categories.

10.3. **2-fold complete Segal spaces.** We finally introduce the model of $(\infty, 2)$-categories given by the 2-*fold complete Segal spaces* first defined by Barwick [Bar05]. In analogy to the strict case where a 2-category is seen as an internal category to categories with a discrete category of objects (see Remark 3.1.8), we want to define an $(\infty, 2)$-category as a "homotopical" internal category to $(\infty, 1)$-categories with "homotopically discrete" $(\infty, 1)$-category of objects. To achieve this, we define an $(\infty, 2)$-category to be a double $(\infty, 1)$-category X whose first column is constant; in particular, we have that the spaces of vertical morphisms and objects are weakly equivalent, i.e., $X_{0,1} \simeq X_{0,0}$, which can be interpreted as the fact that all vertical morphisms are trivial. From this weak equivalence, we also get that $X_{0,1} \simeq X^{\mathrm{heq}}_{1,0}$, and so we can further require the completeness condition in both directions. A priori, there should be another model of double $(\infty, 1)$-categories, closer to the one of 2-fold complete Segal spaces, where there is some (not full) completeness condition between horizontal morphisms and squares; namely, the space of horizontal morphisms should be weakly equivalent to the space of homotopically, vertically invertible squares with "trivial" vertical boundaries. This model might be addressed in future work.

We first introduce the notation for the inclusion maps which will be used to express the fact that the first column is homotopically constant.

Notation 10.3.1. For every $k \geq 0$, we denote by $c_k\colon F[0] \to F[k]$ the inclusion map sending the unique 0-simplex of $F[0]$ to 0. Given $X \in \mathrm{sSet}^{\Delta^{\mathrm{op}}\times\Delta^{\mathrm{op}}}$, it induces a map

$$(c_k)^*\colon X_{0,k} \cong \mathrm{Map}(F[k], X) \to \mathrm{Map}(F[0], X) \cong X_{0,0}.$$

We can now state the definition of a 2-fold complete Segal space.

Definition 10.3.2. A trisimplicial set $X\colon \Delta^{\mathrm{op}} \times \Delta^{\mathrm{op}} \to \mathrm{sSet}$ is a **2-fold complete Segal space** if

(i) it is Reedy fibrant,
(ii) the bisimplicial set $X_{m,-}\colon \Delta^{\mathrm{op}} \to \mathrm{sSet}$ is a complete Segal space, for all $m \geq 0$,
(iii) the bisimplicial set $X_{-,k}\colon \Delta^{\mathrm{op}} \to \mathrm{sSet}$ is a complete Segal space, for all $k \geq 0$,
(iv) the bisimplicial set $X_{0,-}\colon \Delta^{\mathrm{op}} \to \mathrm{sSet}$ is **essentially constant**, i.e., the map $X_{0,k} \xrightarrow{\simeq} X_{0,0}$ of Notation 10.3.1 is a weak equivalence in sSet, for all $k \geq 0$.

Since 2-fold complete Segal spaces are in particular horizontally complete double $(\infty,1)$-categories, we can obtain the model structure for 2-fold complete Segal spaces by localizing the model structure DblCat^h_∞ on $\mathrm{sSet}^{\Delta^{\mathrm{op}}\times\Delta^{\mathrm{op}}}$ for horizontally complete double $(\infty,1)$-categories.

Theorem 10.3.3. *There is a model structure on* $\mathrm{sSet}^{\Delta^{\mathrm{op}}\times\Delta^{\mathrm{op}}}$*, denoted by* 2CSS*, such that*

(i) the cofibrations in 2CSS *are the monomorphisms; in particular, every object is cofibrant,*
(ii) the fibrant objects in 2CSS *are the* 2*-fold complete Segal spaces.*

In particular, it is obtained as a localization of the model structure DblCat^h_∞ *on* $\mathrm{sSet}^{\Delta^{\mathrm{op}}\times\Delta^{\mathrm{op}}}$ *of Theorem 10.2.4 at the set of monomorphisms*

$$\{\mathrm{id}_{R[m]} \times e^F\colon R[m] \cong R[m] \times F[0] \xrightarrow{\simeq} R[m] \times N^F I \mid m \geq 0\} \bigcup \{c_k\colon F[0] \to F[k] \mid k \geq 0\}.$$

Moreover, this model structure is combinatorial, and simplicial for the enrichment of Proposition 10.1.5.

Proof. Since the model structure DblCat^h_∞ on $\mathrm{sSet}^{\Delta^{\mathrm{op}}\times\Delta^{\mathrm{op}}}$ of Theorem 10.2.4 is a combinatorial, simplicial model structure such that all objects are cofibrant, its left Bousfield localization 2CSS at the set $\mathcal{S}$ of monomorphisms

$$\{\mathrm{id}_{R[m]} \times e^F\colon R[m] \cong R[m] \times F[0] \xrightarrow{\simeq} R[m] \times N^F I \mid m \geq 0\} \bigcup \{c_k\colon F[0] \to F[k] \mid k \geq 0\}$$

exists and is again simplicial and combinatorial, by Theorem 5.2.15. Moreover, the cofibrations in 2CSS are the monomorphisms, since they are the cofibrations in DblCat^h_∞. Finally, the fibrant objects are the $\mathcal{S}$-local objects by Proposition 5.2.18, i.e., an object $X \in$ 2CSS is fibrant if and only if it is a horizontally complete double $(\infty,1)$-category and the maps

$$(\mathrm{id}_{R[m]} \times e^F)^*\colon X^{\mathrm{heq}}_{m,1} \cong \mathrm{Map}(R[m] \times N^F I, X) \xrightarrow{\simeq} X_{m,0} \cong \mathrm{Map}(R[m] \times F[0], X),$$

$$c_k^*\colon X_{0,k} \cong \mathrm{Map}(F[k], X) \xrightarrow{\simeq} X_{0,0} \cong \mathrm{Map}(F[0], X)$$

are weak equivalences in sSet, for all $m, k \geq 0$. The first weak equivalences tell us that the Segal space $X_{m,-}$ is a complete Segal space, for all $m \geq 0$, while the second weak equivalences tell us that the bisimplicial set $X_{0,-}$ is essentially constant. This shows that the fibrant objects are precisely the 2-fold complete Segal spaces. □

Finally, since 2CSS is a left Bousfield localization of DblCat^h_∞, the identity adjunction on $\mathrm{sSet}^{\Delta^{\mathrm{op}}\times\Delta^{\mathrm{op}}}$ induces a Quillen reflection embedding the homotopy theory of 2-fold complete Segal spaces into that of horizontally complete double $(\infty,1)$-categories. Hence the functor $\mathrm{id}\colon 2\mathrm{CSS} \to \mathrm{DblCat}^h_\infty$ can be interpreted as the ∞-version of the horizontal embedding $\mathbb{H}\colon 2\mathrm{Cat} \to \mathrm{DblCat}$ (or even $\mathbb{H}^{\simeq}$; see Definitions 3.4.1 and 3.4.11).

Corollary 10.3.4. *The identity adjunction on* $\mathrm{sSet}^{\Delta^{\mathrm{op}}\times\Delta^{\mathrm{op}}}$

$$\begin{tikzcd} 2\mathrm{CSS} \arrow[r, bend right, "\mathrm{id}"'] \arrow[r, phantom, "\perp"] & \mathrm{DblCat}_\infty^h \arrow[l, bend right, "\mathrm{id}"'] \end{tikzcd}$$

is a Quillen reflection.

Proof. This follows directly from Theorem 10.3.3 and Proposition 5.2.14. $\square$

PART V.

NERVE CONSTRUCTIONS

As mentioned in Part IV., an n-category has k-morphisms between $(k-1)$-morphisms, for all $k \leq n$. On the other hand, an (∞, n)-category is a homotopical version of an n-category which has morphisms in all dimensions such that the k-morphisms are all invertible up to higher morphisms, for $k > n$. In particular, an n-category should be a special case of an (∞, n)-category where the k-morphisms, for $k > n$, are all trivial. From this point of view, a (1-)category is a special instance of an $(\infty, 1)$-category, and, for example, in the model of complete Segal spaces, there is a nerve $N_{\mathrm{Rezk}} \colon \mathrm{Cat} \to \mathrm{CSS}$, which constructs from a category a complete Segal space (see Example 9.2.15). This nerve actually embeds the homotopy theory of categories into that of complete Segal spaces in a reflective way, and hence gives the desired inclusion of categories into $(\infty, 1)$-categories.

Going up in dimensions, a 2-category should also be an example of an $(\infty, 2)$-category. Furthermore, we have seen that there is also a model of double $(\infty, 1)$-categories, which should therefore contain its stricter version, namely that of double categories. Hence, we aim to find an inclusion of the 2-dimensional categories into their ∞-analogues which is compatible with the horizontal embedding of 2-categories into double categories. As we have seen in Part IV., $(\infty, 2)$-categories and double $(\infty, 1)$-categories can be modeled by bisimplicial spaces, satisfying certain Segal and completeness conditions, and that the identity on the category $\mathrm{sSet}^{\Delta^{\mathrm{op}} \times \Delta^{\mathrm{op}}}$ of bisimplicial spaces induces an embedding of $(\infty, 2)$-categories into double $(\infty, 1)$-categories. Recall that the model of $(\infty, 2)$-categories considered is that of 2-fold complete Segal spaces. Hence, to achieve our goal, we construct a nerve functor $\mathbb{N} \colon \mathrm{DblCat} \to \mathrm{sSet}^{\Delta^{\mathrm{op}} \times \Delta^{\mathrm{op}}}$, which embeds the homotopy theory of double categories into that of double $(\infty, 1)$-categories, such that it restricts along the horizontal embedding to a nerve functor $2\mathrm{Cat} \to \mathrm{sSet}^{\Delta^{\mathrm{op}} \times \Delta^{\mathrm{op}}}$, which embeds the homotopy theory of 2-categories into that of 2-fold complete Segal spaces.

While the natural model structure for 2-categories is the one constructed by Lack in [Lac02, Lac04], also recalled in Section 6, we constructed in Part III. two different model structures for double categories. Hence we need to choose which homotopy theory of double categories we would like to consider in this context. By looking at the properties of a fibrant nerve, we can see that the nerve of a double category satisfies the Reedy fibrancy condition imposed on a double $(\infty, 1)$-category if and only if the double category considered is weakly horizontally invariant. In particular, since all double categories are fibrant in the first model structure on DblCat, the nerve will not preserve fibrant objects from this model structure, and hence we need to opt for the second model structure on DblCat, in which the fibrant double categories are precisely the weakly horizontally invariant ones.

With this model structure on DblCat for weakly horizontally invariant double categories, we show that the nerve functor $\mathbb{N} \colon \mathrm{DblCat} \to \mathrm{DblCat}^h_\infty$ is right Quillen and homotopically fully faithful into the model structure on $\mathrm{sSet}^{\Delta^{\mathrm{op}} \times \Delta^{\mathrm{op}}}$ for (horizontally complete) double $(\infty, 1)$-categories. Moreover, with this model structure on DblCat, we recall that the horizontal embedding which gives a right Quillen and homotopical fully faithful functor is given by the more homotopical version $\mathbb{H}^{\simeq} \colon 2\mathrm{Cat} \to \mathrm{DblCat}$. We show that the composite of this horizontal embedding with the nerve gives a functor $\mathbb{N}\mathbb{H}^{\simeq} \colon 2\mathrm{Cat} \to 2\mathrm{CSS}$ which is right Quillen and homotopically fully faithful from Lack's model structure on

2Cat into the model structure on $\mathrm{sSet}^{\Delta^{\mathrm{op}}\times\Delta^{\mathrm{op}}}$ for 2-fold complete Segal spaces. To sum up, this yields a commutative square of right Quillen and homotopically full functors as follows.

$$\begin{tikzcd} 2\mathrm{Cat} \arrow[r, "\mathbb{N}\mathbb{H}^{\simeq}"] \arrow[d, "\mathbb{H}^{\simeq}"'] & 2\mathrm{CSS} \arrow[d, "\mathrm{id}"] \\ \mathrm{DblCat} \arrow[r, "\mathbb{N}"'] & \mathrm{DblCat}_\infty^h \end{tikzcd}$$

However, we were hoping to find a nerve that is compatible with the horizontal embedding functor $\mathbb{H}$, but the nerve $\mathbb{N}\mathbb{H}\mathcal{A}$ of a horizontal double category $\mathbb{H}\mathcal{A}$ associated to a 2-category $\mathcal{A}$ is in general not Reedy fibrant since, as we have seen in Remark 8.4.5, a horizontal double category $\mathbb{H}\mathcal{A}$ is in general not weakly horizontally invariant. Hence the nerve $\mathbb{N}\mathbb{H}\mathcal{A}$ does not always give a double $(\infty,1)$-category or a 2-fold complete Segal space. We show that the nerve $\mathbb{N}\mathbb{H}^{\simeq}\mathcal{A}$ actually provides a fibrant replacement of the nerve $\mathbb{N}\mathbb{H}\mathcal{A}$, and hence this yields a diagram of right Quillen and homotopically fully faithful functors

$$\begin{tikzcd} 2\mathrm{Cat} \arrow[r, "\mathbb{N}\mathbb{H}^{\simeq}"] \arrow[d, "\mathbb{H}"'] & 2\mathrm{CSS} \arrow[dd, "\mathrm{id}"] \\ \mathrm{DblCat} \arrow[r, phantom, "\nearrow\!\!\!\!\nearrow \simeq"] & \\ \mathrm{DblCat} \arrow[u, "\mathrm{id}"] \arrow[r, "\mathbb{N}"'] & \mathrm{DblCat}_\infty^h \end{tikzcd}$$

filled with a natural transformation which is level-wise a weak equivalence. This gives the expected compatibility of the nerve $\mathbb{N}$ with the horizontal embedding $\mathbb{H}$.

In Section 11, we first construct the nerve $\mathbb{N}$ for double categories into double $(\infty,1)$-categories and show that it has the expected homotopical properties. Then, in Section 12, we restrict this nerve along the horizontal embedding $\mathbb{H}^{\simeq}$, and show that it gives rise to a homotopically full embedding of 2-categories into 2-fold complete Segal spaces. We also show, as mentioned above, that the nerve $\mathbb{N}\mathbb{H}^{\simeq}$ is a level-wise fibrant replacement of $\mathbb{N}\mathbb{H}$. Finally, in Section 13, we describe the different nerves considered in low dimensions in order to get some intuition about their constructions. The results here are based on the paper [Mos20] by the author.

11. Nerve of double categories

In this section, we give the construction of the nerve functor from double categories to bisimplicial spaces. In Section 11.1, we define the nerve and its left adjoint, and in Section 11.2, we show that they form a Quillen pair between the model structure on DblCat for weakly horizontally invariant double categories, constructed in Theorem 8.1.15, and the model structure DblCat_∞^h for horizontally complete double $(\infty,1)$-categories, constructed in Theorem 10.2.4. Once this fact is established, we prove in Section 11.3 that the nerve functor is homotopically fully faithful, by showing that the (derived) counit of the adjunction is level-wise a weak equivalence in DblCat. Finally, in Section 11.4, we show that the nerve of a double category is almost fibrant; namely, it satisfies all conditions of a horizontally complete double $(\infty,1)$-category except for the Reedy/injective fibrancy condition in the vertical direction. We show that this latter condition is satisfied by the nerve if and only if the double category considered is weakly horizontally invariant.

11.1. **Definition of the nerve.** As mentioned in Section 10.2, a double $(\infty,1)$-category is a trisimplicial set $X\colon \Delta^{\mathrm{op}}\times\Delta^{\mathrm{op}}\to \mathrm{sSet}$ where we can think of $X_{0,0}$ as the space of objects, $X_{1,0}$ as the space of horizontal morphisms, $X_{0,1}$ as the space of vertical morphisms, and $X_{1,1}$ as the space of squares. Hence, given a double category $\mathbb{A}$, we would like to associate to it a trisimplicial set $\mathbb{N}\mathbb{A}$ in such a way that the vertices of the spaces $(\mathbb{N}\mathbb{A})_{0,0}$, $(\mathbb{N}\mathbb{A})_{1,0}$, $(\mathbb{N}\mathbb{A})_{0,1}$, and $(\mathbb{N}\mathbb{A})_{1,1}$ are precisely the objects, horizontal morphisms, vertical morphisms, and squares of $\mathbb{A}$, respectively. In this section, we give the construction of such a nerve functor $\mathbb{N}\colon \mathrm{DblCat}\to \mathrm{sSet}^{\Delta^{\mathrm{op}}\times\Delta^{\mathrm{op}}}$.

To define the nerve functor, we make use of the following proposition, which can also be found in [Cis19, Theorem 1.1.10].

Proposition 11.1.1. *Let $\mathcal{C}$ be a small category and $\mathcal{M}$ be a locally presentable category. Given a functor $F\colon \mathcal{C}\to\mathcal{M}$, its left Kan extension $L\colon \mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}\to\mathcal{M}$ along the Yoneda embedding $\mathcal{C}\to\mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$, which sends an object $C\in\mathcal{C}$ to the representable functor $\mathcal{C}(-,C)\colon\mathcal{C}^{\mathrm{op}}\to\mathrm{Set}$, exists and has a right adjoint $R\colon\mathcal{M}\to\mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$.*

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "F"] \arrow[d] & \mathcal{M} \arrow[dl, bend left, "R"] \\
\mathrm{Set}^{\mathcal{C}^{\mathrm{op}}} \arrow[ur, "L"] \arrow[ur, phantom, shift right=2, "\perp"] &
\end{tikzcd}$$

Moreover, for every object $C\in\mathcal{C}$, we have that $L(\mathcal{C}(-,C))=F(C)$, and, for every pair of objects $A\in\mathcal{M}$ and $C\in\mathcal{C}$, there is an isomorphism $(RA)(C)\cong\mathcal{M}(F(C),A)$, natural in A and C.

Proof. We first construct L. Let $X\colon\mathcal{C}^{\mathrm{op}}\to\mathrm{Set}$ be an object of $\mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$. Then there is an isomorphism in $\mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$
$$X\cong \mathrm{colim}_{x\in X(C),\,C\in\mathcal{C}}\,\mathcal{C}(-,C).$$
We set $LX\coloneqq \mathrm{colim}_{x\in X(C),\,C\in\mathcal{C}}\,F(C)\in\mathcal{M}$. Since $\mathcal{M}$ is locally presentable, it is in particular cocomplete, and this is well-defined. Furthermore, it extends uniquely to a functor $L\colon\mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}\to\mathcal{M}$ by the universal property of colimits. Moreover, by definition, we have that $L(\mathcal{C}(-,C))=F(C)$, for every object $C\in\mathcal{C}$, and that L is the left Kan extension of F, since it is defined with the formula for point-wise left Kan extensions.

We now construct R. Given an object $A\in\mathcal{M}$, we set
$$RA\coloneqq\mathcal{M}(F(-),A)\colon\mathcal{C}^{\mathrm{op}}\to\mathrm{Set}$$
to be the functor which sends an object $C\in\mathcal{C}$ to the set $\mathcal{M}(F(C),A)$. This assignment extends on morphisms of $\mathcal{M}$, and defines a functor $R\colon\mathcal{M}\to\mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$.

Finally, we show that the functors L and R form an adjunction. Let A be an object in $\mathcal{M}$, and $X\colon\mathcal{C}^{\mathrm{op}}\to\mathrm{Set}$ be an object in $\mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$. Then we have isomorphisms
$$\begin{aligned}
\mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}(X,RA) &\cong \mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}(\mathrm{colim}_{x\in X(C),\,C\in\mathcal{C}}\,\mathcal{C}(-,C),RA)\\
&\cong \lim_{x\in X(C),\,C\in\mathcal{C}}\mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}(\mathcal{C}(-,C),RA)\cong \lim_{x\in X(C),\,C\in\mathcal{C}} RA(C)\\
&\cong \lim_{x\in X(C),\,C\in\mathcal{C}}\mathcal{M}(F(C),A)\cong\mathcal{M}(\mathrm{colim}_{x\in X(C),\,C\in\mathcal{C}}\,F(C),A)\\
&=\mathcal{M}(LX,A)
\end{aligned}$$
natural in X and A, where the first isomorphism holds since $X\cong\mathrm{colim}_{x\in X(C),\,C\in\mathcal{C}}\,\mathcal{C}(-,C)$, the second since $\mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}(-,-)$ sends colimits to limits in the first variable, the third by the Yoneda Lemma, the fourth by definition of RA, the fifth since $\mathcal{M}(-,-)$ sends colimits to limits in the first variable, and finally the last one by definition of LX. This shows that $L\dashv R$ is an adjunction. $\square$

To define the nerve we make use of truncated versions of the n-orientals $O(n)$, introduced by Street in [Str87]. More precisely:

Definition 11.1.2. For $n \geq 0$, we define the 2-category $O_2(n)$, called the **2-truncated n-oriental**, as the 2-category described by the following data.

(i) Its set of objects is given by $\{0, \ldots, n\}$,
(ii) For $0 \leq x, x' \leq n$, its hom-category $O_2(n)(x, x')$ is given by the poset

$$O_2(n)(x,x') = \begin{cases} \{I \subseteq [x,x'] \mid x, x' \in I\} & \text{if } x' \leq x, \\ \emptyset & \text{if } x > x', \end{cases}$$

where $[x, x'] = \{y \in \{0, \ldots, n\} \mid x \leq y \leq x'\}$.

We also define the 2-category $O_2^{\simeq}(n)$ as the 2-category obtained from $O_2(n)$ by formally inverting every 2-morphism, and we define the 2-category $\widetilde{O_2(n)}$ as the 2-category obtained from $O_2^{\simeq}(n)$ by formally making every morphism into an adjoint equivalence.

In order to have a better sense of what these 2-categories look like, we describe the lower cases.

Example 11.1.3. For $n = 0$, the 2-categories $O_2(0)$, $O_2^{\simeq}(0)$, and $\widetilde{O_2(0)}$ are all given by the terminal (2-)category $[0]$.

For $n = 1$, the 2-categories $O_2(1)$ and $O_2^{\simeq}(1)$ are both given by the free (2-)category $[1]$ on a morphism, while the 2-category $\widetilde{O_2(1)}$ is the "free-living adjoint equivalence" E_{adj}.

For $n = 2$, the 2-categories $O_2(2)$, $O_2^{\simeq}(2)$, and $\widetilde{O_2(2)}$ are generated, respectively, by the following data,

$$\begin{tikzcd} & 1 \arrow[dr] \arrow[d, phantom, "\Uparrow"] & \\ 0 \arrow[ur] \arrow[rr] & {} & 2 \end{tikzcd} \qquad \begin{tikzcd} & 1 \arrow[dr] \arrow[d, phantom, "\Uparrow\cong"] & \\ 0 \arrow[ur] \arrow[rr] & {} & 2 \end{tikzcd} \qquad \begin{tikzcd} & 1 \arrow[dr, "\simeq"] \arrow[d, phantom, "\Uparrow\cong"] & \\ 0 \arrow[ur, "\simeq"] \arrow[rr, "\simeq"'] & {} & 2 \end{tikzcd}$$

where $\xrightarrow{\simeq}$ denotes the data of an adjoint equivalence.

For $n = 3$, the 2-category $O_2(3)$ is generated by the following data

$$\begin{tikzcd} 1 \arrow[r] & 2 \arrow[d] \\ 0 \arrow[u] \arrow[r] \arrow[ur] \arrow[u, phantom, shift left=4ex, "\Nwarrow"] & 3 \arrow[u, phantom, shift right=4ex, "\Uparrow"] \end{tikzcd} = \begin{tikzcd} 1 \arrow[r] \arrow[dr] & 2 \arrow[d] \\ 0 \arrow[u] \arrow[r] \arrow[u, phantom, shift left=4ex, "\Uparrow"] & 3 \arrow[u, phantom, shift right=4ex, "\Nearrow"] \end{tikzcd}$$

and the 2-category $O_2^{\simeq}(3)$ is generated by the corresponding 2-category with all 2-morphisms invertible, while the 2-category $\widetilde{O_2(3)}$ is generated by the corresponding 2-category with all morphisms being adjoint equivalences and all 2-morphisms being invertible.

We now use Proposition 11.1.1 to construct the nerve functor $\mathbb{N}\colon \mathrm{DblCat} \to \mathrm{Set}^{(\Delta^{\mathrm{op}})^{\times 3}}$, as the right adjoint of the left Kan extension of the following tricosimplicial object in double categories along the Yoneda embedding. To define this tricosimplicial object $\mathbb{X}\colon \Delta \times \Delta \times \Delta \to \mathrm{DblCat}$, we recall that the first and second copies of Δ represents the horizontal and vertical directions, respectively, while the third one represents the space direction of the double $(\infty, 1)$-category. Hence, in the first two copies we want to see horizontal (resp. vertical morphisms) and their composites, while in the last one we would to see all horizontal equivalences and their composites, since double $(\infty, 1)$-categories are assumed to be horizontally complete.

Definition 11.1.4. We define the tricosimplicial double category

$$\mathbb{X}\colon \Delta \times \Delta \times \Delta \to \mathrm{DblCat},$$

$$([m],[k],[n]) \mapsto \mathbb{X}_{m,k,n} := (\mathbb{V}O_2^{\simeq}(k) \otimes O_2^{\simeq}(m)) \otimes \widetilde{O_2(n)},$$

where the cosimplicial maps are induced by the ones of the cosimplicial objects

$$\begin{aligned} \Delta &\to \mathrm{DblCat}, & \Delta &\to 2\mathrm{Cat}, & & \\ [k] &\mapsto \mathbb{V}O_2^{\sim}(k), & [m] &\mapsto O_2^{\sim}(m), \quad \text{and} & [n] &\mapsto \widetilde{O_2(n)}, \end{aligned}$$

and $\otimes\colon \mathrm{DblCat} \times 2\mathrm{Cat} \to \mathrm{DblCat}$ is the tensoring functor introduced in Definition 3.5.1.

Proposition 11.1.5. *The tricosimplicial double category $\mathbb{X}$ induces an adjunction*

$$\begin{tikzcd} \Delta \times \Delta \times \Delta \arrow[r, "\mathbb{X}"] \arrow[d] & \mathrm{DblCat} \arrow[dl, bend left, "\mathbb{N}"] \arrow[dl, phantom, "\perp"] \\ \mathrm{Set}^{(\Delta^{\mathrm{op}})^{\times 3}} \arrow[ur, "\mathbb{C}"] & \end{tikzcd},$$

where $\mathbb{C}$ is the left Kan extension of $\mathbb{X}$ along the Yoneda embedding, and we have that

$$(\mathbb{N}\mathbb{A})_{m,k,n} \cong \mathrm{DblCat}((\mathbb{V}O_2^{\sim}(k) \otimes O_2^{\sim}(m)) \otimes \widetilde{O_2(n)}, \mathbb{A}),$$

for all $\mathbb{A} \in \mathrm{DblCat}$ and all $m, k, n \geq 0$,

Proof. This is a direct application of Proposition 11.1.1, since DblCat is locally presentable by Proposition 3.1.6. □

Remark 11.1.6. As expected, note that the 0-simplices of the simplicial set $(\mathbb{N}\mathbb{A})_{0,0}$ are given by the objects of $\mathbb{A}$, the ones of $(\mathbb{N}\mathbb{A})_{1,0}$ by the horizontal morphisms of $\mathbb{A}$, the ones of $(\mathbb{N}\mathbb{A})_{0,1}$ by the vertical morphisms of $\mathbb{A}$, and the ones of $(\mathbb{N}\mathbb{A})_{1,1}$ by the squares of $\mathbb{A}$. For a description of the 1- and 2-simplices of these simplicial sets, we refer the reader to Section 13.1. For $m \geq 2$ or $k \geq 2$, the simplicial sets $(\mathbb{N}\mathbb{A})_{m,k}$ witness "compositions" in $\mathbb{A}$ of the above data.

Remark 11.1.7. By Proposition 11.1.1, the left adjoint $\mathbb{C}$ of the nerve is given on representables by $\mathbb{C}(F[k] \times R[m] \times \Delta[n]) = \mathbb{X}_{m,k,n}$. In particular, we have that

$$\mathbb{C}(F[k]) = \mathbb{V}O_2^{\sim}(k), \quad \mathbb{C}(R[m]) = \mathbb{H}O_2^{\sim}(m) \;\text{ and }\; \mathbb{C}(\Delta[n]) = \mathbb{H}\widetilde{O_2(n)}.$$

We also introduce a functor $\overline{\mathbb{C}}$, which takes values in 2-categories and coincides with $\mathbb{C}$ in the horizontal and space directions.

Notation 11.1.8. We denote by $\overline{\mathbb{X}}\colon \Delta \times \Delta \times \Delta \to 2\mathrm{Cat}$ the tricosimplicial 2-category given by $\overline{\mathbb{X}}_{m,k,n} \coloneqq O_2^{\sim}(m) \otimes_2 \widetilde{O_2(n)}$, and by $\overline{\mathbb{C}}\colon \mathrm{Set}^{(\Delta^{\mathrm{op}})^{\times 3}} \to 2\mathrm{Cat}$ the left Kan extension of $\overline{\mathbb{X}}$ along the Yoneda embedding, where $\otimes_2\colon 2\mathrm{Cat} \times 2\mathrm{Cat} \to 2\mathrm{Cat}$ is the Gray tensor product introduced in Proposition 2.3.4.

Remark 11.1.9. Note that $\mathbb{X}_{m,0,n} = \mathbb{H}\overline{\mathbb{X}}_{m,0,n}$. Therefore, if $X \in \mathrm{Set}^{(\Delta^{\mathrm{op}})^{\times 3}}$ is constant in the vertical direction, then $\mathbb{C}X = \mathbb{H}\overline{\mathbb{C}}X$. In particular, we have that $\mathbb{C}(R[m]) = \mathbb{H}\overline{\mathbb{C}}(R[m])$ and $\mathbb{C}(\Delta[n]) = \mathbb{H}\overline{\mathbb{C}}(\Delta[n])$, where $\overline{\mathbb{C}}(R[m]) = O_2^{\sim}(m)$ and $\overline{\mathbb{C}}(\Delta[n]) = \widetilde{O_2(n)}$.

11.2. **The nerve $\mathbb{N}$ is right Quillen.** We now want to show that the nerve functor defined in the previous section induces a right Quillen functor from the model structure on DblCat of Theorem 8.1.15 for weakly horizontally invariant double categories and the model structure DblCat_∞^h on $\mathrm{sSet}^{\Delta^{\mathrm{op}} \times \Delta^{\mathrm{op}}}$ of Theorem 10.2.4 for horizontally complete double $(\infty, 1)$-categories. Since the model category DblCat_∞^h is obtained as a left Bousfield localization of the Reedy/injective model structure on $\mathrm{sSet}^{\Delta^{\mathrm{op}} \times \Delta^{\mathrm{op}}}$, we first prove that $\mathbb{N}\colon \mathrm{DblCat} \to \mathrm{sSet}^{\Delta^{\mathrm{op}} \times \Delta^{\mathrm{op}}}$ is right Quillen for the Reedy/injective model structure, and then apply Theorem 5.2.23 to show that it restricts to a right Quillen functor to DblCat_∞^h.

Proposition 11.2.1. *The adjunction*

$$\begin{tikzcd} \mathrm{DblCat} \arrow[r, bend right, "\mathbb{N}"'] \arrow[r, phantom, "\perp"] & \mathrm{sSet}^{\Delta^{\mathrm{op}}\times\Delta^{\mathrm{op}}} \arrow[l, bend right, "\mathbb{C}"'] \end{tikzcd}$$

is a Quillen pair between the model structure on DblCat *of Theorem 8.1.15 and the Reedy/injective model structure on* $\mathrm{sSet}^{\Delta^{\mathrm{op}}\times\Delta^{\mathrm{op}}}$ *of Theorem 10.1.7.*

Proof. By Remark 4.4.4, to prove that $\mathbb{C}$ is left Quillen, it is enough to show that $\mathbb{C}$ sends generating cofibrations and generating trivial cofibrations in the Reedy/injective model structure $\mathrm{sSet}^{\Delta^{\mathrm{op}}\times\Delta^{\mathrm{op}}}$ to cofibrations and trivial cofibrations in DblCat, respectively. Recall from Remark 10.1.8 that a set of generating cofibrations for $\mathrm{sSet}^{\Delta^{\mathrm{op}}\times\Delta^{\mathrm{op}}}$ is given by the pushout-product maps $(\iota^F_k \square_\times \iota^R_m) \square_\times \iota^\Delta_n$, for all $m, k, n \geq 0$, and a set of generating trivial cofibrations is given by the pushout-product maps $(\iota^F_k \square_\times \iota^R_m) \square_\times \ell^\Delta_{n,t}$, for all $m, k \geq 0$, $n \geq 1$, and $0 \leq t \leq n$. Note that the map ι^F_k is constant in the horizontal and space directions, the map ι^R_m is constant in the vertical and space directions, and the maps ι^Δ_n and $\ell^\Delta_{n,t}$ are constant in the horizontal and vertical directions. Therefore, since the functor $\mathbb{C}$ preserves colimits and by Remark 11.1.9, we have that

$$\mathbb{C}((\iota^F_k \square_\times \iota^R_m) \square_\times \iota^\Delta_n) \cong (\mathbb{C}\iota^F_k \square_{\otimes_{\mathrm{G}}} \mathbb{C}\iota^R_m) \square_{\otimes_{\mathrm{G}}} \mathbb{C}\iota^\Delta_n \cong (\mathbb{C}\iota^F_k \square_\otimes \overline{\mathbb{C}}\iota^R_m) \square_\otimes \overline{\mathbb{C}}\iota^\Delta_n,$$

and similarly for $\ell^\Delta_{n,t}$ in place of ι^Δ_n. Since the model structure DblCat is enriched over 2Cat by Remark 8.5.7, pushout-products of cofibrations with respect to $\otimes$ are cofibrations, which are trivial if one of the morphisms involved is a weak equivalence. Therefore, it is enough to show that $\mathbb{C}\iota^F_k$ is a cofibration in DblCat, for all $k \geq 0$, that $\overline{\mathbb{C}}\iota^R_m$ and $\overline{\mathbb{C}}\iota^\Delta_n$ are cofibrations in 2Cat, for all $m, n \geq 0$, and that $\overline{\mathbb{C}}\ell^\Delta_{n,t}$ is a trivial cofibration in 2Cat, for all $n \geq 1$, $0 \leq t \leq n$. These statements are verified in Lemmas 11.2.4 to 11.2.6 below. $\square$

To prove that the boundary and horn inclusions mentioned above are sent to cofibrations in 2Cat and DblCat, we introduce the following definitions of the boundary of $O_2(n)$ and the (n,t)-horn of $O_2(n)$, which will be used to describe the images under $\mathbb{C}$ of the boundary and horn inclusions.

Definition 11.2.2. For $n \geq 0$, we define the **boundary 2-category** $\delta O_2(n)$ as the coequalizer in 2Cat

$$\bigsqcup_{0\leq i<j\leq n} O_2(n-2) \rightrightarrows \bigsqcup_{0\leq i\leq n} O_2(n-1) \longrightarrow \delta O_2(n)\,,$$

where the maps in the (i,j)-copy are induced by the cosimplicial identities $d^i d^j = d^{j-1} d^i$, where $d^r \colon O_2(n-2) \to O_2(n-1)$ and $d^s \colon O_2(n-1) \to O_2(n)$ denote the face maps for $r = i, j$ and $s = i, j-1$. In particular, there is an inclusion $\delta O_2(n) \to O_2(n)$ induced by the face maps $d^i \colon O_2(n-1) \to O_2(n)$ for $0 \leq i \leq n$. More explicitly, these 2-categories are given by the following:

- for $n = 0$, $\delta O_2(0) = \emptyset$ with $\delta O_2(0) = \emptyset \to O_2(0) = [0]$ given by the unique morphism,
- for $n = 1$, $\delta O_2(1) = [0] \sqcup [0]$ with $\delta O_2(1) = [0] \sqcup [0] \to O_2(1) = [1]$ given by including the two copies of $[0]$ as the two endpoints of the morphism in $[1]$,
- for $n = 2$, $\delta O_2(2)$ is the sub-2-category of $O_2(2)$ where the 2-morphism is missing and the inclusion $\delta O_2(2) \to O_2(2)$ is given by the following 2-functor.

$$\begin{tikzcd} & 1 \arrow[dr] & & & & 1 \arrow[dr] & \\ 0 \arrow[ur] \arrow[rr] & & 2 & \longrightarrow & 0 \arrow[ur] \arrow[rr] \arrow[rr, phantom, shift left=3, "\Uparrow"] & & 2 \end{tikzcd}$$

- for $n = 3$, $\delta O_2(3)$ is the sub-2-category of $O_2(3)$ where only the equality between the two pasting diagrams in $O_2(3)$ – as depicted in Example 11.1.3 – is missing,

- for $n \geq 4$, $\delta O_2(n) = O_2(n)$.

Similarly, we define the boundary 2-categories $\delta O_2^{\sim}(n)$ and $\delta\widetilde{O_2(n)}$.

Definition 11.2.3. For $n \geq 1$ and $0 \leq t \leq n$, we define the (n,t)**-horn 2-category** $\Lambda^t O_2(n)$ as the co-equalizer in 2Cat

$$\bigsqcup_{\substack{0\leq i<j\leq n\\ i\neq t, j\neq t}} O_2(n-2) \rightrightarrows \bigsqcup_{\substack{0\leq i\leq n\\ i\neq t}} O_2(n-1) \longrightarrow \Lambda^t O_2(n)\,,$$

where the maps in the (i,j)-copy are induced by the cosimplicial identities $d^i d^j = d^{j-1}d^i$, where $d^r\colon O_2(n-2) \to O_2(n-1)$ and $d^s\colon O_2(n-1) \to O_2(n)$ denote the face maps for $r = i,j$ and $s = i, j-1$. In particular, there is an inclusion $\Lambda^t O_2(n) \to O_2(n)$ induced by the face maps $d^i\colon O_2(n-1) \to O_2(n)$ for $0 \leq i \leq n$, $i \neq t$. More explicitly, these 2-categories are given by the following:

- for $n = 1$, $\Lambda^t O_2(1) = [0]$ with $\Lambda^t O_2(1) = [0] \to O_2(1) = [1]$ given by the inclusion of $[0]$ at the source of the morphism in $[1]$ if $t = 1$ and at the target if $t = 0$,
- for $n = 2$, $\Lambda^2 O_2(2)$, $\Lambda^1 O_2(2)$, and $\Lambda^0 O_2(2)$ are generated, respectively, by the following data

$$\begin{tikzcd} & 1 \arrow[dr] & \\ 0 \arrow[rr] & & 2 \end{tikzcd} \qquad \begin{tikzcd} & 1 \arrow[dr] & \\ 0 \arrow[ur] & & 2 \end{tikzcd} \qquad \begin{tikzcd} & 1 & \\ 0 \arrow[ur] \arrow[rr] & & 2 \end{tikzcd}$$

  with the obvious inclusions into $O_2(2)$,
- for $n = 3$ and $0 \leq t \leq 3$, $\Lambda^t O_2(3)$ is the sub-2-category where the equality between the two pasting diagrams in $O_2(3)$ and the 2-morphism opposite to the object t are missing. For example, when $t = 0$, the inclusion $\Lambda^0 O_2(3) \to O_2(3)$ is given by the following.

$$\begin{tikzcd} 1 \arrow[r] & 2 \arrow[d] \\ 0 \arrow[u] \arrow[ur] \arrow[r] & 3 \arrow[ul, phantom, "\Leftarrow", pos=0.8] \arrow[ul, phantom, "\Uparrow", pos=0.2] \end{tikzcd} \quad \begin{tikzcd} 1 \arrow[r] \arrow[dr] & 2 \arrow[d] \\ 0 \arrow[u] \arrow[r] & 3 \arrow[ul, phantom, "\Uparrow", pos=0.75] \end{tikzcd} \quad \longrightarrow \quad \begin{tikzcd} 1 \arrow[r] & 2 \arrow[d] \\ 0 \arrow[u] \arrow[ur] \arrow[r] & 3 \arrow[ul, phantom, "\Leftarrow", pos=0.8] \arrow[ul, phantom, "\Uparrow", pos=0.2] \end{tikzcd} \quad = \quad \begin{tikzcd} 1 \arrow[r] \arrow[dr] & 2 \arrow[d] \\ 0 \arrow[u] \arrow[r] & 3 \arrow[ul, phantom, "\Uparrow", pos=0.75] \arrow[ul, phantom, "\Rightarrow", pos=0.25] \end{tikzcd}$$

- for $n \geq 4$ and $0 \leq t \leq n$, $\Lambda^t O_2(n) = O_2(n)$.

Similarly, we define the (n,t)-horn 2-categories $\Lambda^t O_2^{\sim}(n)$ and $\Lambda^t \widetilde{O_2(n)}$.

We are now ready to prove the promised lemmas which complete the proof of Proposition 11.2.1. Recall from Notation 8.1.1 the set $\mathcal{I}_w$ of generating cofibrations for the model structure on DblCat of Theorem 8.1.15, and from Notation 6.2.5 the sets $\mathcal{I}_2$ and $\mathcal{J}_2$ of generating cofibrations and generating trivial cofibrations for Lack's model structure on 2Cat of Theorem 6.1.8.

Lemma 11.2.4. *For all $k \geq 0$, the double functor $\mathbb{C}(\iota_k^F)\colon \mathbb{C}(\delta F[k]) \to \mathbb{C}(F[k])$ is a cofibration in* DblCat.

Proof. The boundary $\delta F[k]$ of the representable $F[k]$ can be computed as the following co-equalizer in $\mathrm{sSet}^{\Delta^{\mathrm{op}}\times\Delta^{\mathrm{op}}}$

$$\bigsqcup_{0\leq i<j\leq k} F[k-2] \rightrightarrows \bigsqcup_{0\leq i\leq k} F[k-1] \longrightarrow \delta F[k]\,,$$

where the maps in the (i,j)-copy are induced by the cosimplicial identities $d^i d^j = d^{j-1}d^i$. By construction of $\delta O_2^{\sim}(k)$ (see Definition 11.2.2), by Remark 11.1.7, and since $\mathbb{C}$ preserves colimits, we find that

$$\mathbb{C}(\delta F[k]) = \mathbb{V}\delta O_2^{\sim}(k) \quad \text{and} \quad \mathbb{C}(F[k]) = \mathbb{V}O_2^{\sim}(k),$$

for all $k \geq 0$. Therefore, the double functors $\mathbb{C}(\iota_k^F)$ are given by

- for $k = 0$, the generating cofibration $I_1 \colon \emptyset \to [0]$,
- for $k = 1$, the generating cofibration $I_3' \colon [0] \sqcup [0] \to \mathbb{V}[1]$,
- for $k = 2$, the inclusion

$$\begin{tikzcd}
0 \arrow[r, equal] \arrow[dd, "\bullet" marking] & 0 \arrow[d, "\bullet" marking] & & 0 \arrow[r, equal] \arrow[dd, "\bullet" marking] & 0 \arrow[d, "\bullet" marking] \\
& 1 \arrow[d, "\bullet" marking] & \longrightarrow & {} \arrow[r, phantom, "\cong"] & 1 \arrow[d, "\bullet" marking] \\
2 \arrow[r, equal] & 2 & & 2 \arrow[r, equal] & 2 ,
\end{tikzcd}$$

  which is a cofibration by Corollary 8.1.5 since it is the identity on underlying horizontal and vertical categories,
- for $k = 3$, the inclusion $\mathbb{V}\delta O_2^{\sim}(3) \to \mathbb{V}O_2^{\sim}(3)$, which is a cofibration by Corollary 8.1.5 since it is the identity on underlying horizontal and vertical categories,
- for $k \geq 4$, the identity.

This shows that the double functor $\mathbb{C}(\iota_k^F)$ is a cofibration in DblCat, for all $k \geq 0$. $\square$

Lemma 11.2.5. *For all $m, n \geq 0$, the 2-functors $\overline{\mathbb{C}}(\iota_m^R) \colon \overline{\mathbb{C}}(\delta R[m]) \to \overline{\mathbb{C}}(R[m])$ and $\overline{\mathbb{C}}(\iota_n^\Delta) \colon \overline{\mathbb{C}}(\delta\Delta[n]) \to \overline{\mathbb{C}}(\Delta[n])$ are cofibrations in* 2Cat.

Proof. We first prove the statement for $\overline{\mathbb{C}}(\iota_m^R)$. As in the proof of Lemma 11.2.4 and by Remark 11.1.9, we find that

$$\overline{\mathbb{C}}(\delta R[m]) = \delta O_2^{\sim}(m) \quad \text{and} \quad \overline{\mathbb{C}}(R[m]) = O_2^{\sim}(m),$$

for all $m \geq 0$. Therefore, the 2-functors $\overline{\mathbb{C}}(\iota_m^R)$ are given by

- for $m = 0$, the generating cofibration $i_1 \colon \emptyset \to [0]$,
- for $m = 1$, the generating cofibration $i_2 \colon [0] \sqcup [0] \to [1]$,
- for $m = 2$, the inclusion $\delta O_2^{\sim}(2) \to O_2^{\sim}(2)$, which is a cofibration by Corollary 6.2.3 since it is the identity on underlying categories,
- for $m = 3$, the inclusion $\delta O_2^{\sim}(3) \to O_2^{\sim}(3)$, which is a cofibration by Corollary 6.2.3 since it is the identity on underlying categories,
- for $m \geq 4$, the identity.

Therefore, the 2-functor $\overline{\mathbb{C}}(\iota_m^R)$ is a cofibration in 2Cat, for all $m \geq 0$.

We now prove the statement for $\overline{\mathbb{C}}(\iota_n^\Delta)$. As above, we find that

$$\overline{\mathbb{C}}(\delta\Delta[n]) = \widetilde{\delta O_2(n)} \quad \text{and} \quad \overline{\mathbb{C}}(\Delta[n]) = \widetilde{O_2(n)},$$

for all $n \geq 0$. Therefore the 2-functors $\overline{\mathbb{C}}(\iota_n^\Delta) \colon \widetilde{\delta O_2(n)} \to \widetilde{O_2(n)}$ can be described as the 2-functors $\overline{\mathbb{C}}(\iota_m^R)$ above, but where all the morphisms of the 2-categories in play are adjoint equivalences. Using Corollary 6.2.3, it is then straightforward to see that the 2-functor $\overline{\mathbb{C}}(\iota_n^\Delta)$ is also a cofibration in 2Cat, for all $n \geq 0$. $\square$

Lemma 11.2.6. *For all $n \geq 1$ and $0 \leq t \leq n$, the 2-functor $\overline{\mathbb{C}}(\ell_{n,t}^\Delta) \colon \overline{\mathbb{C}}(\Lambda^t[n]) \to \overline{\mathbb{C}}(\Delta[n])$ is a trivial cofibration in* 2Cat.

Proof. We have that $\Lambda^t[n]$ is defined as the coequalizer in $\mathrm{sSet}^{\Delta^{\mathrm{op}} \times \Delta^{\mathrm{op}}}$

$$\bigsqcup_{\substack{0 \leq i < j \leq n \\ i \neq t, j \neq t}} \Delta[n-2] \rightrightarrows \bigsqcup_{\substack{0 \leq i \leq n \\ i \neq t}} \Delta[n-1] \longrightarrow \Lambda^t[n] \, ,$$

where the maps in the (i,j)-copy are induced by the cosimplicial identities $d^i d^j = d^{j-1} d^i$. By construction of $\Lambda^t \widetilde{O_2(n)}$ (see Definition 11.2.3), by Remark 11.1.9, and since $\overline{\mathbb{C}}$ preserves colimits, we find that

$$\overline{\mathbb{C}}(\Lambda^t[n]) = \Lambda^t\widetilde{O_2(n)} \quad \text{and} \quad \overline{\mathbb{C}}(\Delta[n]) = \widetilde{O_2(n)},$$

for all $n \geq 1$ and $0 \leq t \leq n$. Therefore, the 2-functors $\overline{\mathbb{C}}(\ell^{\Delta}_{n,t})\colon \Lambda^t\widetilde{O_2(n)} \to \widetilde{O_2(n)}$ are given by

- for $n = 1$ and $0 \leq t \leq 1$, the generating trivial cofibration $j_1\colon [0] \to \widetilde{O_2(1)} = E_{\mathrm{adj}}$, including $[0]$ as one of the two end points,
- for $n = 2$ and $0 \leq t \leq 2$, the inclusion $\Lambda^t\widetilde{O_2(2)} \to \widetilde{O_2(2)}$, which is a cofibration by Corollary 6.2.3 since it is given by adding two morphisms $x \to y$ and $y \to x$ freely between objects $x < y \in \{0,1,2\} \setminus \{t\}$ on underlying categories. Moreover, it is a biequivalence, since it is bijective on objects, essentially full on morphisms, and fully faithful on 2-morphisms, where essential fullness on morphisms can be shown using the fact that all the morphisms are adjoint equivalences.
- for $n = 3$ and $0 \leq t \leq 3$, the inclusion $\Lambda^t\widetilde{O_2(3)} \to \widetilde{O_2(3)}$, which is a cofibration by Corollary 6.2.3 since it is the identity on underlying categories. Moreover, it is a biequivalence, since it is bijective on objects and morphisms, and it is fully faithful on 2-morphisms, where fully faithfulness follows from the fact that there is a unique 2-isomorphism filling the triangle of the missing 2-isomorphism and it is given by the obvious composite of the three other 2-isomorphisms.
- for $n \geq 4$ and $0 \leq t \leq n$, the identity.

Therefore, the 2-functor $\overline{\mathbb{C}}(\ell^{\Delta}_{n,t})$ is a trivial cofibration in 2Cat, for all $n \geq 1$ and for all $0 \leq t \leq n$. ☐

We now prove the theorem saying that the nerve functor $\mathbb{N}\colon \mathrm{DblCat} \to \mathrm{DblCat}^h_\infty$ is right Quillen.

Theorem 11.2.7. *The adjunction*

$$\begin{tikzcd} \mathrm{DblCat} \arrow[r, bend right, "\mathbb{N}"'] \arrow[r, phantom, "\perp"] & \mathrm{DblCat}^h_\infty \arrow[l, bend right, "\mathbb{C}"'] \end{tikzcd}$$

is a Quillen pair between the model structure on DblCat *of Theorem 8.1.15 for weakly horizontally invariant double categories and the model structure on* $\mathrm{sSet}^{\Delta^{\mathrm{op}}\times\Delta^{\mathrm{op}}}$ *of Theorem 10.2.4 for horizontally complete double* $(\infty,1)$*-categories.*

Proof. By Theorem 5.2.23 and Proposition 11.2.1, it is enough to show that the cofibrations $g^F_k \times \mathrm{id}_{R[m]}$, $\mathrm{id}_{F[k]} \times q^R_m$, and $\mathrm{id}_{F[k]} \times e^R$, with respect to which we localize the Reedy/injective model structure on $\mathrm{sSet}^{\Delta^{\mathrm{op}}\times\Delta^{\mathrm{op}}}$ in order to obtain the model structure DblCat^h_∞ of Theorem 10.2.4, are sent by $\mathbb{C}$ to weak equivalences in DblCat. By definition of $\mathbb{C}$ and by Remark 11.1.9, we have that

$$\mathbb{C}(g^F_k \times \mathrm{id}_{R[m]}) \cong \mathbb{C}(g^F_k) \otimes \mathrm{id}_{\overline{\mathbb{C}}R[m]} = \mathbb{C}(g^F_k) \,\square_\otimes\, (\emptyset \to \overline{\mathbb{C}}R[m]),$$

and similarly that

$$\mathbb{C}(\mathrm{id}_{F[k]} \times q^R_m) \cong (\emptyset \to \mathbb{C}F[k]) \,\square_\otimes\, \overline{\mathbb{C}}(q^R_m), \quad \mathbb{C}(\mathrm{id}_{F[k]} \times e^R) \cong (\emptyset \to \mathbb{C}F[k]) \,\square_\otimes\, \overline{\mathbb{C}}(e^R).$$

Since $\mathbb{C}$ is left Quillen from the Reedy/injective model structure on $\mathrm{sSet}^{\Delta^{\mathrm{op}}\times\Delta^{\mathrm{op}}}$ in which every object is cofibrant, the unique morphisms $\emptyset \to \mathbb{C}R[m]$ and $\emptyset \to \mathbb{C}F[k]$ are cofibrations DblCat. Hence, the unique morphism $\emptyset \to \overline{\mathbb{C}}R[m]$ is also a cofibration in 2Cat since $\mathbb{C} = \mathbb{H}\overline{\mathbb{C}}$ and the functor $\mathbb{H}$ reflects cofibrations by Remark 8.4.4. Similarly, the morphisms

$\mathbb{C}(g_k^F)$, $\overline{\mathbb{C}}(q_m^R)$ and $\overline{\mathbb{C}}(e^R)$ are cofibrations in DblCat and 2Cat, since they are images of monomorphisms in $\mathrm{sSet}^{\Delta^{\mathrm{op}}\times\Delta^{\mathrm{op}}}$. As the model structure on DblCat is 2Cat-enriched by Remark 8.5.7, it is enough to show that $\mathbb{C}(g_k^F)$ is a weak equivalence in DblCat and that $\overline{\mathbb{C}}(q_m^R)$, and $\overline{\mathbb{C}}(e^R)$ are biequivalences. These statements are the content of Lemmas 11.2.8 and 11.2.9, respectively. □

The following two lemmas complete the proof of Theorem 11.2.7. Recall from Proposition 8.1.18 that double biequivalences, as defined in Definition 7.2.1, are in particular weak equivalences in the model structure on DblCat of Theorem 8.1.15.

Lemma 11.2.8. *For all $k \geq 0$, the double functor $\mathbb{C}(g_k^F)\colon \mathbb{C}(G[k]) \to \mathbb{C}(F[k])$ is a double biequivalence in* DblCat. *In particular, it is a weak equivalence in* DblCat.

Proof. Since $\mathbb{C}$ preserve colimits and $[k] = [1] \sqcup_{[0]} \ldots \sqcup_{[0]} [1]$, we have that

$$\mathbb{C}(G[k]) = \mathbb{V}[k] \quad \text{and} \quad \mathbb{C}(F[k]) = \mathbb{V}O_2^{\sim}(k),$$

for all $k \geq 0$. First note that, when $k = 0, 1$, the double functor $\overline{\mathbb{C}}(g_k^F)$ is an identity. For $k \geq 2$, let us give an example. When $k = 2$, the double functor $\mathbb{C}(g_2^F)$ is given by the inclusion

$$\begin{tikzcd}
0 \arrow[d, "\bullet" marking] & & 0 \arrow[r, equal] \arrow[dd, "\bullet" marking] & 0 \arrow[d, "\bullet" marking] \\
1 \arrow[d, "\bullet" marking] & \longrightarrow & \cong & 1 \arrow[d, "\bullet" marking] \\
2 & & 2 \arrow[r, equal] & 2 .
\end{tikzcd}$$

Having this example in mind, we can see that, for all $k \geq 0$, $\mathbb{C}(g_k^F)\colon \mathbb{V}[k] \to \mathbb{V}O_2^{\sim}(k)$ is the identity on objects and horizontal morphisms, and it is fully faithful on squares, since all squares in $\mathbb{V}[k]$ are trivial. The double functor $\mathbb{C}(g_k^F)$ is also injective on vertical morphisms. Moreover, since every vertical morphism $i \twoheadrightarrow j$ in $\mathbb{V}O_2^{\sim}(k)$ is related by a horizontally invertible square to the composite $i \twoheadrightarrow i+1 \twoheadrightarrow \ldots \twoheadrightarrow j$, then $\mathbb{C}(g_k^F)$ is essentially full on vertical morphisms. This shows that the double functor $\mathbb{C}(g_k^F)$ is a double biequivalence, for all $k \geq 0$. In particular, it is a weak equivalence in DblCat by Proposition 8.1.18. □

Lemma 11.2.9. *For all $m \geq 0$, the* 2*-functors*

$$\overline{\mathbb{C}}(q_m^R)\colon \overline{\mathbb{C}}(Q[m]) \to \overline{\mathbb{C}}(R[m]) \quad \textit{and} \quad \overline{\mathbb{C}}(e^R)\colon \overline{\mathbb{C}}(R[0]) \to \overline{\mathbb{C}}(N^R I)$$

are biequivalences in 2Cat.

Proof. We first show the result for $\overline{\mathbb{C}}(q_m^R)$. As in the proof of Lemma 11.2.8 and by Remark 11.1.9, we have that

$$\overline{\mathbb{C}}(Q[m]) = [m] \quad \text{and} \quad \overline{\mathbb{C}}(R[m]) = O_2^{\sim}(m),$$

for all $m \geq 0$. First note that, when $m = 0, 1$, the 2-functor $\overline{\mathbb{C}}(q_m^R)$ is an identity. For $m \geq 2$, let us give an example. When $m = 3$, the 2-functor $\overline{\mathbb{C}}(q_3^R)$ is given by the inclusion

$$\begin{tikzcd}
1 \arrow[r] & 2 \arrow[d] & & 1 \arrow[r] \arrow[dr, phantom, "\Nwarrow\cong"] & 2 \arrow[d] & & 1 \arrow[r] \arrow[dr] & 2 \arrow[d] \arrow[dl, phantom, "\Nearrow\cong"] \\
0 \arrow[u] & 3 & \longrightarrow & 0 \arrow[u] \arrow[r] \arrow[ur] & 3 \arrow[ul, phantom, "\cong\Uparrow"] & = & 0 \arrow[u] \arrow[r] \arrow[ur, phantom, "\Uparrow\cong"] & 3 .
\end{tikzcd}$$

Having this example in mind, we can see that, for all $m \geq 0$, $\overline{\mathbb{C}}(q_m^R)\colon [m] \to O_2^{\simeq}(m)$ is the identity on objects, and it is fully faithful on 2-morphisms, since all 2-morphisms in $[m]$ are trivial. The 2-functor $\overline{\mathbb{C}}(q_m^R)$ is also injective on morphisms. Moreover, since every morphism $i \to j$ in $O_2^{\simeq}(m)$ is related by a 2-isomorphism to the composite

$$i \to i+1 \to \ldots \to j,$$

then $\overline{\mathbb{C}}(q_m^R)$ is essentially full on morphisms. This shows that the 2-functor $\overline{\mathbb{C}}(q_m^R)$ is a biequivalence, for all $m \geq 0$.

It remains to show that $\overline{\mathbb{C}}(e^R)$ is a biequivalence. We have that $\overline{\mathbb{C}}(R[0]) = [0]$, and we compute $\overline{\mathbb{C}}(N^R I)$. Recall from Example 9.2.12 that m-simplices of the bisimplicial space $N^R I$ constant in the vertical and space directions are given by words of m letters in $\{x, y\}$. Since $\overline{\mathbb{C}}(N^R I)$ is obtained by gluing a copy of $O_2^{\simeq}(m)$ for each m-simplex of $N^R I$, we have that $\overline{\mathbb{C}}(N^R I)$ has

- two objects 0 and 1, given by the 0-simplices x and y,
- two non-trivial morphisms $f\colon 0 \to 1$ and $g\colon 1 \to 0$, given by the 1-simplices xy and yx,
- two non-trivial 2-isomorphisms $\eta\colon \mathrm{id}_x \cong gf$ and $\epsilon\colon \mathrm{id}_y \cong fg$, given by the 2-simplices xyx and yxy,

such that η and ϵ satisfy the triangle identities, expressed by the 3-simplices $yxyx$ and $xyxy$. Higher simplices of $N^R I$ do not add any relations. Therefore, the 2-category $\overline{\mathbb{C}}(N^R I) = E_{\mathrm{adj}}$ is the "free-living adjoint equivalence", and $\overline{\mathbb{C}}(e^R) = j_1\colon [0] \to E_{\mathrm{adj}}$ is a generating trivial cofibration in 2Cat. In particular, it is a biequivalence. □

11.3. **The nerve $\mathbb{N}$ is homotopically fully faithful.** We now prove that the Quillen pair $\mathbb{C} \dashv \mathbb{N}$ is a Quillen reflection which implies that the nerve functor is homotopically fully faithful. For this, we show that the derived counit of the adjunction $\mathbb{C} \dashv \mathbb{N}$ is level-wise a weak equivalence in DblCat. More precisely, we show that it is a trivial fibration as described in Proposition 8.1.2. Note that, since all objects are cofibrant in DblCat_∞^h, the derived counit coincides with the counit.

Theorem 11.3.1. *The components $\epsilon_{\mathbb{A}}\colon \mathbb{C}\mathbb{N}\mathbb{A} \to \mathbb{A}$ of the (derived) counit are trivial fibrations in* DblCat*, for all double categories $\mathbb{A}$. In particular, these are weak equivalences in* DblCat *and therefore the adjunction $\mathbb{C} \dashv \mathbb{N}$ is a Quillen reflection.*

Proof. Let $\mathbb{A}$ be a double category. We first compute the double category $\mathbb{C}\mathbb{N}\mathbb{A}$. By a formula for left Kan extensions, we have that

$$\mathbb{C}\mathbb{N}\mathbb{A} = \mathrm{colim}(\mathcal{Y} \downarrow \mathbb{N}\mathbb{A} \longrightarrow \Delta \times \Delta \times \Delta \xrightarrow{\mathbb{X}} \mathrm{DblCat}),$$

where $\mathcal{Y}\colon \Delta \times \Delta \times \Delta \to \mathrm{Set}^{(\Delta^{\mathrm{op}})^{\times 3}}$ denotes the Yoneda embedding and $\mathcal{Y} \downarrow \mathbb{N}\mathbb{A}$ is the slice category over $\mathbb{N}\mathbb{A}$. An object in $\mathcal{Y} \downarrow \mathbb{N}\mathbb{A}$ is a map $R[m] \times F[k] \times \Delta[n] \to \mathbb{N}\mathbb{A}$, or equivalently, by the adjunction $\mathbb{C} \dashv \mathbb{N}$, a double functor $(\mathbb{V}O_2^{\simeq}(k) \otimes O_2^{\simeq}(m)) \otimes \widetilde{O_2(n)} \to \mathbb{A}$. Therefore, for each double functor $(\mathbb{V}O_2^{\simeq}(k) \otimes O_2^{\simeq}(m)) \otimes \widetilde{O_2(n)} \to \mathbb{A}$, we glue a copy of $\mathbb{X}_{m,k,n} = (\mathbb{V}O_2^{\simeq}(k) \otimes O_2^{\simeq}(m)) \otimes \widetilde{O_2(n)}$ in $\mathbb{C}\mathbb{N}\mathbb{A}$.

The double category $\mathbb{C}\mathbb{N}\mathbb{A}$ is cofibrant, since every object in DblCat_∞^h is cofibrant and $\mathbb{C}$ is left Quillen. Therefore its underlying horizontal and vertical categories are free by Corollary 8.1.6 and it is enough to describe the generating morphisms. First note that $\mathbb{C}\mathbb{N}\mathbb{A}$ has the same objects as $\mathbb{A}$. The horizontal morphisms in $\mathbb{C}\mathbb{N}\mathbb{A}$ are freely generated by

- a horizontal morphism $\overline{f}\colon A \to B$, for each horizontal morphism f of $\mathbb{A}$,
- a horizontal morphism $\widetilde{f}_{(f,g,\eta,\epsilon)}\colon A \to B$ together with a horizontal morphism $\widetilde{g}_{(f,g,\eta,\epsilon)}\colon B \to A$, for each horizontal adjoint equivalence (f, g, η, ϵ) in $\mathbb{A}$.

where $\overline{\mathrm{id}_A}$, $\widetilde{f}_{(\mathrm{id}_A,\mathrm{id}_A,\mathrm{id}_{\mathrm{id}_A},\mathrm{id}_{\mathrm{id}_A})}$, and $\widetilde{g}_{(\mathrm{id}_A,\mathrm{id}_A,\mathrm{id}_{\mathrm{id}_A},\mathrm{id}_{\mathrm{id}_A})}$ are identified with the identity id_A at the object A of $\mathbb{CNA}$. The vertical morphisms in $\mathbb{CNA}$ are freely generated by a vertical morphism $\overline{u}\colon A \,{\bullet\!\!\to}\, A'$, for each vertical morphism u of $\mathbb{A}$, where $\overline{e_A}$ is identified with the identity e_A at the object A of $\mathbb{CNA}$. It remains to identify the squares of $\mathbb{CNA}$. They are generated by:

- vertically invertible squares $\widetilde{\eta}_{(f,g,\eta,\epsilon)}\colon (e_A \;{}^{\mathrm{id}_A}_{\widetilde{g}\,\widetilde{f}}\; e_A)$ and $\widetilde{\epsilon}_{(f,g,\eta,\epsilon)}\colon (e_B \;{}^{\widetilde{f}\,\widetilde{g}}_{\mathrm{id}_B}\; e_B)$ satisfying the triangle identities, for each horizontal adjoint equivalence (f,g,η,ϵ) in $\mathbb{A}$,
- a square $\overline{\alpha}\colon (\overline{u} \;{}^{\overline{f}}_{\overline{f'}}\; \overline{v})$, for each square α in $\mathbb{A}$,
- a square $\widetilde{\alpha}\colon (\overline{u} \;{}^{\widetilde{f}}_{\widetilde{f'}}\; \overline{v})$, for each square α in $\mathbb{A}$ whose horizontal boundaries are horizontal adjoint equivalences (f,g,η,ϵ) and (f',g',η',ϵ'),
- a vertically invertible square $\overline{\theta}_{f,k,g,h}\colon (e_A \;{}^{\widetilde{g}\,\overline{f}}_{\overline{k}\,\widetilde{h}}\; e_C)$, for each vertically invertible square θ in $\mathbb{A}$ as depicted below,

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, equal, "\bullet" marking] & B \arrow[r, "g", "\simeq"'] \arrow[d, phantom, "\theta\ \cong"] & C \arrow[d, equal, "\bullet" marking] \\ A \arrow[r, "\simeq", "h"'] & B' \arrow[r, "k"'] & C \end{tikzcd}$$

  where g and h are horizontal adjoint equivalences,
- a vertically invertible square $\overline{\varphi}_{f,g,h}\colon (e_A \;{}^{\overline{h}}_{\overline{g}\,\overline{f}}\; e_C)$, for each vertically invertible square φ in $\mathbb{A}$ as depicted below,

$$\begin{tikzcd} A \arrow[rr, "h"] \arrow[d, equal, "\bullet" marking] & \arrow[d, phantom, "\varphi\ \cong"] & C \arrow[d, equal, "\bullet" marking] \\ A \arrow[r, "f"'] & B \arrow[r, "g"'] & C \end{tikzcd}$$

- a vertically invertible square $\widetilde{\varphi}_{f,g,h}\colon (e_A \;{}^{\widetilde{h}}_{\widetilde{g}\,\widetilde{f}}\; e_C)$, for each vertically invertible square φ in $\mathbb{A}$ as above, but where the morphisms f, g, and h are all horizontal adjoint equivalences,
- a horizontally invertible square $\overline{\psi}_{u,v,w}\colon (\overline{w} \;{}^{\mathrm{id}_A}_{\mathrm{id}_{A''}}\; \overline{v}\,\overline{u})$, for each horizontally invertible square ψ in $\mathbb{A}$ as depicted below.

$$\begin{tikzcd} A \arrow[r, equal] \arrow[dd, "w"', "\bullet" marking] & A \arrow[d, "u", "\bullet" marking] \\ \arrow[r, phantom, "\psi\ \cong"] & A' \arrow[d, "v", "\bullet" marking] \\ A'' \arrow[r, equal] & A'' \end{tikzcd}$$

Furthermore, these squares are submitted to relations represented by double functors $(\mathbb{V}O_2^{\simeq}(k) \otimes O_2^{\simeq}(m)) \otimes \widetilde{O_2(n)} \to \mathbb{A}$, where $k+m+n \geq 3$. In particular, these relations hold for the squares that represent them in $\mathbb{A}$.

Then the double functor $\epsilon_{\mathbb{A}}\colon \mathbb{CNA} \to \mathbb{A}$ is given by the identity on objects and by sending each horizontal morphism, vertical morphism, and square in $\mathbb{CNA}$ to the horizontal morphism, vertical morphism, and square in $\mathbb{A}$ representing it. This defines a double functor since the underlying horizontal and vertical categories are free, and the relations on squares in $\mathbb{CNA}$ are satisfied by the squares representing them in $\mathbb{A}$. Moreover, it is straightforward to see that this double functor is surjective on objects, full on horizontal, and full on vertical morphisms. Fully faithfulness on squares follows from the fact that, given a boundary in $\mathbb{CNA}$, for each square in $\mathbb{A}$ in the representing boundary, we added a unique square, and the fact that the relations satisfied for squares in $\mathbb{A}$ are also satisfied in $\mathbb{CNA}$. $\square$

Remark 11.3.2. Since all objects are cofibrant in DblCat^h_∞ by Theorem 10.2.4, the functor $\mathbb{C}\colon \mathrm{DblCat}^h_\infty \to \mathrm{DblCat}$ preserves weak equivalences by Ken Brown's Lemma (see Lemma 4.4.5). Therefore, since the components $\epsilon_{\mathbb{A}}\colon \mathbb{CNA} \to \mathbb{A}$ of the counit are weak equivalences by Theorem 11.3.1, for all $\mathbb{A} \in \mathrm{DblCat}$, the nerve $\mathbb{N}\colon \mathrm{DblCat} \to \mathrm{DblCat}^h_\infty$ reflects weak equivalences by 2-out-of-3.

11.4. **Level of fibrancy of nerves of double categories.** The nerve of any double category is almost fibrant in the model structure DblCat^h_∞ of Theorem 10.2.4. Indeed, aside from the vertical Reedy/injective fibrancy condition, the nerve of a double category satisfies the conditions of a horizontally complete double $(\infty,1)$-category. As we will see, vertical Reedy/injective fibrancy for the nerve of a double category $\mathbb{A}$ is satisfied if and only if the double category $\mathbb{A}$ is weakly horizontally invariant.

Let us first summarize the properties that the nerve of a general double category satisfies.

Theorem 11.4.1. *The nerve of a double category $\mathbb{A}$ is such that*

(i) $(\mathbb{NA})_{-,k}\colon \Delta^{\mathrm{op}} \to \mathrm{sSet}$ *is Reedy/injective fibrant, for all $k \geq 0$,*
(ii) $(\mathbb{NA})_{m,-}\colon \Delta^{\mathrm{op}} \to \mathrm{sSet}$ *is satisfies the Segal condition, for all $m \geq 0$,*
(iii) $(\mathbb{NA})_{-,k}\colon \Delta^{\mathrm{op}} \to \mathrm{sSet}$ *is a complete Segal space, for all $k \geq 0$.*

To show this theorem we will need several technical results. The first piece is a Quillen pair between Lack's model structure on 2Cat and the Kan-Quillen model structure on sSet, whose left adjoint is given by the restriction of the functor $\overline{\mathbb{C}}\colon \mathrm{sSet}^{\Delta^{\mathrm{op}}\times\Delta^{\mathrm{op}}} \to 2\mathrm{Cat}$ to its space component.

Definition 11.4.2. We define the cosimplicial 2-category

$$\begin{aligned}\mathcal{X}\colon \Delta &\to 2\mathrm{Cat},\\ [n] &\mapsto \widetilde{O_2(n)}.\end{aligned}$$

Proposition 11.4.3. *The cosimplicial 2-category $\mathcal{X}$ induces an adjunction*

$$\begin{tikzcd} \Delta \arrow[r, "\mathcal{X}"] \arrow[d] & 2\mathrm{Cat} \arrow[dl, bend left, "\mathcal{N}"] \\ \mathrm{Set}^{\Delta^{\mathrm{op}}} \arrow[ur, "\mathcal{C}"] \arrow[r, phantom, "\perp"] & \end{tikzcd},$$

where $\mathcal{C}$ is the left Kan extension of $\mathcal{X}$ along the Yoneda embedding, and we have that

$$(\mathcal{N}\mathcal{A})_n \cong 2\mathrm{Cat}(\widetilde{O_2(n)}, \mathcal{A}),$$

for all $\mathcal{A} \in 2\mathrm{Cat}$ and all $n \geq 0$.

Proof. This is a direct application of Proposition 11.1.1, since 2Cat is locally presentable by Proposition 2.1.6. $\square$

This adjunction gives the desired Quillen pair between the model structures for 2-categories and for Kan complexes.

Proposition 11.4.4. *The adjunction*

$$\begin{tikzcd} 2\mathrm{Cat} \arrow[r, bend right, "\mathcal{N}"'] \arrow[r, phantom, "\perp"] & \mathrm{sSet} \arrow[l, bend right, "\mathcal{C}"'] \end{tikzcd}$$

is a Quillen pair between Lack's model structure on 2Cat *of Theorem 6.1.8 and the Kan-Quillen model structure on* sSet *of Theorem 5.2.7.*

Proof. By Remark 4.4.4, it is enough to show that $\mathcal{C}$ sends generating cofibrations and generating trivial cofibrations in sSet to cofibrations and trivial cofibrations in 2Cat, respectively. Recall from Theorem 5.2.7 that generating cofibrations and trivial cofibrations in sSet are given by the maps $\iota_n^\Delta\colon \delta\Delta[n] \to \Delta[n]$, for $n \geq 0$, and $\ell_{n,t}^\Delta\colon \Lambda^t[n] \to \Delta[n]$, for $n \geq 1$ and $0 \leq t \leq n$, respectively. Note that we have $\mathcal{C}(\iota_n^\Delta) = \overline{\mathbb{C}}(\iota_n^\Delta)$ and $\mathcal{C}(\ell_{n,t}^\Delta) = \overline{\mathbb{C}}(\ell_{n,t}^\Delta)$. Therefore, by Lemmas 11.2.5 and 11.2.6, we see that these are cofibrations and trivial cofibrations in 2Cat, respectively. This shows the result. $\square$

We will reformulate conditions (i-iii) of Theorem 11.4.1, which are for now given in terms of weak equivalences between mapping spaces, using the right Quillen functor $\mathcal{N}$ of the above proposition. This can be done by applying the following lemma.

Lemma 11.4.5. *Let $X \in \mathrm{sSet}^{\Delta^{\mathrm{op}}\times\Delta^{\mathrm{op}}}$ be a trisimplicial set which is constant in the space direction. Then, for every double category $\mathbb{A}$, we have an isomorphism of simplicial sets*

$$\mathrm{Map}(X, \mathbb{N}\mathbb{A}) \cong \mathcal{N}(\mathbf{H}[\mathbb{C}(X), \mathbb{A}]_{\mathrm{ps}})$$

natural in X and $\mathbb{A}$.

Proof. For all $n \geq 0$, we have isomorphisms of sets

$$\begin{aligned} \mathrm{Map}(X, \mathbb{N}\mathbb{A})_n &\cong \mathrm{sSet}^{\Delta^{\mathrm{op}}\times\Delta^{\mathrm{op}}}(X \times \Delta[n], \mathbb{N}\mathbb{A}) \\ &\cong \mathrm{DblCat}(\mathbb{C}(X \times \Delta[n]), \mathbb{A}) \\ &\cong \mathrm{DblCat}(\mathbb{C}(X) \otimes \widetilde{O_2(n)}, \mathbb{A}) \\ &\cong 2\mathrm{Cat}(\widetilde{O_2(n)}, \mathbf{H}[\mathbb{C}(X), \mathbb{A}]_{\mathrm{ps}}) \\ &\cong \mathcal{N}(\mathbf{H}[\mathbb{C}(X), \mathbb{A}]_{\mathrm{ps}})_n \end{aligned}$$

natural in n, X, and $\mathbb{A}$, where the first isomorphism holds by definition of the mapping space (see Proposition 10.1.5), the second by the adjunction $\mathbb{C} \dashv \mathbb{N}$, the third by definition of $\mathbb{C}$ and the fact that X is constant in the space direction, the fourth by the universal property of $\otimes$ (see Proposition 3.5.2), and the last isomorphism by Proposition 11.4.3. In particular, these isomorphisms of sets assemble into an isomorphism $\mathrm{Map}(X, \mathbb{N}\mathbb{A}) \cong \mathcal{N}(\mathbf{H}[\mathbb{C}(X), \mathbb{A}]_{\mathrm{ps}})$ of simplicial sets, which is natural in X and $\mathbb{A}$. $\square$

We now prove Theorem 11.4.1 assuming Lemmas 11.4.6 and 11.4.7 below.

Proof of Theorem 11.4.1. Let $\mathbb{A}$ be a double category. By Lemmas 11.4.6 and 11.4.7, the 2-functor $\mathbf{H}[\mathbb{C}(\mathrm{id}_{F[k]} \times \iota_m^R), \mathbb{A}]_{\mathrm{ps}}$ is a fibration in 2Cat, and the 2-functors

$$\mathbf{H}[\mathbb{C}(\mathrm{id}_{F[k]} \times q_m^R), \mathbb{A}]_{\mathrm{ps}}, \quad \mathbf{H}[\mathbb{C}(\mathrm{id}_{F[k]} \times e^R), \mathbb{A}]_{\mathrm{ps}} \text{ and } \mathbf{H}[\mathbb{C}(g_k^F \times \mathrm{id}_{R[m]}), \mathbb{A}]_{\mathrm{ps}}$$

are trivial fibrations in 2Cat, for all $m, k \geq 0$. As $\mathcal{N}\colon 2\mathrm{Cat} \to \mathrm{sSet}$ is right Quillen by Proposition 11.4.4, these are sent by $\mathcal{N}$ to fibrations and trivial fibrations in sSet, respectively. As the map $\mathrm{id}_{F[k]} \times \iota_m^R$ is constant in the space direction, by Lemma 11.4.5, we have that

$$\mathrm{Map}(\mathrm{id}_{F[k]} \times \iota_m^R, \mathbb{N}\mathbb{A}) \cong \mathcal{N}(\mathbf{H}[\mathbb{C}(\mathrm{id}_{F[k]} \times \iota_m^R), \mathbb{A}]_{\mathrm{ps}}).$$

By the above arguments, this is a fibration in sSet, for all $m, k \geq 0$, which shows (i) saying that $(\mathbb{N}\mathbb{A})_{-,k}$ is Reedy/injective fibrant. Similarly, we have that

$$\mathrm{Map}(\mathrm{id}_{F[k]} \times q_m^R, \mathbb{N}\mathbb{A}) \cong \mathcal{N}(\mathbf{H}[\mathbb{C}(\mathrm{id}_{F[k]} \times q_m^R), \mathbb{A}]_{\mathrm{ps}}),$$

$$\mathrm{Map}(\mathrm{id}_{F[k]} \times e^R, \mathbb{N}\mathbb{A}) \cong \mathcal{N}(\mathbf{H}[\mathbb{C}(\mathrm{id}_{F[k]} \times e^R), \mathbb{A}]_{\mathrm{ps}}),$$
$$\mathrm{Map}(g_k^F \times \mathrm{id}_{R[m]}, \mathbb{N}\mathbb{A}) \cong \mathcal{N}(\mathbf{H}[\mathbb{C}(g_k^F \times \mathrm{id}_{R[m]}), \mathbb{A}]_{\mathrm{ps}}),$$

and these are trivial fibrations in sSet by the above arguments, for all $m, k \geq 0$. The fact that $\mathrm{Map}(\mathrm{id}_{F[k]} \times q_m^R, \mathbb{N}\mathbb{A})$ and $\mathrm{Map}(\mathrm{id}_{F[k]} \times e^R, \mathbb{N}\mathbb{A})$ are in particular weak equivalences in sSet shows that (iii) holds, i.e., we have the Segal and completeness conditions for $(\mathbb{N}\mathbb{A})_{-,k}$, and the fact that $\mathrm{Map}(g_k^F \times \mathrm{id}_{R[m]}, \mathbb{N}\mathbb{A})$ is in particular a weak equivalence in sSet gives (ii), i.e., the Segal condition for $(\mathbb{N}\mathbb{A})_{m,-}$. ☐

The following two lemmas complete the proof of Theorem 11.4.1.

Lemma 11.4.6. *Let $\mathbb{A}$ be a double category. The 2-functor $\mathbf{H}[\mathbb{C}(\mathrm{id}_{F[k]} \times \iota_m^R), \mathbb{A}]_{\mathrm{ps}}$ is a fibration in 2Cat, and the 2-functors $\mathbf{H}[\mathbb{C}(\mathrm{id}_{F[k]} \times q_m^R), \mathbb{A}]_{\mathrm{ps}}$ and $\mathbf{H}[\mathbb{C}(\mathrm{id}_{F[k]} \times e^R), \mathbb{A}]_{\mathrm{ps}}$ are trivial fibrations in 2Cat, for all $m, k \geq 0$.*

Proof. By the universal property of the tensor $\otimes$ in Proposition 3.5.2, we get isomorphisms of 2-categories as in the following commutative square.

$$\begin{tikzcd}
\mathbf{H}[\mathbb{V}O_2^{\sim}(k) \otimes O_2^{\sim}(m), \mathbb{A}]_{\mathrm{ps}} \arrow[r, "{\mathbf{H}[\mathbb{C}(\mathrm{id}_{F[k]} \times \iota_m^R), \mathbb{A}]_{\mathrm{ps}}}"] \arrow[d, "\cong"'] & \mathbf{H}[\mathbb{V}O_2^{\sim}(k) \otimes \delta O_2^{\sim}(m), \mathbb{A}]_{\mathrm{ps}} \arrow[d, "\cong"] \\
{[O_2^{\sim}(m), \mathbf{H}[\mathbb{V}O_2^{\sim}(k), \mathbb{A}]_{\mathrm{ps}}]_{2,\mathrm{ps}}} \arrow[r, "{[\overline{\mathbb{C}}(\iota_m^R), \mathbf{H}[\mathbb{V}O_2^{\sim}(k), \mathbb{A}]_{\mathrm{ps}}]_{2,\mathrm{ps}}}"'] & {[\delta O_2^{\sim}(m), \mathbf{H}[\mathbb{V}O_2^{\sim}(k), \mathbb{A}]_{\mathrm{ps}}]_{2,\mathrm{ps}}}
\end{tikzcd}$$

As every 2-category is fibrant and $\overline{\mathbb{C}}(\iota_m^R)$ is a cofibration in 2Cat by Lemma 11.2.5, the 2-functor $[\overline{\mathbb{C}}(\iota_m^R), \mathbf{H}[\mathbb{V}O_2^{\sim}(k), \mathbb{A}]_{\mathrm{ps}}]_{2,\mathrm{ps}}$ is a fibration in 2Cat by monoidality of Lack's model structure (see Theorem 6.3.5). Hence $\mathbf{H}[\mathbb{C}(\mathrm{id}_{F[k]} \times \iota_m^R), \mathbb{A}]_{\mathrm{ps}}$ is also a fibration in 2Cat.

Similarly, we have isomorphisms $\mathbf{H}[\mathbb{C}(\mathrm{id}_{F[k]} \times q_m^R), \mathbb{A}]_{\mathrm{ps}} \cong [\overline{\mathbb{C}}(q_m^R), \mathbf{H}[\mathbb{V}O_2^{\sim}(k), \mathbb{A}]_{\mathrm{ps}}]_{2,\mathrm{ps}}$ and $\mathbf{H}[\mathbb{C}(\mathrm{id}_{F[k]} \times e^R), \mathbb{A}]_{\mathrm{ps}} \cong [\overline{\mathbb{C}}(e^R), \mathbf{H}[\mathbb{V}O_2^{\sim}(k), \mathbb{A}]_{\mathrm{ps}}]_{2,\mathrm{ps}}$. By Lemma 11.2.9 and since $\overline{\mathbb{C}}$ preserves cofibrations, the 2-functors $\overline{\mathbb{C}}(q_m^R)$ and $\overline{\mathbb{C}}(e^R)$ are trivial cofibrations in 2Cat. Therefore, by monoidality of Lack's model structure, the 2-functors

$$[\overline{\mathbb{C}}(q_m^R), \mathbf{H}[\mathbb{V}O_2^{\sim}(k), \mathbb{A}]_{\mathrm{ps}}]_{2,\mathrm{ps}} \quad \text{and} \quad [\overline{\mathbb{C}}(e^R), \mathbf{H}[\mathbb{V}O_2^{\sim}(k), \mathbb{A}]_{\mathrm{ps}}]_{2,\mathrm{ps}}$$

are trivial fibrations in 2Cat and hence so are

$$\mathbf{H}[\mathbb{C}(\mathrm{id}_{F[k]} \times q_m^R), \mathbb{A}]_{\mathrm{ps}} \quad \text{and} \quad \mathbf{H}[\mathbb{C}(\mathrm{id}_{F[k]} \times e^R), \mathbb{A}]_{\mathrm{ps}}. \qquad ☐$$

For the last piece for the proof of Theorem 11.4.1, we refer the reader to Definitions 3.3.1 and 3.3.3 for a description of the data of the underlying horizontal 2-category $\mathbf{H}[-,-]_{\mathrm{ps}}$ of double functors, horizontal pseudo-natural transformations, and modifications with trivial vertical boundaries.

Lemma 11.4.7. *Let $\mathbb{A}$ be a double category. The 2-functor $\mathbf{H}[\mathbb{C}(g_k^F \times \mathrm{id}_{R[m]}), \mathbb{A}]_{\mathrm{ps}}$ is a trivial fibration in 2Cat, for all $m, k \geq 0$.*

Proof. By the isomorphisms of double categories in Proposition 3.3.5 introducing the Gray tensor $\otimes_{\mathrm{Gr}}$ and by applying $\mathbf{H}$ to these isomorphisms, we get isomorphisms of 2-categories as in the following commutative square.

$$\begin{tikzcd}
\mathbf{H}[\mathbb{V}O_2^{\sim}(k) \otimes O_2^{\sim}(m), \mathbb{A}]_{\mathrm{ps}} \arrow[r, "{\mathbf{H}[\mathbb{C}(g_k^F \times \mathrm{id}_{R[m]}), \mathbb{A}]_{\mathrm{ps}}}"] \arrow[d, "\cong"'] & \mathbf{H}[\mathbb{V}[k] \otimes O_2^{\sim}(m), \mathbb{A}]_{\mathrm{ps}} \arrow[d, "\cong"] \\
\mathbf{H}[\mathbb{V}O_2^{\sim}(k), [\mathbb{H}O_2^{\sim}(m), \mathbb{A}]_{\mathrm{ps}}]_{\mathrm{ps}} \arrow[r, "{\mathbf{H}[\mathbb{C}(g_k^F), [\mathbb{H}O_2^{\sim}(m), \mathbb{A}]_{\mathrm{ps}}]_{\mathrm{ps}}}"'] & \mathbf{H}[\mathbb{V}[k], [\mathbb{H}O_2^{\sim}(m), \mathbb{A}]_{\mathrm{ps}}]_{\mathrm{ps}}
\end{tikzcd}$$

To see that the 2-functor $\mathbf{H}[\mathbb{C}(g_k^F \times \mathrm{id}_{R[m]}), \mathbb{A}]_{\mathrm{ps}}$ is a trivial fibration in 2Cat, it is enough to show that the 2-functor

$$\mathbf{H}[\mathbb{C}(g_k^F), \mathbb{B}]_{\mathrm{ps}} \colon \mathbf{H}[\mathbb{V}O_2^{\sim}(k), \mathbb{B}]_{\mathrm{ps}} \to \mathbf{H}[\mathbb{V}[k], \mathbb{B}]_{\mathrm{ps}}$$

is a trivial fibration in 2Cat, for any $\mathbb{B} \in$ DblCat. Then, by applying this result to $\mathbb{B} = [\mathbb{H}O_2^{\sim}(m), \mathbb{A}]_{\mathrm{ps}}$, we get that $\mathbf{H}[\mathbb{C}(g_k^F), [\mathbb{H}O_2^{\sim}(m), \mathbb{A}]_{\mathrm{ps}}]_{\mathrm{ps}}$ is a trivial fibration in 2Cat. Then, by the commutative square above, we get that $\mathbf{H}[\mathbb{C}(g_k^F \times \mathrm{id}_{R[m]}), \mathbb{A}]_{\mathrm{ps}}$ is also a trivial fibration in 2Cat.

We first describe the double functor $\mathbb{C}(g_k^F) \colon \mathbb{V}[k] \to \mathbb{V}O_2^{\sim}(k)$ on objects and vertical morphisms. Since the horizontal morphisms and squares of $\mathbb{V}[k]$ are all trivial, this describes the image of $\mathbb{C}(g_k^F)$ completely. We denote by $u_i \colon i \to\!\!\!\!\!\bullet\ i+1$, for $0 \leq i < k$, the vertical morphisms generating the double category $\mathbb{V}[k]$. Then the double functor $\mathbb{C}(g_k^F)$ is the identity on objects and sends a generating vertical morphism $u_i \colon i \to\!\!\!\!\!\bullet\ i+1$ of $\mathbb{V}[k]$ to the vertical morphism $i \to\!\!\!\!\!\bullet\ i+1$ of $\mathbb{V}O_2^{\sim}(k)$ represented by $\{i, i+1\}$.

Now let $\mathbb{B}$ be a double category. We show that the 2-functor $\mathbf{H}[\mathbb{C}(g_k^F), \mathbb{B}]_{\mathrm{ps}}$ is a trivial fibration in 2Cat, by verifying that it is surjective on objects, full on morphisms, and fully faithful on 2-morphisms.

Given a double functor $F \colon \mathbb{V}[k] \to \mathbb{B}$, consider the composite

$$\mathbb{V}O_2^{\sim}(k) \xrightarrow{\mathbb{V}\pi} \mathbb{V}[k] \xrightarrow{F} \mathbb{B},$$

where $\pi \colon O_2^{\sim}(k) \to [k]$ is the identity on objects and acts on hom-categories as the unique functor $O_2^{\sim}(k)(i,j) \to [k](i,j) = [0]$. The composite above is a double functor in $\mathbf{H}[\mathbb{V}O_2^{\sim}(k), \mathbb{B}]$ such that $F(\mathbb{V}\pi)\mathbb{C}(g_k^F) = F$, which proves surjectivity on objects.

Let $F, G \colon \mathbb{V}O_2^{\sim}(k) \to \mathbb{B}$ be double functors, and $\varphi \colon F\mathbb{C}(g_k^F) \Rightarrow G\mathbb{C}(g_k^F)$ be a horizontal pseudo-natural transformation in $\mathbf{H}[\mathbb{V}[k], \mathbb{B}]_{\mathrm{ps}}$. We want to define a horizontal pseudo-natural transformation $\overline{\varphi} \colon F \Rightarrow G$ in $\mathbf{H}[\mathbb{V}O_2^{\sim}(k), \mathbb{B}]_{\mathrm{ps}}$ such that $\overline{\varphi}\mathbb{C}(g_k^F) = \varphi$. By (hn2) of Definition 3.2.1, it is enough to define $\overline{\varphi}$ on the generating vertical morphisms of $\mathbb{V}O_2^{\sim}(k)$ which are represented by $\{i,j\}$ for $i < j$. When $j = i+1$, we set $\overline{\varphi}_{\{i,i+1\}} := \varphi_{u_i}$. For $j > i+1$, let θ denote the unique horizontally invertible square in $\mathbb{V}O_2^{\sim}(k)$ from the vertical morphism represented by $\{i,j\}$ to the vertical composite of morphisms represented by $[i,j] = \{l \mid i \leq l \leq j\}$. Then there is a unique way of defining $\overline{\varphi}_{\{i,j\}}$ so that $\overline{\varphi}$ is natural; namely as follows.

$$
\begin{tikzcd}
Fi \arrow[r] \arrow[d, "F_{\{i,j\}}"'] \arrow[dr, phantom, "\overline{\varphi}_{\{i,j\}}"] & Gi \arrow[d, "G_{\{i,j\}}"] \\
Fj \arrow[r] & Gj
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
Fi \arrow[r, equal] \arrow[dddd, "F_{\{i,j\}}"'] \arrow[ddddr, phantom, "F\theta \atop \cong"] & Fi \arrow[r] \arrow[d, "F_{\{i,i+1\}}"'] \arrow[dr, phantom, "\varphi_{u_i}"] & Gi \arrow[r, equal] \arrow[d, "G_{\{i,i+1\}}"] \arrow[ddddr, phantom, "(G\theta)^{-1} \atop \cong"] & Gi \arrow[dddd, "G_{\{i,j\}}"] \\
& F(i+1) \arrow[r] \arrow[d] \arrow[dr, phantom, "\varphi_{u_{i+1}}"] & G(i+1) \arrow[d] & \\
& \vdots \arrow[dr, phantom, "\vdots"] & \vdots & \\
& \bullet \arrow[d] \arrow[dr, phantom, "\varphi_{u_{j-1}}"] & \bullet \arrow[d] & \\
Fj \arrow[r, equal] & Fj \arrow[r] & Gj \arrow[r, equal] & Gj
\end{tikzcd}
$$

This defines a horizontal pseudo-natural transformation $\overline{\varphi} \colon F \Rightarrow G$ which maps to φ via $\mathbf{H}[\mathbb{C}(g_k^F), \mathbb{B}]_{\mathrm{ps}}$, and hence shows fullness on morphisms.

Let $\overline{\varphi}, \overline{\psi} \colon F \Rightarrow G$ be horizontal pseudo-natural transformations in $\mathbf{H}[\mathbb{V}O_2^{\sim}(k), \mathbb{B}]_{\mathrm{ps}}$, and let $\mu \colon \varphi := \overline{\varphi}\mathbb{C}(g_k^F) \Rrightarrow \psi := \overline{\psi}\mathbb{C}(g_k^F)$ be a 2-morphism in $\mathbf{H}[\mathbb{V}[k], \mathbb{B}]_{\mathrm{ps}}$, i.e., a modification with trivial vertical boundaries. The modification μ comprises the data of squares $\mu_i \colon (e_{Fi} \ {}^{\varphi_i}_{\psi_i} \ e_{Gi})$, for $0 \leq i \leq k$, natural with respect to the square components of φ

and ψ. By the relations between the square components of $\overline{\varphi}$ and φ, and the ones of $\overline{\psi}$ and ψ as indicated in the pasting equality above, one can show that the squares μ_i of μ are also natural with respect to the square components of $\overline{\varphi}$ and $\overline{\psi}$. Therefore μ also defines a 2-morphism $\mu\colon \overline{\varphi} \Rrightarrow \overline{\psi}$ in $\mathbf{H}[\mathbb{V}O_2^{\simeq}(k),\mathbb{B}]_{\mathrm{ps}}$. As it is the unique such 2-morphism in $\mathbf{H}[\mathbb{V}O_2^{\simeq}(k),\mathbb{B}]_{\mathrm{ps}}$ that maps to μ via $\mathbf{H}[\mathbb{C}(g_k^F),\mathbb{B}]_{\mathrm{ps}}$, this shows fully faithfulness on 2-morphisms. $\square$

Finally, we show that the nerve of a double category satisfies the missing condition of a horizontally complete double $(\infty,1)$-category in the list of Theorem 11.4.1, namely the Reedy/injective fibrancy in the vertical direction, precisely when the double category is weakly horizontally invariant. Recall that the weakly horizontally invariant double categories are the fibrant objects in the model structure on DblCat of Theorem 8.1.15.

Theorem 11.4.8. *The nerve of a double category $\mathbb{A}$ is such that $(\mathbb{N}\mathbb{A})_{m,-}\colon \Delta^{\mathrm{op}} \to \mathrm{sSet}$ is Reedy/injective fibrant, for all $m \geq 0$, if and only if the double category $\mathbb{A}$ is weakly horizontally invariant.*

Proof. Let $\mathbb{A}$ be a double category. Suppose that $\mathbb{A}$ is weakly horizontally invariant, then $\mathbb{N}\mathbb{A}$ is a horizontally complete double $(\infty,1)$-category since $\mathbb{N}\colon \mathrm{DblCat} \to \mathrm{DblCat}_\infty^h$ is right Quillen. In particular, this says that $(\mathbb{N}\mathbb{A})_{m,-}\colon \Delta^{\mathrm{op}} \to \mathrm{sSet}$ is Reedy/injective fibrant, for all $m \geq 0$.

Conversely, suppose that $(\mathbb{N}\mathbb{A})_{m,-}\colon \Delta^{\mathrm{op}} \to \mathrm{sSet}$ is Reedy/injective fibrant, for all $m \geq 0$. Then $(\mathbb{N}\mathbb{A})_{0,-}$ is Reedy/injective fibrant and therefore the map

$$(\iota_1^F)^*\colon (\mathbb{N}\mathbb{A})_{0,1} \cong \mathrm{Map}(F[1],\mathbb{N}\mathbb{A}) \to \mathrm{Map}(\delta F[1],\mathbb{N}\mathbb{A}) \cong (\mathbb{N}\mathbb{A})_{0,0} \times (\mathbb{N}\mathbb{A})_{0,0}.$$

is a fibration in sSet. In particular, it has the right lifting property with respect to $\ell_{1,1}^\Delta\colon \Delta[0] \to \Delta[1]$, i.e., there is a lift in every commutative diagram as below.

$$\begin{tikzcd} \Delta[0] \arrow[r, "v"] \arrow[d, "\ell_{1,1}^\Delta"'] & (\mathbb{N}\mathbb{A})_{0,1} \arrow[d, "(\iota_1^F)^*"] \\ \Delta[1] \arrow[r, "{(f,f')}"'] \arrow[ur, dashed] & (\mathbb{N}\mathbb{A})_{0,0} \times (\mathbb{N}\mathbb{A})_{0,0} \end{tikzcd}$$

By Descriptions 13.1.2 and 13.1.4, the upper map v is the data of a vertical morphism $v\colon B \twoheadrightarrow B'$ in $\mathbb{A}$, while the bottom map (f,f') is the data of a pair of horizontal adjoint equivalences $(f\colon A \xrightarrow{\simeq} B, f'\colon A' \xrightarrow{\simeq} B')$ in $\mathbb{A}$. Therefore, the existence of a lift in each diagram as above corresponds to the existence of a weakly horizontally invertible square in $\mathbb{A}$ of the form

$$\begin{tikzcd} A \arrow[r, "f", "\simeq"'] \arrow[d, "u"'] \arrow[dr, phantom, "\simeq"] & B \arrow[d, "v"] \\ A' \arrow[r, "\simeq", "f'"'] & B' \end{tikzcd},$$

for each such data (v,f,f'). In other words, this says that $\mathbb{A}$ is weakly horizontally invariant. $\square$

Remark 11.4.9. In particular, since a horizontal double category is not generally weakly horizontally invariant (see Remark 8.4.5), the nerve $\mathbb{N}\mathbb{H}\mathcal{A}$ of a 2-category $\mathcal{A}$ is not generally fibrant in DblCat_∞^h. Since every 2-category is fibrant in Lack's model structure on 2Cat, this shows that the composite $\mathbb{N}\mathbb{H}$ is not right Quillen from 2Cat to DblCat_∞^h. Therefore, we will need to define the nerve for 2-categories differently in the next section.

12. Nerve of 2-categories

As 2-categories are horizontally embedded in double categories, we hope that the nerve functor $\mathbb{N}\colon \mathrm{DblCat} \to \mathrm{DblCat}^h_\infty$ restricts to a nerve functor $2\mathrm{Cat} \to 2\mathrm{CSS}$. Since the nerve of a double category $\mathbb{H}\mathcal{A}$ associated to a 2-category $\mathcal{A}$ is not generally fibrant, as explained in Remark 11.4.9, we need to define the nerve of a 2-category as the nerve of the fibrant replacement of $\mathbb{H}\mathcal{A}$ given by $\mathbb{H}^{\simeq}\mathcal{A}$ in DblCat; see Theorem 8.4.9. In Section 12.1, we show that the composite of the Quillen pairs $L^{\simeq} \dashv \mathbb{H}^{\simeq}$ and $\mathbb{C} \dashv \mathbb{N}$ restrict to a Quillen pair between 2Cat and 2CSS. The (derived) counit of the composite of these adjunctions is also level-wise a biequivalence, and we get a homotopically full embedding of 2Cat into 2CSS. As all objects are fibrant in 2Cat, the nerve $\mathbb{N}\mathbb{H}^{\simeq}$ preserves weak equivalences, and we can further show in Section 12.2 that Lack's model on 2Cat is right-induced from 2CSS along $\mathbb{N}\mathbb{H}^{\simeq}$. In particular, as the weak equivalences and fibrations are determined through their images under $\mathbb{N}\mathbb{H}^{\simeq}$, this says that the homotopy theory of 2-categories is created by that of 2-fold complete Segal spaces. In Section 12.3, we compare the nerve of the double categories $\mathbb{H}\mathcal{A}$ and $\mathbb{H}^{\simeq}\mathcal{A}$, by showing that the nerve of the latter is a fibrant replacement of the nerve of the former in 2CSS, and hence also in DblCat^h_∞.

12.1. **The nerve $\mathbb{N}\mathbb{H}^{\simeq}$ is right Quillen and homotopically fully faithful.** We consider the composite of the Quillen pairs

$$\begin{tikzcd} 2\mathrm{Cat} \arrow[r, bend right, "\mathbb{H}^{\simeq}"'] \arrow[r, phantom, "\perp"] & \mathrm{DblCat} \arrow[l, bend right, "L^{\simeq}"'] \arrow[r, bend right, "\mathbb{N}"'] \arrow[r, phantom, "\perp"] & \mathrm{DblCat}^h_\infty \arrow[l, bend right, "\mathbb{C}"'] \end{tikzcd},$$

and show that this gives a Quillen pair between Lack's model structure on 2Cat and the model structure 2CSS on $\mathrm{sSet}^{\Delta^{\mathrm{op}}\times\Delta^{\mathrm{op}}}$ for 2-fold complete Segal spaces. Since this latter is obtained as a left Bousfield localization of DblCat^h_∞ by Theorem 10.3.3, we can again apply Theorem 5.2.23.

Theorem 12.1.1. *The adjunction*

$$\begin{tikzcd} 2\mathrm{Cat} \arrow[r, bend right, "\mathbb{N}\mathbb{H}^{\simeq}"'] \arrow[r, phantom, "\perp"] & 2\mathrm{CSS} \arrow[l, bend right, "L^{\simeq}\mathbb{C}"'] \end{tikzcd}$$

is a Quillen pair between Lack's model structure on 2Cat *and the model structure on* $\mathrm{sSet}^{\Delta^{\mathrm{op}}\times\Delta^{\mathrm{op}}}$ *for* 2*-fold complete Segal spaces, i.e.,* $(\infty, 2)$*-categories.*

Remark 12.1.2. Note that the functor $L^{\simeq}\colon \mathrm{DblCat} \to 2\mathrm{Cat}$ does not preserve tensors. For example, the 2-category $L^{\simeq}(\mathbb{V}[1] \otimes [1])$ is generated by a non-invertible 2-morphism as below left, while the 2-category $L^{\simeq}(\mathbb{V}[1]) \otimes_2 [1]$ is generated by a 2-isomorphism as below right.

$$\begin{tikzcd} 0 \arrow[r] \arrow[d, "\simeq"'] \arrow[dr, phantom, "\swArrow"] & 1 \arrow[d, "\simeq"] \\ 0' \arrow[r] & 1' \end{tikzcd} \qquad \begin{tikzcd} 0 \arrow[r] \arrow[d, "\simeq"'] \arrow[dr, phantom, "\overset{\cong}{\swArrow}"] & 1 \arrow[d, "\simeq"] \\ 0' \arrow[r] & 1' \end{tikzcd}$$

However, the fact that the left-hand 2-morphism is not invertible in a square coming from a pair of a vertical morphism and a horizontal morphism is the only difference between $L^{\simeq}(-\otimes -)$ and $L^{\simeq}(-) \otimes_2 L^{\simeq}(-)$.

Proof. First note that the adjunction $L^{\simeq}\mathbb{C} \dashv \mathbb{N}\mathbb{H}^{\simeq}$ is a Quillen pair between 2Cat and DblCat^h_∞, since it is a composite of the two Quillen pairs $L^{\simeq} \dashv \mathbb{H}^{\simeq}$ of Theorem 8.4.7 and $\mathbb{C} \dashv \mathbb{N}$ of Theorem 11.2.7. By Theorem 5.2.23, it is enough to show that the functor $L^{\simeq}\mathbb{C}$

sends the cofibrations $e^F \times \mathrm{id}_{R[m]}$ and c_k, with respect to which we localize DblCat^h_∞ to obtain 2CSS in Theorem 10.3.3, to weak equivalences in 2Cat.

We first show that $L^{\simeq}\mathbb{C}(e^F \times \mathrm{id}_{R[m]})$ is a biequivalence. By a similar computation to the one of $\mathbb{C}(N^R I)$ in the proof of Lemma 11.2.9, we obtain that

$$L^{\simeq}\mathbb{C}(N^F I \times R[m]) \cong L^{\simeq}(\mathbb{V}\widetilde{O_2(k)} \otimes O_2^{\simeq}(m)).$$

Then the squares in the tensor $\mathbb{V}\widetilde{O_2(k)} \otimes O_2^{\simeq}(m)$ induced from vertical morphisms in $\mathbb{V}\widetilde{O_2(k)}$ and morphisms in $O_2^{\simeq}(m)$ must be weakly vertically invertible, since all vertical morphisms in $\mathbb{V}\widetilde{O_2(k)}$ are vertical equivalences, and these correspond to 2-isomorphisms in $L^{\simeq}(\mathbb{V}\widetilde{O_2(k)} \otimes O_2^{\simeq}(m))$, by a dual version of Lemma 3.6.9. By Remark 12.1.2, we deduce that $L^{\simeq}$ preserves this tensor:

$$L^{\simeq}(\mathbb{V}\widetilde{O_2(k)} \otimes O_2^{\simeq}(m)) \cong \widetilde{O_2(k)} \otimes_2 O_2^{\simeq}(m) \cong L^{\simeq}\mathbb{C}(N^F I) \otimes_2 L^{\simeq}\mathbb{C}(R[m]).$$

Therefore, $L^{\simeq}\mathbb{C}(e^F \times \mathrm{id}_{R[m]}) \cong L^{\simeq}\mathbb{C}(e^F)\,\square_{\otimes_2}\,(\emptyset \to L^{\simeq}\mathbb{C}R[m])$. Both morphisms in this pushout-product are cofibrations in 2Cat since $L^{\simeq}\mathbb{C}$ is left Quillen from DblCat^h_∞, and therefore, by monoidality of the model structure on 2Cat (see Theorem 6.3.5), it is enough to show that $L^{\simeq}\mathbb{C}(e^F)$ is a biequivalence. But this is clear since the 2-functor $L^{\simeq}\mathbb{C}(e^F)\colon L^{\simeq}\mathbb{C}(F[0]) \to L^{\simeq}\mathbb{C}(N^F I)$ can be identified with the generating trivial cofibration $j_1\colon [0] \to E_{\mathrm{adj}}$ in 2Cat.

We now show that the 2-functor $L^{\simeq}\mathbb{C}(c_k)\colon L^{\simeq}\mathbb{C}(F[0]) \to L^{\simeq}\mathbb{C}(F[k])$ is a biequivalence. It is given by the inclusion $[0] \to \widetilde{O_2(k)}$ at 0. First note that for $k = 0$, this is the identity. For $k \geq 1$, it is a biequivalence since it is

- bi-essentially surjective on objects as every object in $\widetilde{O_2(k)}$ is related by an adjoint equivalence to the object 0,
- essentially full on morphisms since every composite of adjoint equivalences $0 \to 0$ in $\widetilde{O_2(k)}$ is related by a 2-isomorphism to id_0, which is given by a pasting of units and counits of the adjoint equivalences,
- fully faithful on 2-morphisms since the only 2-morphism $\mathrm{id}_0 \Rightarrow \mathrm{id}_0$ in $\widetilde{O_2(k)}$ is the identity.

This proves the theorem. $\square$

As in the double categorical case, the adjunction $L^{\simeq}\mathbb{C} \dashv \mathbb{N}\mathbb{H}^{\simeq}$ is a Quillen reflection and hence the nerve $\mathbb{N}\mathbb{H}^{\simeq}$ is homotopically fully faithful. Again, since all objects in 2CSS are cofibrant, the derived counit of the adjunction $L^{\simeq}\mathbb{C} \dashv \mathbb{N}\mathbb{H}^{\simeq}$ coincide with the counit, and we show that it is level-wise a biequivalence.

Theorem 12.1.3. *The components $\epsilon_{\mathcal{A}}\colon L^{\simeq}\mathbb{C}\mathbb{N}\mathbb{H}^{\simeq}\mathcal{A} \to \mathcal{A}$ of the (derived) counit are biequivalences, for all 2-categories $\mathcal{A}$. In particular, the adjunction $L^{\simeq}\mathbb{C} \dashv \mathbb{N}\mathbb{H}^{\simeq}$ is a Quillen reflection.*

Proof. This follows from the fact that the (derived) counits of the adjunctions $\mathbb{C} \dashv \mathbb{N}$ and $L^{\simeq} \dashv \mathbb{H}^{\simeq}$ are weak equivalences, by Theorems 11.3.1 and 8.4.7, respectively. $\square$

Remark 12.1.4. We recall the adjunction $P \dashv D$ between Cat and 2Cat introduced in Proposition 2.1.13. By Theorem 6.1.14, these functors form a Quillen reflection between the canonical model structure on Cat and Lack's model structure on 2Cat. By composing with the Quillen reflection of Theorems 12.1.1 and 12.1.3, we obtain a Quillen reflection

$$\begin{tikzcd} \mathrm{Cat} \arrow[r, bend right, "D"'] \arrow[r, phantom, "\perp"] & 2\mathrm{Cat} \arrow[l, bend right, "P"'] \arrow[r, bend right, "\mathbb{N}\mathbb{H}^{\simeq}"'] \arrow[r, phantom, "\perp"] & 2\mathrm{CSS} \arrow[l, bend right, "L^{\simeq}\mathbb{C}"'] \end{tikzcd}$$

between the canonical model structure on Cat and the model structure on $\mathrm{sSet}^{\Delta^{\mathrm{op}}\times\Delta^{\mathrm{op}}}$ for 2-fold complete Segal spaces, i.e., $(\infty,2)$-categories.

12.2. 2Cat **is right-induced from** 2CSS **along** $\mathbb{NH}^{\simeq}$**.** We now show that Lack's model structure on 2Cat is right-induced from the model structure 2CSS for 2-fold complete Segal spaces along the nerve $\mathbb{NH}^{\simeq}$. In particular, this says that the homotopy theory of 2-categories is determined by the homotopy theory of 2-fold complete Segal spaces through its image under $\mathbb{NH}^{\simeq}$.

Theorem 12.2.1. *Lack's model structure on* 2Cat *of Theorem 6.1.8 is right-induced along the adjunction*

$$\begin{tikzcd} \mathrm{2Cat} \arrow[r, bend right, "\mathbb{NH}^{\simeq}"'] \arrow[r, phantom, "\perp"] & \mathrm{2CSS} \arrow[l, bend right, "L^{\simeq}\mathbb{C}"'] \end{tikzcd},$$

where 2CSS *denotes the model structure on* $\mathrm{sSet}^{\Delta^{\mathrm{op}}\times\Delta^{\mathrm{op}}}$ *of Theorem 10.3.3 for 2-fold complete Segal spaces.*

Proof. It is enough to show that a 2-functor F is a weak equivalence (resp. fibration) in 2Cat if and only if $\mathbb{NH}^{\simeq}F$ is a weak equivalence (resp. fibration) in 2CSS, as model structures are uniquely determined by their classes of weak equivalences and fibrations.

Since the functor $\mathbb{NH}^{\simeq}$ is right Quillen, it preserves fibrations. Moreover, since all objects are fibrant in 2Cat, the functor $\mathbb{NH}^{\simeq}$ also preserves weak equivalences by Ken Brown's Lemma (see Lemma 4.4.5). This shows that, if F is a weak equivalence (resp. fibration) in 2Cat, then $\mathbb{NH}^{\simeq}F$ is a weak equivalence (resp. fibration) in 2CSS.

Now let $F\colon \mathcal{A}\to\mathcal{B}$ be a 2-functor such that $\mathbb{NH}^{\simeq}F\colon \mathbb{NH}^{\simeq}\mathcal{A}\to\mathbb{NH}^{\simeq}\mathcal{B}$ is a weak equivalence in 2CSS. Since all objects are cofibrant in 2CSS, by Ken Brown's Lemma (see Lemma 4.4.5), the left Quillen functor $L^{\simeq}\mathbb{C}$ preserves weak equivalences. Therefore, the 2-functor $L^{\simeq}\mathbb{C}\mathbb{NH}^{\simeq}F$ is a biequivalence. We then have a commutative square

$$\begin{tikzcd} L^{\simeq}\mathbb{C}\mathbb{NH}^{\simeq}\mathcal{A} \arrow[r, "L^{\simeq}\mathbb{C}\mathbb{NH}^{\simeq}F", "\sim"'] \arrow[d, "\epsilon_{\mathcal{A}}"', "\wr"] & L^{\simeq}\mathbb{C}\mathbb{NH}^{\simeq}\mathcal{B} \arrow[d, "\epsilon_{\mathcal{B}}", "\wr"'] \\ \mathcal{A} \arrow[r, "F"'] & \mathcal{B} \end{tikzcd},$$

where the vertical 2-functors are biequivalences by Theorem 12.1.3. By 2-out-of-3, we get that F is also a biequivalence.

Finally, let $F\colon \mathcal{A}\to\mathcal{B}$ be a 2-functor such that $\mathbb{NH}^{\simeq}F\colon \mathbb{NH}^{\simeq}\mathcal{A}\to\mathbb{NH}^{\simeq}\mathcal{B}$ is a fibration in 2CSS. We show that F has the right lifting property with respect to the generating trivial cofibrations $j_1\colon [0]\to E_{\mathrm{adj}}$ and $j_2\colon [1]\to C_{\mathrm{inv}}$ in 2Cat as described in Notation 6.2.5, where E_{adj} denotes the "free-living adjoint equivalence" and C_{inv} denotes the "free-living 2-isomorphism". First note that, if $\mathbb{NH}^{\simeq}F$ is a fibration, then $(\mathbb{NH}^{\simeq}F)_{m,k}$ is a fibration in sSet for all $m,k\geq 0$, since fibrations between fibrant objects in 2CSS are in particular level-wise fibrations (see Proposition 5.2.20 and Remark 10.1.9).

By taking $m=k=0$, as $(\mathbb{NH}^{\simeq}F)_{0,0}$ is a fibration in sSet, there is a lift in every commutative diagram as below left.

$$\begin{tikzcd} \Delta[0] \arrow[r] \arrow[d, "\ell^{\Delta}_{1,1}"'] & (\mathbb{NH}^{\simeq}\mathcal{A})_{0,0} \arrow[d, "(\mathbb{NH}^{\simeq}F)_{0,0}"] \\ \Delta[1] \arrow[r] \arrow[ur, dashed] & (\mathbb{NH}^{\simeq}\mathcal{B})_{0,0} \end{tikzcd} \qquad \begin{tikzcd} {[0]} \arrow[r] \arrow[d, "j_1"'] & \mathcal{A} \arrow[d, "F"] \\ E_{\mathrm{adj}} \arrow[r] \arrow[ur, dashed] & \mathcal{B} \end{tikzcd}$$

By Description 13.2.1, a 0-simplex in $(\mathbb{N}\mathbb{H}^{\simeq}\mathcal{A})_{0,0}$ is an object of $\mathcal{A}$, and a 1-simplex in $(\mathbb{N}\mathbb{H}^{\simeq}\mathcal{A})_{0,0}$ is an adjoint equivalence in $\mathcal{A}$. Therefore, the existence of a lift in each diagram as above left corresponds to the existence of a lift in each diagram as above right. This shows that F has the right lifting property with respect to j_1.

Now take $m = 1$ and $k = 0$. As $(\mathbb{N}\mathbb{H}^{\simeq}\mathcal{A})_{1,0}$ is a fibration in sSet, there is a lift in every commutative diagram as below left.

$$\begin{tikzcd} \Delta[0] \arrow[r] \arrow[d, "\ell^{\Delta}_{1,1}"'] & (\mathbb{N}\mathbb{H}^{\simeq}\mathcal{A})_{1,0} \arrow[d, "(\mathbb{N}\mathbb{H}^{\simeq}F)_{1,0}"] \\ \Delta[1] \arrow[r] \arrow[ur, dashed] & (\mathbb{N}\mathbb{H}^{\simeq}\mathcal{B})_{1,0} \end{tikzcd} \qquad \begin{tikzcd} {[1]} \arrow[r] \arrow[d, "j'_2"'] & \mathcal{A} \arrow[d, "F"] \\ {[1]} \otimes_2 E_{\mathrm{adj}} \arrow[r] \arrow[ur, dashed] & \mathcal{B} \end{tikzcd}$$

By Description 13.2.2, a 0-simplex in $(\mathbb{N}\mathbb{H}^{\simeq}\mathcal{A})_{1,0}$ is a morphism of $\mathcal{A}$, and a 1-simplex in $(\mathbb{N}\mathbb{H}^{\simeq}\mathcal{A})_{1,0}$ is a 2-isomorphism in $\mathcal{A}$, as depicted in Description 13.2.3 (1). Therefore, the existence of a lift in each diagram as above left corresponds to the existence of a lift in each diagram as above right. Now the generating trivial cofibration $j_2\colon [1] \to C_{\mathrm{inv}}$ is a retract of j'_2 of the following form

$$\begin{tikzcd} {[1]} \arrow[r, equal] \arrow[d, "j_2"'] & {[1]} \arrow[r, equal] \arrow[d, "j'_2"'] & {[1]} \arrow[d, "j_2"] \\ C_{\mathrm{inv}} \arrow[r, "i"'] \arrow[rr, equal, bend right] & {[1]} \otimes_2 E_{\mathrm{adj}} \arrow[r, "r"'] & C_{\mathrm{inv}} \end{tikzcd},$$

where i sends the 2-isomorphism of C_{inv} to the 2-isomorphism of $[1] \otimes_2 E_{\mathrm{adj}}$, and r sends the adjoint equivalences of E_{adj} to identities, and the 2-isomorphism of $[1] \otimes_2 E_{\mathrm{adj}}$ to the 2-isomorphism of C_{inv}. Therefore, since F has the right lifting property with respect to j'_2, then F also has the right lifting property with respect to j_2. This shows that F is a fibration in 2Cat and concludes the proof. $\square$

12.3. **Comparison between the nerves $\mathbb{N}\mathbb{H}$ and $\mathbb{N}\mathbb{H}^{\simeq}$.** We now want to compare the nerves $\mathbb{N}\mathbb{H}\mathcal{A}$ and $\mathbb{N}\mathbb{H}^{\simeq}\mathcal{A}$ of a 2-category $\mathcal{A}$. For this, we will construct a homotopy equivalence between the spaces $(\mathbb{N}\mathbb{H}\mathcal{A})_{m,k}$ and $(\mathbb{N}\mathbb{H}^{\simeq}\mathcal{A})_{m,k}$. Their sets of n-simplices are given by

$$(\mathbb{N}\mathbb{H}\mathcal{A})_{m,k,n} = \mathrm{DblCat}(\mathbb{X}_{m,k,n}, \mathbb{H}\mathcal{A}) \cong 2\mathrm{Cat}(L\mathbb{X}_{m,k,n}, \mathcal{A})$$

and

$$(\mathbb{N}\mathbb{H}^{\simeq}\mathcal{A})_{m,k,n} = \mathrm{DblCat}(\mathbb{X}_{m,k,n}, \mathbb{H}^{\simeq}\mathcal{A}) \cong 2\mathrm{Cat}(L^{\simeq}\mathbb{X}_{m,k,n}, \mathcal{A}).$$

Let us first describe the 2-categories $L^{\simeq}\mathbb{X}_{m,k,n}$ and $L\mathbb{X}_{m,k,n}$.

Description 12.3.1. The 2-category $L\mathbb{X}_{m,k,n}$ is obtained from the double category

$$\mathbb{X}_{m,k,n} = (\mathbb{V}O_2^{\sim}(k) \otimes O_2^{\sim}(m)) \otimes \widetilde{O_2(n)}$$

by identifying the objects $(x, y, z) \sim (x, y', z)$, for all $0 \leq x \leq m$, $0 \leq y, y' \leq k$, and $0 \leq z \leq n$, and by identifying the vertical morphisms $(x, g, z)\colon (x, y, z) \to (x, y', z)$, where $g \in O_2^{\sim}(k)(y, y')$, with the identity at $(x, y, z) \sim (x, y', z)$. We denote by $[x, z]$ the equivalence class $\{(x, y, z) \mid 0 \leq y \leq k\}$. Then, the 2-category $L\mathbb{X}_{m,k,n}$ has

- objects $[x, z]$ for all $0 \leq x \leq m$ and $0 \leq z \leq n$,
- morphisms freely generated by
  - a morphism $(f, y, z)\colon [x, z] \to [x', z]$ where $f \in O_2^{\sim}(m)(x, x')$ is represented by the set $\{x, x'\}$, for all $0 \leq x, x' \leq m$, $0 \leq y \leq k$, and $0 \leq z \leq n$,

  – a morphism $(x,y,h)\colon [x,z]\to[x,z']$ where $h\in \widetilde{O_2(n)}(z,z')$ is represented by the set $\{z,z'\}$, for all $0\leq x\leq m$, $0\leq y\leq k$, and $0\leq z,z'\leq n$,
- a 2-morphism $\alpha\colon \overline{f}\Rightarrow\overline{f'}$ for each square $\alpha\colon (u\,\frac{f}{f'}\,v)$ in $\mathbb{X}_{m,k,n}$.

Description 12.3.2. The 2-category $L^{\simeq}\mathbb{X}_{m,k,n}$ has

- the same objects as the double category $\mathbb{X}_{m,k,n}=(\mathbb{V}O_2^{\simeq}(k)\otimes O_2^{\simeq}(m))\otimes\widetilde{O_2(n)}$, i.e., triples (x,y,z) with $0\leq x\leq m$, $0\leq y\leq k$, $0\leq z\leq n$,
- morphisms generated by
  – a morphism $(f,y,z)\colon (x,y,z)\to(x',y,z)$ where $f\in O_2^{\simeq}(m)(x,x')$ is represented by the set $\{x,x'\}$, for all $0\leq x,x'\leq m$, $0\leq y\leq k$, and $0\leq z\leq n$,
  – a morphism $(x,y,h)\colon (x,y,z)\to(x,y,z')$ where $h\in\widetilde{O_2(n)}(z,z')$ is represented by the set $\{z,z'\}$, for all $0\leq x\leq m$, $0\leq y\leq k$, and $0\leq z,z'\leq n$,
  – an adjoint equivalence $(x,g,z)\colon (x,y,z)\xrightarrow{\simeq}(x,y',z)$ where $g\in O_2^{\simeq}(k)(y,y')$ is represented by the set $\{y,y'\}$, for all $0\leq x\leq m$, $0\leq y,y'\leq k$, and $0\leq z\leq n$,
- a 2-morphism $\alpha\colon v\overline{f}\Rightarrow\overline{f}'u$ for each square $\alpha\colon (u\,\frac{f}{f'}\,v)$ in $\mathbb{X}_{m,k,n}$.

Example 12.3.3. We compute these 2-categories in the case where $m=1$, $k=1$, and $n=0$. Let us denote by $u\colon 0'\to 1'$ the vertical morphism in $\mathbb{V}[1]$ and by $f\colon 0\to 1$ the morphism in $[1]$. We have that $L(\mathbb{V}[1]\otimes[1])$ is the free 2-category on a 2-morphism as depicted below left, while $L^{\simeq}(\mathbb{V}[1]\otimes[1])$ is the 2-category as depicted below right. We omit the z-component here since it is always 0.

$$\begin{tikzcd}
{[0]} \arrow[r, bend left, "{(f,0')}"] \arrow[r, bend right, "{(f,1')}"'] \arrow[r, phantom, "\Downarrow (f,u)"] & {[1]}
\end{tikzcd}
\qquad
\begin{tikzcd}
(0,0') \arrow[r, "{(f,0')}"] \arrow[d, "{(0,u)}"', "\simeq"] \arrow[dr, phantom, "\swarrow (f,u)"] & (1,0') \arrow[d, "{(1,u)}", "\simeq"'] \\
(0,1') \arrow[r, "{(f,1')}"'] & (1,1')
\end{tikzcd}$$

Remark 12.3.4. Using these descriptions, we can see that the 0-simplices of the simplicial sets $(\mathbb{NH}\mathcal{A})_{0,0}$ and $(\mathbb{NH}^{\simeq}\mathcal{A})_{0,0}$ are the objects of $\mathcal{A}$, and the ones of $(\mathbb{NH}\mathcal{A})_{1,0}$ and $(\mathbb{NH}^{\simeq}\mathcal{A})_{1,0}$ the morphisms of $\mathcal{A}$. The 0-simplices in $(\mathbb{NH}\mathcal{A})_{1,1}$ are the 2-morphisms of $\mathcal{A}$ as in the above left diagram of Example 12.3.3, while the ones of $(\mathbb{NH}^{\simeq}\mathcal{A})_{1,1}$ are the 2-morphisms of $\mathcal{A}$ as in the above right diagram of Example 12.3.3. Finally, the 0-simplices in $(\mathbb{NH}\mathcal{A})_{0,1}$ are just objects of $\mathcal{A}$, while the ones of $(\mathbb{NH}^{\simeq}\mathcal{A})_{0,1}$ are adjoint equivalences in $\mathcal{A}$. We describe these simplicial sets in greater detail in Sections 13.2 and 13.3.

There is a comparison morphism $\pi_{m,k,n}\colon L^{\simeq}\mathbb{X}_{m,k,n}\to L\mathbb{X}_{m,k,n}$ which sends an object (x,y,z) to the object $[x,z]$, morphisms (f,y,z) and (x,y,h) to the morphisms (f,y,z) and (x,y,h), the adjoint equivalences (x,g,z) to the identity at $[x,z]$, and a 2-morphism $\alpha\colon v\overline{f}\Rightarrow\overline{f}'u$ to the corresponding 2-morphism $\alpha\colon\overline{f}\Rightarrow\overline{f}'$. Note that this is a 2-functor since the adjoint equivalences are sent to identities. Moreover, this 2-functor is clearly surjective on objects, full on morphisms, and fully faithful on 2-morphisms. By constructing an inverse 2-functor up to pseudo-natural equivalence (see Definition 2.4.6) to this comparison morphism $\pi_{m,k,n}$, we obtain the following result.

Theorem 12.3.5. *Let $\mathcal{A}$ be a 2-category. The map $\pi^*\colon\mathbb{NH}\mathcal{A}\to\mathbb{NH}^{\simeq}\mathcal{A}$ induced by the comparison maps $\pi_{m,k,n}\colon L^{\simeq}\mathbb{X}_{m,k,n}\to L\mathbb{X}_{m,k,n}$ is level-wise a homotopy equivalence in* $\mathrm{sSet}^{\Delta^{\mathrm{op}}\times\Delta^{\mathrm{op}}}$. *In particular, this exhibits $\mathbb{NH}^{\simeq}\mathcal{A}$ as a fibrant replacement of $\mathbb{NH}\mathcal{A}$ in* 2CSS *(or in* DblCat_∞^h*).*

Proof. We first construct an inverse 2-functor up to pseudo-natural equivalence

$$\iota_{m,k,n} \colon L\mathbb{X}_{m,k,n} \to L^{\simeq}\mathbb{X}_{m,k,n}$$

to the 2-functor $\pi_{m,k,n}$ such that the composite $\pi_{m,k,n}\iota_{m,k,n}$ is the identity at $L\mathbb{X}_{m,k,n}$. It sends an object $[x,z]$ to the object $(x,0,z)$, a generating morphism $(f,y,z)\colon [x,z] \to [x',z]$ with $f \in O_2^{\simeq}(m)(x,x')$ represented by the set $\{x,x'\}$ to the composite

$$(x,0,z) \xrightarrow[\simeq]{(x,g,z)} (x,y,z) \xrightarrow{(f,y,z)} (x',y,z) \xrightarrow[\simeq]{(x',g',z)} (x',0,z),$$

and a generating morphism $(x,y,h)\colon [x,z] \to [x,z']$ with $h \in \widetilde{O_2(n)}(z,z')$ represented by the set $\{z,z'\}$ to the composite

$$(x,0,z) \xrightarrow[\simeq]{(x,g,z)} (x,y,z) \xrightarrow{(x,y,h)} (x,y,z') \xrightarrow[\simeq]{(x,g',z')} (x,0,z'),$$

where $g \in \widetilde{O_2(k)}(0,y)$ is represented by the set $\{0,y\}$ and $g' \in \widetilde{O_2(k)}(y,0)$ is its weak inverse. The assignment on 2-morphisms is uniquely determined by these assignments on objects and morphisms, since the 2-functor $\pi_{m,k,n}$ is fully faithful on 2-morphisms and we imposed that $\pi_{m,k,n}\iota_{m,k,n} = \mathrm{id}_{L\mathbb{X}_{m,k,n}}$. In particular, since the morphisms in the 2-category $L\mathbb{X}_{m,k,n}$ are freely generated by the morphisms (f,y,z) and (x,y,h), this defines a 2-functor $\iota_{m,k,n} \colon L\mathbb{X}_{m,k,n} \to L^{\simeq}\mathbb{X}_{m,k,n}$.

We now construct a pseudo-natural adjoint equivalence

$$\epsilon_{m,k,n} \colon \iota_{m,k,n}\pi_{m,k,n} \Rightarrow \mathrm{id}_{L^{\simeq}\mathbb{X}_{m,k,n}}.$$

At an object $(x,y,z) \in L^{\simeq}\mathbb{X}_{m,k,n}$, we define $\epsilon_{(x,y,z)}$ to be the morphism

$$\epsilon_{(x,y,z)} \coloneqq (x,g,z) \colon (x,0,z) \xrightarrow{\simeq} (x,y,z),$$

where $g \in \widetilde{O_2(k)}(0,y)$ is represented by the set $\{0,y\}$. Note that the morphism $\epsilon_{(x,y,z)}$ as defined above is an adjoint equivalence. Given a morphism $(f,y,z)\colon (x,y,z) \to (x',y,z)$, we define $\epsilon_{(f,y,z)}$ to be the following 2-isomorphism

$$\begin{tikzcd}
(x,0,z) \arrow[r, "\epsilon_{(x,y,z)} = (x,g,z)", "\simeq"'] \arrow[d, "(x,g,z)"', "\simeq"] & (x,y,z) \arrow[ddd, "(f,y,z)"] \\
(x,y,z) \arrow[d, "(f,y,z)"'] \arrow[r, phantom, "="] & {} \\
(x',y,z) \arrow[d, "(x',g',z)"', "\simeq"] \arrow[dr, equal] & {} \\
(x',0,z) \arrow[r, "\simeq", "\epsilon_{(x',y,z)} = (x',g,z)"'] \arrow[ur, phantom, "\Nearrow\cong"] & (x',y,z)
\end{tikzcd}$$

induced by the counit $gg' \cong \mathrm{id}_y$ of the adjoint equivalence (g,g'). We define $\epsilon_{(x,y,h)}$ for a morphism $(x,y,h)\colon (x,y,z) \to (x,y,z')$ similarly. This defines a pseudo-natural adjoint equivalence $\epsilon_{m,k,n} \colon \iota_{m,k,n}\pi_{m,k,n} \Rightarrow \mathrm{id}_{L^{\simeq}\mathbb{X}_{m,k,n}}$, which can be represented by a 2-functor $\widetilde{O_2(1)} \to [L^{\simeq}\mathbb{X}_{m,k,n}, L^{\simeq}\mathbb{X}_{m,k,n}]_{2,\mathrm{ps}}$ since it corresponds to an adjoint equivalence in the pseudo-hom 2-category Proposition 2.4.5. By definition of the Gray tensor product $\otimes_2$ (see Proposition 2.3.4), this pseudo-natural adjoint equivalence can equivalently be seen as a 2-functor

$$\begin{tikzcd}
L^{\simeq}\mathbb{X}_{m,k,n} \arrow[d, "\mathrm{id}\otimes_2 d^0"'] \arrow[dr, "\iota_{m,k,n}\circ\pi_{m,k,n}"] & \\
L^{\simeq}\mathbb{X}_{m,k,n}\otimes_2 \widetilde{O_2(1)} \arrow[r, "\epsilon_{m,k,n}"] & L^{\simeq}\mathbb{X}_{m,k,n}\,. \\
L^{\simeq}\mathbb{X}_{m,k,n} \arrow[u, "\mathrm{id}\otimes_2 d^1"] \arrow[ur, equal] &
\end{tikzcd}$$

We claim that these 2-functors $\epsilon_{m,k,n}$ induce a homotopy $\epsilon^*_{m,k}$ as in

$$\begin{tikzcd}
(\mathbb{N}\mathbb{H}^{\simeq}\mathcal{A})_{m,k} \arrow[d, "\mathrm{id}\times d^0"'] \arrow[dr, "\pi^*_{m,k}\circ\iota^*_{m,k}"] & \\
(\mathbb{N}\mathbb{H}^{\simeq}\mathcal{A})_{m,k}\times\Delta[1] \arrow[r, "\epsilon^*_{m,k}"] & (\mathbb{N}\mathbb{H}^{\simeq}\mathcal{A})_{m,k}\,, \\
(\mathbb{N}\mathbb{H}^{\simeq}\mathcal{A})_{m,k} \arrow[u, "\mathrm{id}\times d^1"] \arrow[ur, equal] &
\end{tikzcd}$$

where the nth component of $\epsilon^*_{m,k}$ is obtained by applying the functor $2\mathrm{Cat}(-,\mathcal{A})$ to $\epsilon_{m,k,n}$, for all $n\geq 0$.

For each $F\in(\mathbb{N}\mathbb{H}^{\simeq}\mathcal{A})_{m,k,n}$, we want to describe the corresponding $(\Delta[n]\times\Delta[1])$-prism of the homotopy, which coincide with $F\iota_{m,k,n}\pi_{m,k,n}$ at $0\in\Delta[1]$ and with F at $1\in\Delta[1]$. Note that a $(\Delta[n]\times\Delta[1])$-prism in $(\mathbb{N}\mathbb{H}^{\simeq}\mathcal{A})_{m,k}$ corresponds to a 2-functor

$$L^{\simeq}((\mathbb{V}O_2^{\sim}(k)\otimes O_2^{\sim}(m))\otimes(\widetilde{O_2(n)}\otimes_2\widetilde{O_2(1)}))\longrightarrow\mathcal{A}.$$

The squares induced by vertical morphisms in $\mathbb{V}O_2^{\sim}(k)$ and morphisms in $\widetilde{O_2(1)}$ must be weakly horizontally invertible in $(\mathbb{V}O_2^{\sim}(k)\otimes O_2^{\sim}(m))\otimes(\widetilde{O_2(n)}\otimes_2\widetilde{O_2(1)})$, since the morphisms in $\widetilde{O_2(1)}$ are adjoint equivalences. It follows from Lemma 3.6.9 that the corresponding 2-morphisms in $L^{\simeq}((\mathbb{V}O_2^{\sim}(k)\otimes O_2^{\sim}(m))\otimes(\widetilde{O_2(n)}\otimes_2\widetilde{O_2(1)}))$ are invertible and therefore, by Remark 12.1.2, we get that

$$\begin{aligned}
L^{\simeq}((\mathbb{V}O_2^{\sim}(k)\otimes O_2^{\sim}(m))&\otimes(\widetilde{O_2(n)}\otimes_2\widetilde{O_2(1)}))\\
&\cong L^{\simeq}((\mathbb{V}O_2^{\sim}(k)\otimes O_2^{\sim}(m))\otimes\widetilde{O_2(n)})\otimes_2\widetilde{O_2(1)}\\
&= L^{\simeq}\mathbb{X}_{m,k,n}\otimes_2\widetilde{O_2(1)}.
\end{aligned}$$

This says that a $(\Delta[n]\times\Delta[1])$-simplex in $(\mathbb{N}\mathbb{H}^{\simeq}\mathcal{A})_{m,k}$ corresponds to a 2-functor

$$L^{\simeq}\mathbb{X}_{m,k,n}\otimes_2\widetilde{O_2(1)}\to\mathcal{A}.$$

We can therefore define the component of the homotopy at $F\in(\mathbb{N}\mathbb{H}^{\simeq}\mathcal{A})_{m,k,n}$ to be $F\epsilon_{m,k,n}$. This shows the claim.

Since $\iota^*_{m,k}\circ\pi^*_{m,k}=\mathrm{id}_{(\mathbb{N}\mathbb{H}\mathcal{A})_{m,k}}$ and by the above homotopy, we see that $\iota^*_{m,k}$ and $\pi^*_{m,k}$ give a homotopy equivalence between $(\mathbb{N}\mathbb{H}\mathcal{A})_{m,k}$ and $(\mathbb{N}\mathbb{H}^{\simeq}\mathcal{A})_{m,k}$, for all $m,k\geq 0$. These assemble into maps ι^* and π^* of $\mathrm{sSet}^{\Delta^{\mathrm{op}}\times\Delta^{\mathrm{op}}}$ which give a level-wise weak equivalence between $\mathbb{N}\mathbb{H}\mathcal{A}$ and $\mathbb{N}\mathbb{H}^{\simeq}\mathcal{A}$. This is in particular a weak equivalence in 2CSS and in DblCat^h_∞. Since $\mathbb{N}\mathbb{H}^{\simeq}\mathcal{A}$ is fibrant in 2CSS and in DblCat^h_∞, we conclude that it gives a fibrant replacement of $\mathbb{N}\mathbb{H}\mathcal{A}$. □

Remark 12.3.6. Recall from Theorem 6.1.14 the Quillen pair $P \dashv D$ between Cat and 2Cat and let $\mathcal{C}$ be a category. We compute the nerve of the double category $\mathbb{H}D\mathcal{C}$:

$$(\mathbb{N}\mathbb{H}D\mathcal{C})_{m,k,n} = 2\mathrm{Cat}(L\mathbb{X}_{m,k,n}, D\mathcal{C}) \cong \mathrm{Cat}(PL\mathbb{X}_{m,k,n}, \mathcal{C}),$$

for all $m,k,n \geq 0$. By applying the functor P to the 2-category $L\mathbb{X}_{m,k,n}$ as given in Description 12.3.1, we can see that $PL\mathbb{X}_{m,k,n} \cong [m] \times I[n]$, where $I[n]$ is the category with object set $\{0,\dots,n\}$ and a unique isomorphism between any two objects. Therefore,

$$(\mathbb{N}\mathbb{H}D\mathcal{C})_{m,k,n} \cong \mathrm{Cat}([m] \times I[n], \mathcal{C}) = N_{\mathrm{Rezk}}(\mathcal{C})_{m,n}$$

is given by the Rezk nerve (see Example 9.2.15) constant in the vertical direction. Similarly, we compute the nerve of $\mathbb{H}^{\simeq}D\mathcal{C}$ and find that

$$\begin{aligned}(\mathbb{N}\mathbb{H}^{\simeq}D\mathcal{C})_{m,k,n} &= 2\mathrm{Cat}(L^{\simeq}\mathbb{X}_{m,k,n}, D\mathcal{C}) \\ &\cong \mathrm{Cat}(PL^{\simeq}\mathbb{X}_{m,k,n}, \mathcal{C}) \cong \mathrm{Cat}((I[k] \times [m]) \times I[n], \mathcal{C}).\end{aligned}$$

By Theorem 12.3.5, we get a level-wise homotopy equivalence $\mathbb{N}\mathbb{H}D\mathcal{C} \to \mathbb{N}\mathbb{H}^{\simeq}D\mathcal{C}$ which exhibits $\mathbb{N}\mathbb{H}^{\simeq}D\mathcal{C}$ as a fibrant replacement of the Rezk nerve of $\mathcal{C}$ in 2CSS (or $\mathrm{DblCat}_{\infty}^{h}$).

13. Explicit description of the nerves in lower dimensions

In this last section, we describe the nerves of the different double categories considered in this paper in lower dimensions; namely, for $0 \leq m,k \leq 1$ and $0 \leq n \leq 2$. The aim of these descriptions is to give the intuition that the space of the nerve at $(m,k) = (0,0)$ models the *space of objects*, the one at $(m,k) = (1,0)$ models the *space of horizontal morphisms*, the one at $(m,k) = (0,1)$ models the *space of vertical morphisms*, and the one at $(m,k) = (1,1)$ models the *space of squares* of the corresponding double category. In Section 13.1, we first describe the nerve $\mathbb{N}$ of a general double category. Then, in Section 13.2, we describe the nerve $\mathbb{N}\mathbb{H}^{\simeq}$ of a 2-category. Finally, in Section 13.3, we also describe the nerve $\mathbb{N}\mathbb{H}$ of a 2-category, in order to compare it with its fibrant replacement $\mathbb{N}\mathbb{H}^{\simeq}$.

13.1. **Nerve of a double category.**
Let $\mathbb{A}$ be a double category. We want to describe the 0-, 1-, and 2-simplices of the space $(\mathbb{N}\mathbb{A})_{m,k}$ for $0 \leq m,k \leq 1$.

Description 13.1.1. By definition of $\mathbb{N}$, we have that

$$\begin{aligned}(\mathbb{N}\mathbb{A})_{m,k,n} &= \mathrm{DblCat}((\mathbb{V}O_2^{\sim}(k) \otimes O_2^{\sim}(m)) \otimes \widetilde{O_2(n)}, \mathbb{N}\mathbb{A}) \\ &\cong 2\mathrm{Cat}(\widetilde{O_2(n)}, \mathbf{H}[\mathbb{V}O_2^{\sim}(k) \otimes O_2^{\sim}(m), \mathbb{A}]_{\mathrm{ps}}).\end{aligned}$$

Therefore we can describe the 0-, 1-, and 2-simplices of the space $(\mathbb{N}\mathbb{A})_{m,k}$ as follows.

(0) A 0-simplex in $(\mathbb{N}\mathbb{A})_{m,k}$ is a double functor $F \colon \mathbb{V}O_2^{\sim}(k) \otimes O_2^{\sim}(m) \to \mathbb{A}$.
(1) A 1-simplex in $(\mathbb{N}\mathbb{A})_{m,k}$ is an adjoint equivalence in $\mathbf{H}[\mathbb{V}O_2^{\sim}(k) \otimes O_2^{\sim}(m), \mathbb{A}]_{\mathrm{ps}}$, i.e., by Proposition 3.6.10, a horizontal pseudo-natural transformation

$$\begin{tikzcd} \mathbb{V}O_2^{\sim}(k) \otimes O_2^{\sim}(m) \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow\varphi"] & \mathbb{A} \end{tikzcd}$$

such that, the horizontal morphism $\varphi_i \colon Fi \to Gi$ is a horizontal adjoint equivalence in $\mathbb{A}$, for each object $i \in \mathbb{V}O_2^{\sim}(k) \otimes O_2^{\sim}(m)$, and the square $\varphi_u \colon (Fu \,{}^{\varphi_i}_{\varphi_{i'}}\, Gu)$ is weakly horizontally invertible, for each vertical morphism u in $\mathbb{V}O_2^{\sim}(k) \otimes O_2^{\sim}(m)$. Recall from Definition 3.6.11 that such a φ is called a **horizontal pseudo-natural adjoint equivalence** and we denote it by $\varphi \colon F \overset{\simeq}{\Rightarrow} G$.
(2) A 2-simplex is the data of three horizontal pseudo-natural adjoint equivalences $\varphi \colon F \overset{\simeq}{\Rightarrow} G$, $\psi \colon G \overset{\simeq}{\Rightarrow} H$, and $\theta \colon F \overset{\simeq}{\Rightarrow} H$ together with an invertible modification μ as follows.

$$\begin{tikzcd} & G \arrow[dr, Rightarrow, "\psi"] & \\ F \arrow[ur, Rightarrow, "\varphi"] \arrow[rr, Rightarrow, "\theta"'] \arrow[urr, phantom, "\mu \Uparrow \cong"] & & H \end{tikzcd}$$

We first compute the space $(\mathbb{NA})_{0,0}$, which is given by the *space of objects*. As expected from the completeness condition being in the horizontal direction, its 0-simplices are given by the objects, and its 1-simplices by the horizontal adjoint equivalences.

Description 13.1.2 ($m = 0$, $k = 0$)**.** We describe the space $(\mathbb{NA})_{0,0}$. First note that the double category $\mathbb{V}O_2^{\simeq}(0) \otimes O_2^{\simeq}(0) = [0]$ is the terminal (double) category.

(0) A 0-simplex in $(\mathbb{NA})_{0,0}$ is a double functor $A\colon [0] \to \mathbb{A}$, i.e., the data of an object $A \in \mathbb{A}$.
(1) A 1-simplex in the space $(\mathbb{NA})_{0,0}$ is a horizontal pseudo-natural adjoint equivalence $\varphi\colon A \overset{\simeq}{\Rightarrow} B$, i.e., the data of a horizontal adjoint equivalence $\varphi\colon A \xrightarrow{\simeq} C$ in $\mathbb{A}$.
(2) A 2-simplex in $(\mathbb{NA})_{0,0}$ is an invertible modification $\mu\colon \theta \cong \psi\varphi$ between such horizontal pseudo-natural adjoint equivalences, i.e., the data of a vertically invertible square in $\mathbb{A}$

$$\begin{tikzcd} A \arrow[rr, "\theta", "\simeq"'] \arrow[d, equal] & & E \arrow[d, equal] \\ A \arrow[r, "\simeq", "\varphi"'] \arrow[urr, phantom, "\mu \ \cong"] & C \arrow[r, "\simeq", "\psi"'] & E \end{tikzcd} .$$

We now turn our attention to the *space of horizontal morphisms* $(\mathbb{NA})_{1,0}$. We observe that the squares appearing as n-simplices of this space all have trivial vertical boundaries. In particular, this prevents a completeness condition for $(\mathbb{NA})_{1,-}$. However, this still looks like a degenerate completeness condition, which could be added to the definition of a double $(\infty, 1)$-category, as mentioned at the beginning of Theorem 10.3.3.

Description 13.1.3 ($m = 1$, $k = 0$)**.** We describe the space $(\mathbb{NA})_{1,0}$. First note that $\mathbb{V}O_2^{\simeq}(0) \otimes O_2^{\simeq}(1) = \mathbb{H}[1]$ is the free double category on a horizontal morphism.

(0) A 0-simplex in $(\mathbb{NA})_{1,0}$ is a double functor $f\colon \mathbb{H}[1] \to \mathbb{A}$, i.e., the data of a horizontal morphism $f\colon A \to B$ in $\mathbb{A}$.
(1) A 1-simplex in the space $(\mathbb{NA})_{1,0}$ is a horizontal pseudo-natural adjoint equivalence $\varphi\colon f \overset{\simeq}{\Rightarrow} g$, i.e., the data of two horizontal adjoint equivalences $\varphi_0\colon A \xrightarrow{\simeq} C$ and $\varphi_1\colon B \xrightarrow{\simeq} D$ together with a vertically invertible square in $\mathbb{A}$

$$\begin{tikzcd} A \arrow[r, "\varphi_0", "\simeq"'] \arrow[d, equal] & C \arrow[r, "g"] & D \arrow[d, equal] \\ A \arrow[r, "f"'] \arrow[urr, phantom, "\varphi \ \cong"] & B \arrow[r, "\simeq", "\varphi_1"'] & D \end{tikzcd} .$$

(2) A 2-simplex in $(\mathbb{NA})_{1,0}$ is an invertible modification $\mu\colon \theta \cong \psi\varphi$ between such horizontal pseudo-natural adjoint equivalences, i.e., the data of two vertically invertible squares μ_0 and μ_1 in $\mathbb{A}$ satisfying the following pasting equality.

$$
\begin{tikzcd}
A \arrow[rr, "\theta_0", "\simeq"'] \arrow[d, equal, "\bullet" marking] & & E \arrow[r, "h"] \arrow[d, equal, "\bullet" marking] & F \arrow[d, equal, "\bullet" marking] \\
A \arrow[r, "\simeq", "\varphi_0"'] \arrow[d, equal, "\bullet" marking] \arrow[urr, phantom, "\mu_0 \cong"] & C \arrow[r, "\simeq", "\psi_0"'] \arrow[d, equal, "\bullet" marking] & E \arrow[r, "h"'] \arrow[ur, phantom, "e_h"] & F \arrow[d, equal, "\bullet" marking] \\
A \arrow[r, "\simeq", "\varphi_0"'] \arrow[d, equal, "\bullet" marking] \arrow[ur, phantom, "e_{\varphi_0}"] & C \arrow[r, "g"'] & D \arrow[r, "\simeq", "\psi_1"'] \arrow[d, equal, "\bullet" marking] \arrow[ul, phantom, "\psi \cong"] & F \arrow[d, equal, "\bullet" marking] \\
A \arrow[r, "f"'] \arrow[urr, phantom, "\varphi \cong"] & B \arrow[r, "\simeq", "\varphi_1"'] & D \arrow[r, "\simeq", "\psi_1"'] \arrow[ur, phantom, "e_{\psi_1}"] & F
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
A \arrow[rr, "\theta_0", "\simeq"'] \arrow[d, equal, "\bullet" marking] & & E \arrow[r, "h"] & F \arrow[d, equal, "\bullet" marking] \\
A \arrow[r, "f"] \arrow[d, equal, "\bullet" marking] \arrow[urr, phantom, "\theta \cong"] & B \arrow[rr, "\theta_1", "\simeq"'] \arrow[d, equal, "\bullet" marking] & & E \arrow[d, equal, "\bullet" marking] \\
A \arrow[r, "f"'] \arrow[ur, phantom, "e_f"] & B \arrow[r, "\simeq", "\varphi_1"'] \arrow[urr, phantom, "\mu_1 \cong"] & D \arrow[r, "\simeq", "\psi_1"'] & F
\end{tikzcd}
$$

We now compute the lower simplices of the space $(\mathbb{N}\mathbb{A})_{0,1}$ – the *space of vertical morphisms*. As expected from the horizontal completeness condition, its 0-simplices are given by the vertical morphisms, and its 1-simplices by the weakly horizontally invertible squares.

Description 13.1.4 ($m = 0$, $k = 1$)**.** We describe the space $(\mathbb{N}\mathbb{A})_{0,1}$. First note that $\mathbb{V}O_2^{\simeq}(1) \otimes O_2^{\simeq}(0) = \mathbb{V}[1]$ is the free double category on a vertical morphism.

(0) A 0-simplex in $(\mathbb{N}\mathbb{A})_{0,1}$ is a double functor $u\colon \mathbb{V}[1] \to \mathbb{A}$, i.e., the data of a vertical morphism $u\colon A \to\!\!\!\!\bullet\; A'$ in $\mathbb{A}$.

(1) A 1-simplex in the space $(\mathbb{N}\mathbb{A})_{0,1}$ is a horizontal pseudo-natural adjoint equivalence $\varphi\colon u \overset{\simeq}{\Rightarrow} w$, i.e., the data of two horizontal adjoint equivalences $\varphi\colon A \xrightarrow{\simeq} C$ and $\varphi'\colon A' \xrightarrow{\simeq} C'$ together with a weakly horizontally invertible square in $\mathbb{A}$

$$
\begin{tikzcd}
A \arrow[r, "\varphi", "\simeq"'] \arrow[d, "u"', "\bullet" marking] \arrow[dr, phantom, "\widetilde{\varphi} \simeq"] & C \arrow[d, "w", "\bullet" marking] \\
A' \arrow[r, "\simeq", "\varphi'"'] & C' .
\end{tikzcd}
$$

(2) A 2-simplex in $(\mathbb{N}\mathbb{A})_{0,1}$ is an invertible modification $\mu\colon \theta \cong \psi\varphi$ between such horizontal pseudo-natural adjoint equivalences, i.e., the data of two vertically invertible squares μ and μ' in $\mathbb{A}$ satisfying the following pasting equality.

$$
\begin{tikzcd}
A \arrow[rr, "\theta", "\simeq"'] \arrow[d, equal, "\bullet" marking] & & E \arrow[d, equal, "\bullet" marking] \\
A \arrow[r, "\varphi", "\simeq"'] \arrow[d, "u"', "\bullet" marking] \arrow[urr, phantom, "\mu \cong"] & C \arrow[r, "\psi", "\simeq"'] \arrow[d, "w"', "\bullet" marking] & E \arrow[d, "y", "\bullet" marking] \\
A' \arrow[r, "\simeq", "\varphi'"'] \arrow[ur, phantom, "\widetilde{\varphi} \simeq"] & C' \arrow[r, "\simeq", "\psi'"'] \arrow[ur, phantom, "\widetilde{\psi} \simeq"] & E'
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
A \arrow[rr, "\theta", "\simeq"'] \arrow[d, "u"', "\bullet" marking] & & E \arrow[d, "y", "\bullet" marking] \\
A' \arrow[rr, "\theta'", "\simeq"'] \arrow[d, equal, "\bullet" marking] \arrow[urr, phantom, "\widetilde{\theta} \simeq"] & & E' \arrow[d, equal, "\bullet" marking] \\
A' \arrow[r, "\simeq", "\varphi'"'] \arrow[urr, phantom, "\mu' \cong"] & C' \arrow[r, "\simeq", "\psi'"'] & E'
\end{tikzcd}
$$

Finally, we consider the *space of squares* $(\mathbb{N}\mathbb{A})_{1,1}$.

Description 13.1.5 ($m = 1$, $k = 1$)**.** We describe the space $(\mathbb{N}\mathbb{A})_{1,1}$. First note that $\mathbb{V}O_2^{\simeq}(1) \otimes O_2^{\simeq}(1) = \mathbb{V}[1] \times \mathbb{H}[1]$ is the free double category on a square.

(0) A 0-simplex in $(\mathbb{N}\mathbb{A})_{1,1}$ is a double functor $\alpha\colon \mathbb{V}[1] \times \mathbb{H}[1] \to \mathbb{A}$, i.e., the data of a square α in $\mathbb{A}$

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "u"', "\bullet" marking] \arrow[dr, phantom, "\alpha"] & B \arrow[d, "v", "\bullet" marking] \\
A' \arrow[r, "f'"'] & B' .
\end{tikzcd}$$

(1) A 1-simplex in the space $(\mathbb{N}\mathbb{A})_{1,1}$ is a horizontal pseudo-natural adjoint equivalence $\varphi\colon \alpha \overset{\simeq}{\Rrightarrow} \beta$, i.e., the data of four horizontal adjoint equivalences φ_0, φ_1, φ_0', and φ_1', two vertically invertible squares φ and φ', and two weakly horizontally invertible squares $\widetilde{\varphi_0}$ and $\widetilde{\varphi_1}$ in $\mathbb{A}$ fitting in the following pasting equality.

$$\begin{tikzcd}
A \arrow[r, "\varphi_0", "\simeq"'] \arrow[d, equal, "\bullet" marking] & C \arrow[r, "g"] \arrow[d, phantom, "\varphi\ \cong"] & D \arrow[d, equal, "\bullet" marking] \\
A \arrow[r, "f"] \arrow[d, "u"', "\bullet" marking] \arrow[dr, phantom, "\alpha"] & B \arrow[r, "\varphi_1", "\simeq"'] \arrow[d, "v"', "\bullet" marking] \arrow[dr, phantom, "\widetilde{\varphi_1}\ \simeq"] & D \arrow[d, "x", "\bullet" marking] \\
A' \arrow[r, "f'"'] & B' \arrow[r, "\simeq", "\varphi_1'"'] & D'
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
A \arrow[r, "\varphi_0", "\simeq"'] \arrow[d, "u"', "\bullet" marking] \arrow[dr, phantom, "\widetilde{\varphi_0}\ \simeq"] & C \arrow[r, "g"] \arrow[d, "w", "\bullet" marking] \arrow[dr, phantom, "\beta"] & D \arrow[d, "x", "\bullet" marking] \\
A' \arrow[r, "\simeq", "\varphi_0'"'] \arrow[d, equal, "\bullet" marking] & C' \arrow[r, "g'"'] \arrow[d, phantom, "\varphi'\ \cong"] & D' \arrow[d, equal, "\bullet" marking] \\
A' \arrow[r, "f'"'] & B' \arrow[r, "\simeq", "\varphi_1'"'] & D'
\end{tikzcd}$$

(2) A 2-simplex in $(\mathbb{N}\mathbb{A})_{1,1}$ is an invertible modification $\mu\colon \theta \cong \psi\varphi$ between such horizontal pseudo-natural adjoint equivalences, i.e., the data of four vertically invertible squares in $\mathbb{A}$

$$\begin{tikzcd}
A \arrow[rr, "\theta_0", "\simeq"'] \arrow[d, equal, "\bullet" marking] & \arrow[d, phantom, "\mu_0\ \cong"] & E \arrow[d, equal, "\bullet" marking] \\
A \arrow[r, "\simeq", "\varphi_0"'] & C \arrow[r, "\simeq", "\psi_0"'] & E
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[rr, "\theta_1", "\simeq"'] \arrow[d, equal, "\bullet" marking] & \arrow[d, phantom, "\mu_1\ \cong"] & E \arrow[d, equal, "\bullet" marking] \\
A \arrow[r, "\simeq", "\varphi_1"'] & C \arrow[r, "\simeq", "\psi_1"'] & E
\end{tikzcd}$$

$$\begin{tikzcd}
A \arrow[rr, "\theta_0'", "\simeq"'] \arrow[d, equal, "\bullet" marking] & \arrow[d, phantom, "\mu_0'\ \cong"] & E \arrow[d, equal, "\bullet" marking] \\
A \arrow[r, "\simeq", "\varphi_0'"'] & C \arrow[r, "\simeq", "\psi_0'"'] & E
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[rr, "\theta_1'", "\simeq"'] \arrow[d, equal, "\bullet" marking] & \arrow[d, phantom, "\mu_1'\ \cong"] & E \arrow[d, equal, "\bullet" marking] \\
A \arrow[r, "\simeq", "\varphi_1'"'] & C \arrow[r, "\simeq", "\psi_1'"'] & E
\end{tikzcd}$$

such that

- (μ_0, μ_1) satisfies the pasting equality as in Description 13.1.3 (2) with respect to φ, ψ, and θ,
- (μ_0', μ_1') satisfies the pasting equality as in Description 13.1.3 (2) with respect to φ', ψ', and θ',
- (μ_0, μ_0') satisfies the pasting equality as in Description 13.1.4 (2) with respect to $\widetilde{\varphi_0}$, $\widetilde{\psi_0}$, and $\widetilde{\theta_0}$,
- (μ_1, μ_1') satisfies the pasting equality as in Description 13.1.4 (2) with respect to $\widetilde{\varphi_1}$, $\widetilde{\psi_1}$, and $\widetilde{\theta_1}$.

13.2. **Nerve of a 2-category.** By computing the nerve of a 2-category, we expect to see the *space of objects* at $(m,k) = (0,0)$, the *space of morphisms* at $(m,k) = (1,0)$, and the *space of 2-morphisms* at $(m,k) = (1,1)$, while the space at $(m,k) = (0,1)$ should be weakly equivalent to the space of objects, since the first column of 2-fold complete Segal space is essentially constant.

Let $\mathcal{A}$ be a 2-category. Recall that its nerve is given by the nerve of its associated double category $\mathbb{H}^{\simeq}\mathcal{A}$. We therefore translate Descriptions 13.1.2 to 13.1.5 to this setting. In particular, we first obtain the *space of objects* $(\mathbb{N}\mathbb{H}^{\simeq}\mathcal{A})_{0,0}$, whose 0-simplices are the objects, and whose 1-simplices are the adjoint equivalences of $\mathcal{A}$, as expected by the completeness condition.

Description 13.2.1 ($m = 0$, $k = 0$)**.** We describe the space $(\mathbb{N}\mathbb{H}^{\simeq}\mathcal{A})_{0,0}$.

(0) A 0-simplex in $(\mathbb{N}\mathbb{H}^{\simeq}\mathcal{A})_{0,0}$ is the data of an object $A \in \mathcal{A}$.

(1) A 1-simplex in $(\mathbb{N}\mathbb{H}^{\simeq}\mathcal{A})_{0,0}$ is the data of an adjoint equivalence $A \xrightarrow{\simeq} C$ in $\mathcal{A}$.

(2) A 2-simplex in $(\mathbb{N}\mathbb{H}^{\simeq}\mathcal{A})_{0,0}$ is the data of a 2-isomorphism as in the following diagram.

$$\begin{tikzcd}
A \arrow[rr, "\simeq"] \arrow[dr, "\simeq"'] & \arrow[d, phantom, "\Downarrow\cong"] & E \\
& C \arrow[ur, "\simeq"'] &
\end{tikzcd}$$

As for the *space of morphisms* $(\mathbb{N}\mathbb{H}^{\simeq}\mathcal{A})_{1,0}$, we can see that the completeness condition is now satisfied for $(\mathbb{N}\mathbb{H}^{\simeq}\mathcal{A})_{1,-}$, since vertical morphisms are now adjoint equivalences in $\mathcal{A}$ and they therefore also appear in the horizontal direction.

Description 13.2.2 ($m = 1$, $k = 0$)**.** We describe the space $(\mathbb{N}\mathbb{H}^{\simeq}\mathcal{A})_{1,0}$.

(0) A 0-simplex in $(\mathbb{N}\mathbb{H}^{\simeq}\mathcal{A})_{1,0}$ is the data of a morphism $f\colon A \to B$ in $\mathcal{A}$.

(1) A 1-simplex in $(\mathbb{N}\mathbb{H}^{\simeq}\mathcal{A})_{1,0}$ is the data of two adjoint equivalences and a 2-isomorphism in $\mathcal{A}$ as in the following diagram.

$$\begin{tikzcd}
A \arrow[r, "\simeq"] \arrow[d, "f"'] \arrow[dr, phantom, "\overset{\cong}{\swarrow}"] & C \arrow[d, "g"] \\
B \arrow[r, "\simeq"'] & D
\end{tikzcd}$$

(2) A 2-simplex in $(\mathbb{N}\mathbb{H}^{\simeq}\mathcal{A})_{1,0}$ is the data of two 2-isomorphisms filling the triangles of the following pasting equality.

$$\begin{tikzcd}
A \arrow[rr, "\simeq"] \arrow[dr, "\simeq"'] \arrow[dd, "f"'] & \arrow[d, phantom, "\Downarrow\cong"] & E \arrow[dd, "h"] \\
\arrow[r, phantom, "\overset{\Leftarrow}{\cong}"] & C \arrow[ur, "\simeq"'] \arrow[dd, "g"] \arrow[r, phantom, "\swarrow\cong"] & \\
B \arrow[dr, "\simeq"'] & & F \\
& D \arrow[ur, "\simeq"'] &
\end{tikzcd}
=
\begin{tikzcd}
A \arrow[rr, "\simeq"] \arrow[d, "f"'] & \arrow[d, phantom, "\overset{\cong}{\swarrow}"] & E \arrow[d, "h"] \\
B \arrow[rr, "\simeq"] \arrow[dr, "\simeq"'] & \arrow[d, phantom, "\Downarrow\cong"] & F \\
& D \arrow[ur, "\simeq"'] &
\end{tikzcd}$$

The space $(\mathbb{N}\mathbb{H}^{\simeq}\mathcal{A})_{0,1}$ is actually given by the *space of adjoint equivalences.* Since the "free-living adjoint equivalence" is biequivalent to the point, this space can be interpreted as "homotopically the same" as the space of objects.

Description 13.2.3 ($m = 0$, $k = 1$)**.** We describe the space $(\mathbb{N}\mathbb{H}^{\simeq}\mathcal{A})_{0,1}$.

(0) A 0-simplex in $(\mathbb{N}\mathbb{H}^{\simeq}\mathcal{A})_{0,1}$ is the data of an adjoint equivalence $u\colon A \xrightarrow{\simeq} A'$ in $\mathcal{A}$.

(1) A 1-simplex in $(\mathbb{N}\mathbb{H}^{\simeq}\mathcal{A})_{0,1}$ is the data of a 2-isomorphism as in the following diagram, by Lemma 3.6.8.

$$\begin{tikzcd}
A \arrow[r, "\simeq"] \arrow[d, "u"', "\wr"] \arrow[dr, phantom, "\overset{\cong}{\swarrow}"] & C \arrow[d, "w", "\wr"'] \\
A' \arrow[r, "\simeq"'] & C'
\end{tikzcd}$$

(2) A 2-simplex in $(\mathbb{N}\mathbb{H}^{\simeq}\mathcal{A})_{0,1}$ is the data of two 2-isomorphisms filling the triangles of the following pasting equality.

$$
\begin{tikzcd}[column sep=small]
A \arrow[rr, "\simeq"] \arrow[dr, "\simeq"'] \arrow[dd, "u"', "\wr"] & \arrow[d, phantom, "\Downarrow\cong"] & E \arrow[dd, "y", "\wr"'] \\
& C \arrow[ur, "\simeq"'] \arrow[dd, "w"', "\wr"] \arrow[dl, phantom, "\Leftarrow\cong"] \arrow[dr, phantom, "\Swarrow\cong"] & \\
A' \arrow[dr, "\simeq"'] & & E' \\
& C' \arrow[ur, "\simeq"'] &
\end{tikzcd}
=
\begin{tikzcd}[column sep=small]
A \arrow[rr, "\simeq"] \arrow[dd, "u"', "\wr"] & \arrow[d, phantom, "\Swarrow\cong"] & E \arrow[dd, "y", "\wr"'] \\
& {} & \\
A' \arrow[rr, "\simeq"] \arrow[dr, "\simeq"'] & \arrow[d, phantom, "\Downarrow\cong"] & E' \\
& C' \arrow[ur, "\simeq"'] &
\end{tikzcd}
$$

Finally, we compute the *space of 2-morphisms* $(\mathbb{N}\mathbb{H}^{\simeq}\mathcal{A})_{1,1}$. Although its 0-simplices are not precisely the 2-morphisms of $\mathcal{A}$, homotopically they give the right notion as the vertical morphisms u and v in the square below are adjoint equivalences.

Description 13.2.4. We describe the space $(\mathbb{N}\mathbb{H}^{\simeq}\mathcal{A})_{1,1}$.

(0) A 0-simplex in $(\mathbb{N}\mathbb{H}^{\simeq}\mathcal{A})_{1,1}$ is the data of a 2-morphism in $\mathcal{A}$ as in the following diagram.

$$
\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "u"', "\wr"] \arrow[dr, phantom, "\overset{\alpha}{\Swarrow}"] & B \arrow[d, "v", "\wr"'] \\
A' \arrow[r, "f'"'] & B'
\end{tikzcd}
$$

(1) A 1-simplex in $(\mathbb{N}\mathbb{H}^{\simeq}\mathcal{A})_{1,1}$ is the data of four adjoint equivalences and four 2-isomorphisms in $\mathcal{A}$ as in the following diagram.

$$
\begin{tikzcd}[column sep=small]
& C \arrow[dr, "g"] \arrow[d, phantom, "\Downarrow\cong"] & \\
A \arrow[ur, "\simeq"] \arrow[dr, "f"] \arrow[dd, "u"', "\wr"] & {} & D \arrow[dd, "x", "\wr"'] \\
& B \arrow[ur, "\simeq"'] \arrow[dd, "v"', "\wr"] \arrow[dl, phantom, "\Leftarrow\alpha"] \arrow[dr, phantom, "\Swarrow\cong"] & \\
A' \arrow[dr, "f'"'] & & D' \\
& B' \arrow[ur, "\simeq"'] &
\end{tikzcd}
=
\begin{tikzcd}[column sep=small]
& C \arrow[dr, "g"] \arrow[d, "w"', "\wr"] & \\
A \arrow[ur, "\simeq"] \arrow[dd, "u"', "\wr"] \arrow[r, phantom, "\Swarrow\cong"] & C' \arrow[dr, "g'"] \arrow[r, phantom, "\overset{\Leftarrow}{\beta}"] & D \arrow[dd, "x", "\wr"'] \\
& {} \arrow[d, phantom, "\Downarrow\cong"] & \\
A' \arrow[uur, "\simeq"] \arrow[dr, "f'"'] & {} & D' \\
& B' \arrow[ur, "\simeq"'] &
\end{tikzcd}
$$

(2) A 2-simplex in $(\mathbb{N}\mathbb{H}^{\simeq}\mathcal{A})_{1,1}$ is the data of four 2-isomorphisms filling triangles satisfying relations as described in Description 13.2.2 (2) an Description 13.2.3 (2).

13.3. **Nerve of a horizontal double category.** Finally, we compute the nerve of a horizontal double category $\mathbb{H}\mathcal{A}$ in lower dimensions, where $\mathcal{A}$ is a 2-category, in order to compare it with the nerve $\mathbb{N}\mathbb{H}^{\simeq}\mathcal{A}$ described above. Since $\mathbb{H}\mathcal{A}$ and $\mathbb{H}^{\simeq}\mathcal{A}$ have the same underlying horizontal 2-category, namely $\mathcal{A}$ itself, then the spaces $(\mathbb{N}\mathbb{H}\mathcal{A})_{0,0}$ and $(\mathbb{N}\mathbb{H}\mathcal{A})_{1,0}$ are equal to the spaces $(\mathbb{N}\mathbb{H}^{\simeq}\mathcal{A})_{0,0}$ and $(\mathbb{N}\mathbb{H}^{\simeq}\mathcal{A})_{1,0}$ and they can therefore be described as in Descriptions 13.2.1 and 13.2.2, respectively. In particular, they are the desired *space of objects* and *space of morphisms*.

We now turn our attention to the space $(\mathbb{N}\mathbb{H}\mathcal{A})_{0,1}$. Unlike $(\mathbb{N}\mathbb{H}^{\simeq}\mathcal{A})_{0,1}$, this space has as 0-simplices the objects of $\mathcal{A}$. This prohibits a completeness condition in the vertical direction since equalities are not homotopically good enough.

Description 13.3.1 ($m = 0$, $k = 1$)**.** We describe the space $(\mathbb{N}\mathbb{H}\mathcal{A})_{0,1}$.

(0) A 0-simplex in $(\mathbb{N}\mathbb{H}\mathcal{A})_{0,1}$ is the data of an object $A \in \mathcal{A}$.
(1) A 1-simplex in $(\mathbb{N}\mathbb{H}\mathcal{A})_{0,1}$ is the data of a 2-isomorphism as in the following diagram, by Proposition 3.6.7.

$$\begin{tikzcd}
A \arrow[r, bend left, "\simeq"] \arrow[r, bend right, "\simeq"'] \arrow[r, phantom, "\Downarrow\cong"] & C
\end{tikzcd}$$

(2) A 2-simplex in $(\mathbb{N}\mathbb{H}\mathcal{A})_{0,1}$ is the data of two 2-isomorphisms filling the triangles of the following pasting equality.

$$\begin{tikzcd}[column sep=small]
A \arrow[rr, "\simeq"] \arrow[dr, "\simeq"] \arrow[dr, bend right=60, "\simeq"'] \arrow[dr, phantom, bend right=30, "\Swarrow\cong"] & \arrow[d, phantom, "\Downarrow\cong"] & E \\
& C \arrow[ur, "\simeq"'] \arrow[ur, bend right=60, "\simeq"'] \arrow[ur, phantom, bend right=30, "\Searrow\cong"] &
\end{tikzcd}
=
\begin{tikzcd}[column sep=small]
A \arrow[rr, bend left=40, "\simeq"] \arrow[rr, phantom, bend left=20, "\Downarrow\cong"] \arrow[rr, "\simeq"] \arrow[dr, "\simeq"'] & \arrow[d, phantom, "\Downarrow\cong"] & E \\
& C \arrow[ur, "\simeq"'] &
\end{tikzcd}$$

Finally, we compute the *space of* 2*-morphisms* $(\mathbb{N}\mathbb{H}\mathcal{A})_{1,1}$, which appears to have precisely the 2-morphisms of $\mathcal{A}$ as 0-simplices. However, as explained above, this description is not homotopically well-behaved, since we would also need to consider adjoint equivalences in the vertical direction.

Description 13.3.2. We describe the space $(\mathbb{N}\mathbb{H}\mathcal{A})_{1,1}$.

(0) A 0-simplex in $(\mathbb{N}\mathbb{H}\mathcal{A})_{1,1}$ is the data of a 2-morphism in $\mathcal{A}$

$$\begin{tikzcd}
A \arrow[r, bend left, "f"] \arrow[r, bend right, "f'"'] \arrow[r, phantom, "\Downarrow\alpha"] & B .
\end{tikzcd}$$

(1) A 1-simplex in $(\mathbb{N}\mathbb{H}\mathcal{A})_{1,1}$ is the data of four adjoint equivalences and four 2-isomorphisms in $\mathcal{A}$ as in the following diagram.

$$\begin{tikzcd}[column sep=small]
& C \arrow[dr, "g"] & \\
A \arrow[ur, "\simeq"] \arrow[dr, "f"] \arrow[dr, bend right=60, "f'"'] \arrow[dr, phantom, bend right=30, "\Swarrow\alpha"] & \Downarrow\cong & D \\
& B \arrow[ur, "\simeq"'] \arrow[ur, bend right=60, "\simeq"'] \arrow[ur, phantom, bend right=30, "\Searrow\cong"] &
\end{tikzcd}
=
\begin{tikzcd}[column sep=small]
& C \arrow[dr, "g'"'] \arrow[dr, bend left=60, "g"] \arrow[dr, phantom, bend left=30, "\Swarrow\beta"] & \\
A \arrow[ur, "\simeq"'] \arrow[ur, bend left=60, "\simeq"] \arrow[ur, phantom, bend left=30, "\Searrow\cong"] \arrow[dr, "f'"'] & \Downarrow\cong & D \\
& B \arrow[ur, "\simeq"'] &
\end{tikzcd}$$

(2) A 2-simplex in $(\mathbb{N}\mathbb{H}\mathcal{A})_{1,1}$ is the data of four 2-isomorphisms filling triangles satisfying relations as described in Description 13.2.2 (2) an Description 13.3.1 (2).

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UPHESS BMI FSV, Ecole Polytechnique Fédérale de Lausanne, Station 8, 1015 Lausanne, Switzerland

Email address: `lyne.moser@epfl.ch`