LECTURE NOTES: SYNTHETIC CATEGORY THEORY

Taught in parallel at Universität Hamburg and Universität Regensburg

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Contents

Introduction 3
What is an ∞-category? 3
Axiomatic approach to ∞-categories 3
Course overview 5
I. Strict category theory 6
1. First definitions 6
1.1. Size issues 6
1.2. Strict categories 7
1.3. Functors between strict categories 8
1.4. Natural transformations between functors 9
2. Yoneda lemma 11
3. Adjunctions and equivalences 13
3.1. Adjunctions 13
3.2. Equivalences of strict categories 14
3.3. Stability under isomorphism and invariance under equivalence 15
4. Universal constructions in strict categories 17
4.1. Terminal and initial objects 17
4.2. Products and coproducts 18
4.3. Pullback squares 21
5. Subcategories, localizations, and strict groupoids 24
5.1. Subcategories 24
5.2. Localizations 25
5.3. Strict groupoids 26
6. Discrete right fibrations and cartesian fibrations 27
6.1. Discrete right fibrations 28
6.2. Cartesian fibrations 32
II. Synthetic category theory - Axioms 36
7. Different approaches to foundations 36
8. Basic structure of clans for category theory 37
8.1. Clans for category theory 37
8.2. Families over a category as clans for category theory 38
9. First structural axioms 42
9.1. Equality families 42
9.2. Equivalences 47
9.3. Anima of sections and functor extensionality 49
9.4. Mapping animae and homotopies 52

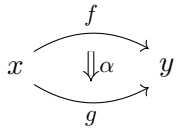
9.5. Homotopy pullbacks 55
References 60

Introduction

This course is an introduction to ∞-*category theory*. ∞-Categories are homotopical generalisations of ordinary categories. They provide a convenient language for organising mathematical objects and their relationships, and for formulating and proving theorems. Over the past few decades, they have become an important tool in areas of algebraic topology, algebraic geometry, number theory, homological algebra, and mathematical physics.

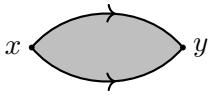
What is an ∞-category?

Informally, an ∞-category consists of

- objects $x, y, \ldots$;
- morphisms $f\colon x \to y$ between objects;
- 2-morphisms  between parallel morphisms
- 3-morphisms between parallel 2-morphisms;
- and so on;

such that all k-morphisms for $k \geq 2$ are invertible, in a suitably weak sense. Composition is associative and unital up to higher invertible morphisms, which themselves satisfy compatibility conditions in all higher dimensions.

A motivating example comes from topology. Given a topological space X, consider the ∞-category whose:

- objects are points of X;
- morphisms are paths between points in X; 
- 2-morphisms are homotopies between paths, fixing the endpoints; 
- 3-morphisms are homotopies between homotopies, fixing the boundaries;
- and so on.

Composition is given by concatenating paths and homotopies. Concatenation of paths is associative and unital up to homotopy: different ways of composing the same paths need not give exactly the same parametrised path, but they can be continuously deformed into one another. Moreover, reversing a path gives an inverse up to homotopy. Thus, in this example, even the morphisms are invertible in the appropriate sense. Such a structure is called an ∞-*groupoid*.

Axiomatic approach to ∞-categories

Traditionally, ∞-categories have been studied through concrete *models*, such as quasi-categories [Joy08, Lur09] and complete Segal spaces [Rez01, Ras23]. These models make the informal description above precise using constructions within set-theoretic foundations. Encoding ∞-categories in set-theoretic terms can make them difficult to work with: questions about intrinsic categorical properties may become hard combinatorial problems involving the chosen model.

In this course, we will take an axiomatic approach to ∞-category theory. Our approach differs from the model-based one described above in the following aspects: the foundations, including the structure identity principle, and the synthetic approach.

Homotopy type theory. The first aspect is a change of foundations. In ordinary set-based mathematics, given a set X and elements $x, y \in X$, the expression $x = y$ is a proposition: it is either true or false, and carries no further structure describing an identification between x and y.

In *homotopy type theory*, our basic objects are types. We write $x : X$ to say that x is a term, or element, of the type X. Given terms $x, y : X$, the expression

$$x = y$$

is itself a type, called an *identity type*. Its terms can be thought of as paths identifying x and y. Given two such terms $p, q : x = y$, we can form another identity type

$$p = q,$$

whose terms can be thought of as homotopies between these paths. This process continues indefinitely.

A type therefore already carries higher-dimensional structure: terms, identifications between terms, identifications between identifications, and so on. These identifications behave like paths and higher homotopies in a space, so types naturally behave as ∞-groupoids. The higher structure is built into our foundations.

The structure identity principle. When studying mathematical structures in classical foundations, one must distinguish between properties that are invariant under isomorphism and those that depend on the chosen presentation. For example, given a group G, the property "G contains an element of order 2" is invariant under group isomorphism, whereas "the identity element of G is the number 0" is not. Indeed, the groups $(\{0, 1\}, +\text{mod } 2)$ and $(\{1, -1\}, \cdot)$ are isomorphic, but the latter property holds only for the first. Both properties can be expressed in set-theoretic foundations, but only the first describes the abstract group structure.

An inbuild feature of equality, both in set-theoretic foundations and in homotopy type theory, is the *indiscernibility principle*: if $x = y$, then any property of x also holds for y, and any additional structure on x can be transported to y, and vice-versa.

Homotopy type theory allows for an additional feature that is inconceivable in classical foundations: namely the *structure identity principle*. It connects equality with the mathematical notion of having the same structure. For example, given groups G and H, the terms of the identity type $G = H$ correspond precisely to group isomorphisms from G to H. Thus, giving an identification of two groups amounts to giving an isomorphism between them. The identity type records not just whether the groups can be identified, but also how they are identified.

Combined with indiscernibility, this implies that every property or construction of groups expressible in this framework is invariant under isomorphism. Invariance under isomorphism is therefore built into the foundations.

Synthetic category theory. The second aspect is the synthetic viewpoint itself. Rather than beginning with a concrete construction of an ∞-category, we will specify axioms describing how ∞-categories and the relationships between them behave. Our emphasis will therefore shift from the question "What is an ∞-category?" to "How does the collection of ∞-categories behave?" This requires a sufficiently rich axiomatic setting, but allows us to develop the theory directly from its structural properties.

To do this, we extend homotopy type theory so that we can also talk about non-invertible morphisms. In this framework, ∞-categories are basic types, and we retain the core principles of homotopy type theory, such as the structure identity principle.

Course overview

We will begin with a brief introduction to ordinary category theory, which we will call *strict category theory* throughout these notes. The aim is to become familiar with examples and basic constructions before encountering their ∞-categorical counterparts. We will then survey the foundations of homotopy type theory, following [Rij25, Uni13]. Next, we will pass to an extension of this type theory, which is inspired from [Riv25] and formulate the axioms of synthetic category theory in this extended type theory. This approach to the foundations differs from the one presented in [CCNW21]. We will then develop category theory in this synthetic setting, following [CCNW21].

We plan to cover adjunctions, localisations, left and cocartesian fibrations, the Yoneda lemma, left and right Kan extensions, and limits and colimits. These names need not be familiar at this stage: the course will introduce them progressively, explaining both their meaning and their role.

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I. Strict category theory

In this first part, we introduce or recall various definitions and constructions in ordinary category theory that will later be generalized to the ∞-setting. Our aim is to build intuition for the kinds of structures and results we will encounter, so that the abstract setting introduced later feels less intimidating. We will also highlight some of the limitations of strict categories and explain why they can be too rigid to do category theory. For instance, working with *sets* of objects and morphisms encourages us to formulate notions in terms of equality of objects, which can lead to constructions that are not invariant under equivalences of strict categories. Understanding this distinction, and the difficulties it creates, will motivate the synthetic approach to category theory developed later in the course.

The topics covered in this part have been chosen specifically with the axioms of synthetic category theory in mind.

1. First definitions

We start by introducing strict categories, functors, and natural transformations, together with some examples. Throughout the course, we refer to ordinary categories as *strict categories*, reserving the term *categories* for their ∞-counterparts, namely synthetic categories, which will play a much more central role in this course.

1.1. Size issues

Russell's paradox tells us that the collection of all sets is not a set. To handle this issue, we will work with three nested universes of sets.

Definition 1.1. A **set-universe** is a set $\mathcal{U}$ whose elements are sets satisfying the following properties:

- (Infinite) the set of natural numbers $\mathbb{N}$ belongs to $\mathcal{U}$.
- (Subset closure) if $A \in \mathcal{U}$ and $B \subseteq A$, then $B \in \mathcal{U}$;
- (Power set closure) if $A \in \mathcal{U}$, then its power set $\mathcal{P}(A) \in \mathcal{U}$;
- (Union closure) if $I \in \mathcal{U}$ and $A_i \in \mathcal{U}$, for each $i \in I$, then $\bigcup_{i \in I} A_i \in \mathcal{U}$.

Remark 1.2. A similar proof to Russell's paradox applies here to show that $\mathcal{U} \notin \mathcal{U}$. By contradiction, suppose that $\mathcal{U} \in \mathcal{U}$. Then we can form the subset of $\mathcal{U}$

$$R \coloneqq \{A \in \mathcal{U} \mid A \notin A\} \subseteq \mathcal{U}$$

and, by hypothesis, we have that $R \in \mathcal{U}$. But this gives a contradiction since we have that $R \in R$ if and only if $R \notin R$.

Convention 1.3. We will assume that there exist set-universes

$$\mathcal{U}_{\text{small}} \subseteq \mathcal{U}_{\text{large}} \subseteq \mathcal{U}_{\text{huge}}$$

such that $\mathcal{U}_{\text{small}} \in \mathcal{U}_{\text{large}}$ and $\mathcal{U}_{\text{large}} \in \mathcal{U}_{\text{huge}}$.

Definition 1.4. We introduce the following terminology.

- A **set** or a **small set** is an element of $\mathcal{U}_{\text{small}}$.
- A **large set** is an element of $\mathcal{U}_{\text{large}}$.
- A **huge set** is an element of $\mathcal{U}_{\text{huge}}$.

Remark 1.5. Our definitions, statements, and constructions are relative to a chosen universe. We will sometimes apply these at a larger universe level, regarding "large" as the new "small" and "huge" as the new "large".

1.2. **Strict categories**

Definition 1.6. A **strict category** $\mathcal{C}$ consists of

- a large set $\mathrm{Ob}(\mathcal{C})$ of **objects** $x, y, z, \ldots$;
- for all objects $x, y \in \mathcal{C}$, a large set $\mathcal{C}(x, y)$ of **morphisms** $f\colon x \to y$ from x to y;
- for each object $x \in \mathcal{C}$, an **identity morphism** $\mathrm{id}_x\colon x \to x$;
- for all morphisms $f\colon x \to y$ and $g\colon y \to z$ in $\mathcal{C}$, a **composite morphism** $g \circ f\colon x \to z$; we also write simply gf for the composite;

satisfying the following axioms:

- composition is unital: for each morphism $f\colon x \to y$,
$$f \circ \mathrm{id}_x = f \quad \text{and} \quad \mathrm{id}_y \circ f = f;$$
- composition is associative: for all morphisms $f\colon x \to y$, $g\colon y \to z$, and $h\colon z \to t$ in $\mathcal{C}$,
$$(h \circ g) \circ f = h \circ (g \circ f).$$

Remark 1.7. By convention, a strict category has a large set of objects, which we refer to simply as its **object set**, and large sets of morphisms, which we refer to simply as its **hom sets**.

We will also consider **huge strict categories** with huge sets of objects and of morphisms.

Definition 1.8. A strict category $\mathcal{C}$ is

- **locally small** if, for all objects $x, y \in \mathcal{C}$, the hom set $\mathcal{C}(x, y)$ is small;
- **small** if its object set is small and it is locally small.

Example 1.9. We have the following examples of strict categories.

- The strict category $\mathcal{S}et$ of sets and functions.
- The strict category $\mathcal{S}et_\bullet$ of pointed sets and basepoint-preserving functions.
- For a field k, the strict category $\mathcal{V}ect_k$ of k-vector spaces and k-linear maps.
- The strict category $\mathcal{T}op$ of topological spaces and continuous functions.

All these strict categories are locally small, but not small.

Definition 1.10. Let $\mathcal{C}$ be a strict category. A morphism $f\colon x \to y$ in $\mathcal{C}$ is an **isomorphism** if there is a morphism $g\colon y \to x$ in $\mathcal{C}$ such that $g \circ f = \mathrm{id}_x$ and $f \circ g = \mathrm{id}_y$. We write $f\colon x \xrightarrow{\cong} y$ or $f\colon x \cong y$. We call g an **inverse** of f.

We say that two objects x and y in $\mathcal{C}$ are **isomorphic** if there is an isomorphism $f\colon x \xrightarrow{\cong} y$ in $\mathcal{C}$. We write $x \cong y$.

Example 1.11. The strict categories in Example 1.9 have the following isomorphisms.

- An isomorphism in $\mathcal{S}et$ is a bijection.
- An isomorphism in $\mathcal{S}et_\bullet$ is a basepoint-preserving bijection.
- An isomorphism in $\mathcal{V}ect_k$ is a linear bijection.
- An isomorphism in $\mathcal{T}op$ is a homeomorphism.

Proposition 1.12. *Let $\mathcal{C}$ be a strict category. An isomorphism $f\colon x \to y$ in $\mathcal{C}$ has a unique inverse, which we denote by $f^{-1}\colon y \to x$.*

Proof. Suppose that $g, h\colon y \to x$ are two inverses of f in $\mathcal{C}$. Then
$$g = g \circ \mathrm{id}_y = g \circ (f \circ h) = (g \circ f) \circ h = \mathrm{id}_x \circ h = h. \qquad \square$$

1.3. **Functors between strict categories**

Definition 1.13. Given strict categories $\mathcal{C}$ and $\mathcal{D}$, a **functor** $F\colon \mathcal{C} \to \mathcal{D}$ consists of:

- a function $\mathrm{Ob}(\mathcal{C}) \to \mathrm{Ob}(\mathcal{D})$ sending an object $x \in \mathcal{C}$ to an object $Fx \in \mathcal{D}$;
- for all objects $x, y \in \mathcal{C}$, a function $\mathcal{C}(x,y) \to \mathcal{D}(Fx, Fy)$ sending a morphism $f\colon x \to y$ in $\mathcal{C}$ to a morphism $Ff\colon Fx \to Fy$ in $\mathcal{D}$;

satisfying the following axioms:

- F preserves identities: for each object $x \in \mathcal{C}$,
$$F(\mathrm{id}_x) = \mathrm{id}_{Fx};$$
- F preserves composites: for all morphisms $f\colon x \to y$ and $g\colon y \to z$ in $\mathcal{C}$,
$$F(g \circ f) = Fg \circ Ff.$$

Example 1.14. We have the following examples of functors.

- The forgetful functors $U\colon \mathcal{S}et_\bullet \to \mathcal{S}et$, $U\colon \mathcal{V}ect_k \to \mathcal{S}et$ and $U\colon \mathcal{T}op \to \mathcal{S}et$ sending a pointed set, a k-vector space, or a topological space to its underlying set.
- The functor $(-)_+\colon \mathcal{S}et \to \mathcal{S}et_\bullet$ which freely adjoins a basepoint to a set S, namely $S_+ \coloneqq (S \amalg \{*\}, *)$.
- The functor $\bigoplus_{(-)} k\colon \mathcal{S}et \to \mathcal{V}ect_k$ which sends a set S to the k-vector space with basis S, namely $\bigoplus_S k$.
- The discrete topology functor $D\colon \mathcal{S}et \to \mathcal{T}op$ equipping a set S with the discrete topology, namely $(S, \mathcal{P}(S))$, where $\mathcal{P}(S)$ is the set of subsets of S.
- The indiscrete topology functor $I\colon \mathcal{S}et \to \mathcal{T}op$ equipping a set S with the indiscrete topology, namely $(S, \{\emptyset, S\})$.

Proposition 1.15. *Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor between strict categories. Then F preserves isomorphisms, i.e., if f is an isomorphism in $\mathcal{C}$, then Ff is an isomorphism in $\mathcal{D}$.*

Proof. Exercise. □

Construction 1.16. There is a strict category $\mathcal{C}at_s$ with

- object set $\mathrm{Ob}(\mathcal{C}at_s)$ consisting of the small strict categories;
- for all objects $\mathcal{C}, \mathcal{D} \in \mathcal{C}at_s$, hom set $\mathcal{C}at_s(\mathcal{C}, \mathcal{D})$ consisting of the functors $\mathcal{C} \to \mathcal{D}$;
- for each object $\mathcal{C} \in \mathcal{C}at_s$, **identity functor** $\mathrm{id}_{\mathcal{C}}\colon \mathcal{C} \to \mathcal{C}$ acting as the identity on objects and on morphisms;
- for all morphisms $F\colon \mathcal{C} \to \mathcal{D}$ and $G\colon \mathcal{D} \to \mathcal{E}$ in $\mathcal{C}at_s$, **composite functor** $GF\colon \mathcal{C} \to \mathcal{E}$ sending
  - an object $x \in \mathcal{C}$ to the object $G(F(x)) \in \mathcal{E}$;
  - a morphism $f\colon x \to y$ in $\mathcal{C}$ to the morphism $G(F(f))\colon G(F(x)) \to G(F(y))$ in $\mathcal{E}$.

This is a well-defined strict category (Exercise).

Similarly, there is a huge strict category $\mathcal{CAT}_s$ of (large) strict categories and functors.

Remark 1.17. The strict category $\mathcal{C}at_s$ of small strict categories and functors defines an object of $\mathcal{CAT}_s$. As for the other examples, the strict category $\mathcal{C}at_s$ is locally small, but not small.

Construction 1.18. Given a strict category $\mathcal{C}$, there is a strict category $\mathcal{C}^{\mathrm{op}}$, called the **opposite strict category** of $\mathcal{C}$, with

- object set $\mathrm{Ob}(\mathcal{C})$;

- for all objects $x, y \in \mathcal{C}$, hom set
$$\mathcal{C}^{\mathrm{op}}(x, y) \coloneqq \mathcal{C}(y, x);$$
given a morphism $f\colon y \to x$ in $\mathcal{C}$, we write $f^{\mathrm{op}}\colon x \to y$ for the corresponding morphism in $\mathcal{C}^{\mathrm{op}}$;
- for each object $x \in \mathcal{C}$, identity morphism $(\mathrm{id}_x)^{\mathrm{op}}$;
- for all morphisms $f^{\mathrm{op}}\colon x \to y$ and $g^{\mathrm{op}}\colon y \to z$ in $\mathcal{C}^{\mathrm{op}}$, composite morphism
$$g^{\mathrm{op}} \circ f^{\mathrm{op}} \coloneqq (f \circ g)^{\mathrm{op}}\colon x \to z.$$

This is a well-defined strict category (Exercise).

Remark 1.19. A **contravariant functor** from $\mathcal{C}$ to $\mathcal{D}$ is a functor $F\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{D}$. It consists of

- a function $\mathrm{Ob}(\mathcal{C}) \to \mathrm{Ob}(\mathcal{D})$ sending an object $x \in \mathcal{C}$ to an object $Fx \in \mathcal{D}$;
- for all objects $x, y \in \mathcal{C}$, a function $\mathcal{C}(x, y) \to \mathcal{D}(Fy, Fx)$ sending a morphism $f\colon x \to y$ in $\mathcal{C}$ to a morphism $Ff\colon Fy \to Fx$ in $\mathcal{D}$;

satisfying the following axioms:

- F preserves identities;
- F preserves compositions: for all morphisms $f\colon x \to y$ and $g\colon y \to z$ in $\mathcal{C}$,
$$F(g \circ f) = Ff \circ Fg.$$

Construction 1.20. Let $\mathcal{C}$ be a locally small strict category and $c \in \mathcal{C}$ be an object. There is a functor
$$\mathcal{C}(-, c)\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{S}et,$$
called the **representable functor** at c, sending:

- an object $x \in \mathcal{C}$ to the (small) set of morphisms $\mathcal{C}(x, c)$;
- a morphism $f\colon x \to y$ in $\mathcal{C}$ to the function
$$f^*\colon \mathcal{C}(y, c) \to \mathcal{C}(x, c), \quad (g\colon y \to c) \mapsto (gf\colon x \to c).$$

This is a well-defined functor (Exercise).

1.4. Natural transformations between functors

Definition 1.21. Given functors $F, G\colon \mathcal{C} \to \mathcal{D}$ between strict categories, a **natural transformation** $\alpha\colon F \Rightarrow G$ consists of:

- for each object $x \in \mathcal{C}$, a morphism $\alpha_x\colon Fx \to Gx$ in $\mathcal{D}$, called the **component** of α at x;

satisfying the following axiom:

- α is natural: for each morphism $f\colon x \to y$ in $\mathcal{C}$, the following square commutes
$$\begin{array}{ccc} Fx & \xrightarrow{\alpha_x} & Gx \\ {\scriptstyle Ff}\downarrow & & \downarrow{\scriptstyle Gf} \\ Fy & \xrightarrow[\alpha_y]{} & Gy \end{array}$$
i.e., $Gf \circ \alpha_x = \alpha_y \circ Ff$.

Construction 1.22. Let $\mathcal{C}$ be a locally small strict category and $f\colon c \to d$ be a morphism in $\mathcal{C}$. There is a natural transformation
$$\mathcal{C}(-, f)\colon \mathcal{C}(-, c) \Rightarrow \mathcal{C}(-, d)$$

whose component at an object $x \in \mathcal{C}$ is the function

$$f_* \colon \mathcal{C}(x,c) \to \mathcal{C}(x,d), \quad (g \colon x \to c) \mapsto (fg \colon x \to d).$$

This is well-defined (Exercise).

Definition 1.23. Given functors $F, G \colon \mathcal{C} \to \mathcal{D}$ between strict categories, a natural transformation $\alpha \colon F \Rightarrow G$ is a **natural isomorphism** if, for each object $x \in \mathcal{C}$, its component $\alpha_x \colon Fx \to Gx$ is an isomorphism in $\mathcal{D}$.

We sometimes say that, for all objects $x \in \mathcal{C}$, there is an isomorphism $Fx \cong Gx$ which is **natural** in x.

Construction 1.24. Given strict categories $\mathcal{C}$ and $\mathcal{D}$, there is a strict category $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$, called the **functor strict category**, with

- object set $\mathrm{Ob}(\mathrm{Fun}(\mathcal{C}, \mathcal{D})) \coloneqq \mathcal{CAT}_s(\mathcal{C}, \mathcal{D})$ consisting of the functors $\mathcal{C} \to \mathcal{D}$;
- for all objects $F, G \in \mathrm{Fun}(\mathcal{C}, \mathcal{D})$, hom set $\mathrm{Fun}(\mathcal{C}, \mathcal{D})(F, G)$ consisting of the natural transformations $F \Rightarrow G$;
- for each object $F \in \mathrm{Fun}(\mathcal{C}, \mathcal{D})$, **identity natural transformation** $\mathrm{id}_F \colon F \Rightarrow F$ having component at an object $c \in \mathcal{C}$ given by
$$(\mathrm{id}_F)_c \coloneqq \mathrm{id}_{Fc} \colon Fc \to Fc;$$
- for all morphisms $\alpha \colon F \Rightarrow G$ and $\beta \colon G \Rightarrow H$ in $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$, **composite natural transformation** $\beta \circ \alpha \colon F \Rightarrow H$ having component at an object $c \in \mathcal{C}$ given by the composite
$$(\beta \circ \alpha)_c \coloneqq \beta_c \circ \alpha_c \colon Fc \to Hc.$$

This is a well-defined strict category (Exercise).

Proposition 1.25. *Given strict categories $\mathcal{C}$ and $\mathcal{D}$, a natural transformation is an isomorphism in $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ if and only if it is a natural isomorphism.*

Proof. Exercise. □

Construction 1.26. The **terminal strict category** is the strict category $[0]$ with a single object $*$ and a single morphism id_*.

Proposition 1.27. *Given a strict category $\mathcal{C}$, there is an isomorphism of strict categories*

$$\mathrm{Fun}([0], \mathcal{C}) \cong \mathcal{C}$$

which is natural in $\mathcal{C}$.

Proof. Exercise. □

Remark 1.28. As a consequence of this result, we have that an object $c \in \mathcal{C}$ corresponds to a unique functor $c \colon [0] \to \mathcal{C}$, which we also call c.

Construction 1.29. Given a functor $F \colon \mathcal{C} \to \mathcal{D}$ between strict categories and a strict category $\mathcal{E}$, there is a functor

$$F^* \colon \mathrm{Fun}(\mathcal{D}, \mathcal{E}) \to \mathrm{Fun}(\mathcal{C}, \mathcal{E}),$$

called the **precomposition functor** with F, sending

- a functor $G \colon \mathcal{D} \to \mathcal{E}$ to the composite $GF \colon \mathcal{C} \to \mathcal{E}$;
- a natural transformation $\alpha \colon G \Rightarrow H$ between functors $G, H \colon \mathcal{D} \to \mathcal{E}$ to the natural transformation $\alpha_F \colon GF \Rightarrow HF$ whose component at an object $x \in \mathcal{C}$ is the morphism in $\mathcal{E}$
$$\alpha_{Fx} \colon G(Fx) \to H(Fx).$$

There is also a functor

$$F_*\colon \mathrm{Fun}(\mathcal{E},\mathcal{C}) \to \mathrm{Fun}(\mathcal{E},\mathcal{D}),$$

called the **postcomposition functor** with F, sending

- a functor $G\colon \mathcal{E} \to \mathcal{C}$ to the composite $FG\colon \mathcal{E} \to \mathcal{D}$;
- a natural transformation $\alpha\colon G \Rightarrow H$ between functors $G, H\colon \mathcal{E} \to \mathcal{C}$ to the natural transformation $F\alpha\colon FG \Rightarrow FH$ whose component at an object $x \in \mathcal{E}$ is the morphism in $\mathcal{D}$

$$F(\alpha_x)\colon F(Gx) \to F(Hx).$$

These are well-defined functors (Exercise).

2. Yoneda lemma

We now turn to one of the fundamental results of category theory, the *Yoneda lemma*. An important consequence of this result is that an object in a strict category is determined, up to isomorphism, by its representable functor. Hence we can understand an object (up to isomorphism) through its relationships with other objects. It also highlights the central role of the strict category $\mathcal{S}et$ in strict category theory. For instance, we can detect isomorphisms in a strict category by proving that certain functions between sets are bijections.

Theorem 2.1 (Yoneda lemma)**.** *Let $\mathcal{C}$ be a locally small strict category. For all objects $c \in \mathcal{C}$ and functors $F\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{S}et$, there is a bijection*

$$\mathrm{Fun}(\mathcal{C}^{\mathrm{op}},\mathcal{S}et)(\mathcal{C}(-,c),F) \cong Fc$$

which is natural in both c and F.

Proof. Evaluation at the identity morphism defines a function

$$\mathrm{Fun}(\mathcal{C}^{\mathrm{op}},\mathcal{S}et)(\mathcal{C}(-,c),F) \to Fc, \quad (\alpha\colon \mathcal{C}(-,c) \Rightarrow F) \mapsto \alpha_c(\mathrm{id}_c),$$

where $\alpha_c\colon \mathcal{C}(c,c) \to Fc$ is the function given by the component of α at c.

Next, consider the function

$$Fc \to \mathrm{Fun}(\mathcal{C}^{\mathrm{op}},\mathcal{S}et)(\mathcal{C}(-,c),F), \quad u \mapsto \alpha_u$$

sending an element $u \in Fc$ to $\alpha_u\colon \mathcal{C}(-,c) \Rightarrow F$ with component at an object $x \in \mathcal{C}$ given by the function

$$(\alpha_u)_x\colon \mathcal{C}(x,c) \to Fx, \quad (f\colon x \to c) \mapsto Ff(u),$$

where $Ff\colon Fc \to Fx$ is the function given by the image of f under F. This is a well-defined natural transformation (Exercise).

These functions are mutually inverse and natural in c and F (Exercise). □

Construction 2.2. Let $\mathcal{C}$ be a locally small strict category. There is a functor

$$\mathbf{y}_{\mathcal{C}}\colon \mathcal{C} \to \mathrm{Fun}(\mathcal{C}^{\mathrm{op}},\mathcal{S}et)$$

called the **Yoneda functor**, sending

- an object $c \in \mathcal{C}$ to the representable functor $\mathcal{C}(-,c)\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{S}et$;
- a morphism $f\colon c \to d$ in $\mathcal{C}$ to the natural transformation $\mathcal{C}(-,f)\colon \mathcal{C}(-,c) \Rightarrow \mathcal{C}(-,d)$.

This is well-defined (Exercise).

Definition 2.3. A functor $F\colon \mathcal{C} \to \mathcal{D}$ between strict categories is **fully faithful** if, for all objects $x, y \in \mathcal{C}$, the induced function

$$\mathcal{C}(x,y) \to \mathcal{D}(Fx,Fy)$$

is a bijection.

Corollary 2.4. *Let $\mathcal{C}$ be a locally small strict category. The Yoneda functor*

$$\mathbf{y}_{\mathcal{C}} \colon \mathcal{C} \to \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}et)$$

is fully faithful.

Proof. Exercise. □

Proposition 2.5. *Let $F \colon \mathcal{C} \to \mathcal{D}$ be a fully faithful functor. Then F reflects isomorphisms, i.e., if f is a morphism in $\mathcal{C}$ such that Ff is an isomorphism in $\mathcal{D}$, then f is an isomorphism in $\mathcal{C}$.*

Moreover, two objects $x, y \in \mathcal{C}$ are isomorphic if and only if their images $Fx, Fy \in \mathcal{D}$ are isomorphic.

Proof. Exercise. □

Corollary 2.6. *Let $\mathcal{C}$ be a locally small strict category. The following are equivalent for a morphism $f \colon c \to d$ in $\mathcal{C}$.*

(i) The morphism f is an isomorphism in $\mathcal{C}$.
(ii) For all objects $x \in \mathcal{C}$, the induced function

$$f_* \colon \mathcal{C}(x, c) \to \mathcal{C}(x, d)$$

is a bijection.
(iii) For all objects $x \in \mathcal{C}$, the induced function

$$f^* \colon \mathcal{C}(d, x) \to \mathcal{C}(c, x)$$

is a bijection.

Proof. Since the Yoneda functor $\mathbf{y}_{\mathcal{C}} \colon \mathcal{C} \to \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}et)$ is fully faithful, it preserves and reflects isomorphisms. Hence the morphism $f \colon c \to d$ is an isomorphism in $\mathcal{C}$ if and only if the natural transformation $\mathcal{C}(-, f) \colon \mathcal{C}(-, c) \Rightarrow \mathcal{C}(-, d)$ is an isomorphism in $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}et)$. By Proposition 1.25, this holds if and only if it is a natural isomorphism, i.e., for every object $x \in \mathcal{C}$, the induced function

$$f_* \colon \mathcal{C}(x, c) \to \mathcal{C}(x, d)$$

is a bijection. This proves that (i) and (ii) are equivalent.

Next, observe that f is an isomorphism in $\mathcal{C}$ if and only if $f^{\mathrm{op}} \colon d \to c$ is an isomorphism in $\mathcal{C}^{\mathrm{op}}$ (Exercise). Then the equivalence between (i) and (ii) for $\mathcal{C}^{\mathrm{op}}$ implies that f^{op} is an isomorphism in $\mathcal{C}^{\mathrm{op}}$ if and only if, for every object $x \in \mathcal{C}$, the induced function

$$f^{\mathrm{op}}_* = f^* \colon \mathcal{C}^{\mathrm{op}}(x, d) = \mathcal{C}(d, x) \to \mathcal{C}(c, x) = \mathcal{C}^{\mathrm{op}}(x, c)$$

is a bijection. This proves that (i) and (iii) are equivalent. □

Corollary 2.7. *Let $\mathcal{C}$ be a locally small strict category. Two objects $x, y \in \mathcal{C}$ are isomorphic if and only if their representable functors $\mathcal{C}(-, x)$ and $\mathcal{C}(-, y)$ are naturally isomorphic.*

Proof. This follows from the preceding proposition applied to the fully faithful Yoneda functor, together with Proposition 1.25. □

3. Adjunctions and equivalences

We now turn to two important concepts in strict category theory, namely *adjunctions* and *equivalences*. Adjunctions provide a way to describe universal constructions and organize them into functors. Equivalences express when two strict categories should be thought of as being the "same".

From a categorical point of view, two isomorphic objects should be interchangeable, even if they are not equal. However, an isomorphism of strict categories requires a bijection between the sets of objects, making it too restrictive for this purpose. Equivalences provide a weakening of the notion of isomorphism between strict categories and capture the intended notion of sameness: they detect objects only up to isomorphism, rather than up to equality.

A fundamental difficulty with the set-theoretic description of strict categories is that equality and isomorphism of objects remain distinct notions. Consequently, properties and constructions can depend on literal equalities between objects and fail to be well-behaved with respect to isomorphisms. We will illustrate this with examples of properties and constructions that are not invariant under replacing objects by isomorphic ones or strict categories by equivalent ones. Further examples will appear in the following sections.

The synthetic approach will address this foundational issue by not allowing a distinction between equality and isomorphism of objects, so that invariance under isomorphism is built into the theory. Similarly, the appropriate notion of isomorphism between synthetic categories will generalize that of an equivalence between strict categories.

3.1. **Adjunctions**

Definition 3.1. Given strict categories $\mathcal{C}$ and $\mathcal{D}$, an **adjunction** between $\mathcal{C}$ and $\mathcal{D}$ consists of:

- a functor $L\colon \mathcal{C} \to \mathcal{D}$, called the **left adjoint**;
- a functor $R\colon \mathcal{D} \to \mathcal{C}$, called the **right adjoint**;
- a natural transformation $\eta\colon \mathrm{id}_{\mathcal{C}} \Rightarrow RL$, called the **unit**;
- a natural transformation $\varepsilon\colon LR \Rightarrow \mathrm{id}_{\mathcal{D}}$, called the **counit**;

such that, for all objects $x \in \mathcal{C}$ and $y \in \mathcal{D}$, the following triangles commute in $\mathcal{D}$ and $\mathcal{C}$, respectively:



We write $L\colon \mathcal{C} \rightleftarrows \mathcal{D} : R$ or $L \dashv R$ for such an adjunction.

Example 3.2. We have the following examples of adjunctions between strict categories.

- The free-forgetful adjunctions

$$(-)_+\colon \mathcal{S}et \rightleftarrows \mathcal{S}et_\bullet : U \quad \text{and} \quad \textstyle\bigoplus_{(-)} k\colon \mathcal{S}et \rightleftarrows \mathcal{V}ect_k : U.$$

- The discrete-forgetful and forgetful-indiscrete adjunctions

$$D\colon \mathcal{S}et \rightleftarrows \mathcal{T}op : U \quad \text{and} \quad U\colon \mathcal{T}op \rightleftarrows \mathcal{S}et : I.$$

See Example 1.14 for the definition of the functors involved.

Theorem 3.3 (Adjunctions representably)**.** *Given functors $L\colon \mathcal{C} \to \mathcal{D}$ and $R\colon \mathcal{D} \to \mathcal{C}$ between strict categories, there is a unit and a counit making the pair (L, R) into an adjunction $L \dashv R$ if and only if, for all objects $x \in \mathcal{C}$ and $y \in \mathcal{D}$, there is a bijection*

$$\mathcal{D}(Lx, y) \cong \mathcal{C}(x, Ry)$$

which is natural in x and y.

Proof. Given an adjunction $L \dashv R$, the unit $\eta\colon \mathrm{id}_{\mathcal{C}} \Rightarrow RL$ induces a function

$$\mathcal{D}(Lx, y) \xrightarrow{R} \mathcal{C}(RLx, Ry) \xrightarrow{(\eta_x)^*} \mathcal{C}(x, Ry), \quad (g\colon Lx \to y) \mapsto R(g) \circ \eta_x,$$

and the counit $\varepsilon\colon LR \Rightarrow \mathrm{id}_{\mathcal{D}}$ induces a function

$$\mathcal{C}(x, Ry) \xrightarrow{L} \mathcal{D}(Lx, LRy) \xrightarrow{(\varepsilon_y)_*} \mathcal{D}(Lx, y), \quad (f\colon x \to Ry) \mapsto \varepsilon_y \circ L(f).$$

These are mutually inverse and natural in x and y (Exercise).

Conversely, suppose that, for all objects $x \in \mathcal{C}$ and $y \in \mathcal{D}$, we are given natural bijections

$$\varphi_{x,y}\colon \mathcal{D}(Lx, y) \cong \mathcal{C}(x, Ry).$$

Define $\eta\colon \mathrm{id}_{\mathcal{C}} \Rightarrow RL$ whose component at an object $x \in \mathcal{C}$ is given by

$$\eta_x \coloneqq \varphi_{x,Lx}(\mathrm{id}_{Lx})\colon x \to RLx.$$

Define $\varepsilon\colon LR \Rightarrow \mathrm{id}_{\mathcal{D}}$ whose component at an object $y \in \mathcal{D}$ is given by the morphism

$$\varepsilon_y \coloneqq \varphi^{-1}_{Ry,y}(\mathrm{id}_{Ry})\colon LRy \to y.$$

These are well-defined natural transformations satisfying the triangle identities (Exercise). This proves that $L \dashv R$ is an adjunction. □

3.2. **Equivalences of strict categories**

Recall that a functor $F\colon \mathcal{C} \to \mathcal{D}$ between strict categories is an isomorphism in $\mathcal{CAT}_s$ if there is a functor $G\colon \mathcal{D} \to \mathcal{C}$ such that $GF = \mathrm{id}_{\mathcal{C}}$ and $FG = \mathrm{id}_{\mathcal{D}}$.

Since we have a notion of morphisms between functors, namely the natural transformations, we can weaken the notion of sameness between strict categories as follows.

Definition 3.4. A functor $F\colon \mathcal{C} \to \mathcal{D}$ between strict categories is an **equivalence** if there is a functor $G\colon \mathcal{D} \to \mathcal{C}$ and natural isomorphisms $\eta\colon \mathrm{id}_{\mathcal{C}} \overset{\cong}{\Longrightarrow} GF$ and $\varepsilon\colon FG \overset{\cong}{\Longrightarrow} \mathrm{id}_{\mathcal{D}}$.

Example 3.5. The following classes of functors give special cases of equivalences.

- Any isomorphism in $\mathcal{CAT}_s$ is an equivalence of strict categories.
- Given an adjunction $L \dashv R$ such that the unit and the counit are natural isomorphisms, both L and R are equivalences of strict categories; we refer to this as an **adjoint equivalence**.

Fact: every equivalence of strict categories can be promoted to an adjoint equivalence; see [Rie16, Corollary 4.3.6]. We won't use this fact in this course.

Definition 3.6. A functor $F\colon \mathcal{C} \to \mathcal{D}$ between strict categories is **essentially surjective** if, for all objects $d \in \mathcal{D}$, there is an object $c \in \mathcal{C}$ and an isomorphism $\varphi\colon Fc \xrightarrow{\cong} d$ in $\mathcal{D}$.

Theorem 3.7 (Fundamental Theorem)**.** *A functor $F\colon \mathcal{C} \to \mathcal{D}$ between strict categories is an equivalence if and only if it is essentially surjective on objects and fully faithful.*

Proof. Suppose that F is an equivalence so that we have a functor $G\colon \mathcal{D} \to \mathcal{C}$ and natural isomorphisms $\eta\colon \mathrm{id}_{\mathcal{C}} \xRightarrow{\cong} GF$ and $\varepsilon\colon FG \xRightarrow{\cong} \mathrm{id}_{\mathcal{D}}$. Given an object $d \in \mathcal{D}$, the object $Gd \in \mathcal{C}$ and the isomorphism in $\mathcal{D}$

$$\varepsilon_d\colon FGd \xrightarrow{\cong} d$$

provide the desired data for F to be essentially surjective. Next, given objects $x, y \in \mathcal{C}$, we show that the induced function

$$\mathcal{C}(x,y) \to \mathcal{D}(Fx, Fy)$$

is a bijection. Its inverse is given by the function

$$\mathcal{D}(Fx,Fy) \to \mathcal{C}(x,y), \quad (g\colon Fx \to Fy) \mapsto (x \xrightarrow{\eta_x} GFx \xrightarrow{Gg} GFy \xrightarrow{\eta_y^{-1}} y).$$

Naturality of $\eta\colon \mathrm{id}_{\mathcal{C}} \Rightarrow GF$ and $\varepsilon\colon FG \Rightarrow \mathrm{id}_{\mathcal{D}}$ implies that these functions are mutual inverses (Exercise). This proves that F is fully faithful.

Conversely, suppose that F is essentially surjective on objects and fully faithful. We construct a functor $G\colon \mathcal{D} \to \mathcal{C}$ and a natural isomorphism $\varepsilon\colon FG \xRightarrow{\cong} \mathrm{id}_{\mathcal{D}}$ simultaneously. Let $d \in \mathcal{D}$ be an object. Since F is essentially surjective, there is an object $c \in \mathcal{C}$ and an isomorphism $\varphi\colon Fc \xrightarrow{\cong} d$. We set $Gd \coloneqq c$ and $\varepsilon_d \coloneqq \varphi\colon FGd \xrightarrow{\cong} d$. Next, let $g\colon x \to y$ be a morphism in $\mathcal{D}$ and consider the composite in $\mathcal{D}$

$$FGx \xrightarrow{\varepsilon_x} x \xrightarrow{g} y \xrightarrow{\varepsilon_y^{-1}} FGy.$$

Since F is fully faithful, there is a unique morphism $f\colon Gx \to Gy$ in $\mathcal{C}$ with $Ff = \varepsilon_y^{-1} \circ g \circ \varepsilon_x$. We set $Gg \coloneqq f$. Then G is a well-defined functor and the isomorphisms ε_d assemble into a natural transformation (Exercise).

Finally, we construct the natural isomorphism $\eta\colon \mathrm{id}_{\mathcal{C}} \xRightarrow{\cong} GF$. Let $c \in \mathcal{C}$ be an object. Consider the morphism in $\mathcal{D}$

$$\varepsilon_{Fc}^{-1}\colon Fc \xrightarrow{\cong} FGFc.$$

Since F is fully faithful, there is a unique isomorphism $f\colon c \xrightarrow{\cong} GFc$ in $\mathcal{C}$ with $Ff = \varepsilon_{Fc}^{-1}$. We set $\eta_c \coloneqq f$. Then the isomorphisms η_c assemble into a natural transformation (Exercise). □

Construction 3.8. The **walking isomorphism** is the strict category $\mathcal{I} \coloneqq \{0 \cong 1\}$ with two objects 0 and 1, whose only morphisms are the identities and mutually inverse isomorphisms $f\colon 0 \to 1$ and $f^{-1}\colon 1 \to 0$. Observe that the hom sets of $\mathcal{I}$ are all singletons.

Proposition 3.9. *The functor* $0\colon [0] \to \mathcal{I}$ *is an equivalence of strict categories.*

Proof. It is surjective on objects up to isomorphisms, since $0 \cong 1$ in $\mathcal{I}$, and it is fully faithful since, for the unique object $* \in [0]$, we have a bijection

$$[0](*,*) = \{\mathrm{id}_*\} \xrightarrow{\cong} \{\mathrm{id}_0\} = \mathcal{I}(0,0).$$ □

3.3. **Stability under isomorphism and invariance under equivalence**

As mentioned in the section's introduction, we want to study objects in a strict category up to isomorphism and, accordingly, strict categories up to equivalence. Some notions that we can define at the point-set level may fail to be invariant under isomorphisms between objects or equivalences between strict categories and are therefore unsuitable as intrinsic notions of category theory. We call a notion *iso-invariant* if it is invariant under replacing an object by an isomorphic one, and *iso-sensitive* otherwise. Similarly, we call a notion *equiv-invariant*

if it is invariant under replacing a strict category by an equivalent one, and *equiv-sensitive* otherwise. The following examples illustrate these distinctions.

Construction 3.10. The **walking morphism** is the strict category $[1]$ with two objects 0 and 1, whose only morphisms are the identities and a single morphism $0 \to 1$.

Example 3.11 (Iso-sensitive notion). Let $\mathcal{C}$ be a strict category containing a non-identity isomorphism $f\colon x \xrightarrow{\cong} y$. Consider the arrow strict category $\mathrm{Fun}([1], \mathcal{C})$, whose objects are morphisms in $\mathcal{C}$ and whose morphisms are commutative squares in $\mathcal{C}$. The property for an object of $\mathrm{Fun}([1], \mathcal{C})$ to be an identity morphism in $\mathcal{C}$ is not stable under isomorphisms in $\mathrm{Fun}([1], \mathcal{C})$. Indeed, we have a commutative square in $\mathcal{C}$

$$\begin{array}{ccc} x & \xrightarrow{f} & y \\ {\scriptstyle \mathrm{id}_x}\downarrow{\scriptstyle \cong} & & {\scriptstyle f^{-1}}\downarrow{\scriptstyle \cong} \\ x & \xrightarrow[\mathrm{id}_x]{} & x \end{array}$$

where both vertical morphisms are isomorphisms, which provides an isomorphism between f and id_x in $\mathrm{Fun}([1], \mathcal{C})$. However, the morphism f is not an identity morphism in $\mathcal{C}$.

In the next section, we will consider universal properties for objects, which will all be iso-invariant.

We now turn to an example of a property of a strict category that is equiv-sensitive.

Construction 3.12. Every large set S can be regarded as a strict category, also denoted by S, with object set S and only identity morphisms.

Example 3.13 (Equiv-sensitive notion). Consider the property for a strict category $\mathcal{C}$ to be a (large) set, i.e., all morphisms in $\mathcal{C}$ are identities. This property is not invariant under equivalence of strict categories. Indeed, the functor $0\colon [0] \to \mathcal{I}$ is an equivalence of strict categories. However, the strict category $[0]$ is clearly a set, whereas $\mathcal{I}$ is not, as it has non-identity morphisms f and f^{-1}.

We will see in the exercises that the correct replacement of the property of "being a (large) set" is that of a strict category whose hom sets are either empty or singletons and whose morphisms are all isomorphisms.

However, if we restrict the condition above to endomorphisms, i.e., morphisms of the form $x \to x$, then this property becomes equiv-invariant.

Example 3.14 (Equiv-invariant notion). Consider the property for a strict category $\mathcal{C}$ that all endomorphisms in $\mathcal{C}$ are identities. Unlike the one above, this property is invariant under equivalence. Indeed, if $F\colon \mathcal{C} \xrightarrow{\simeq} \mathcal{D}$ is an equivalence between strict categories, then, for every object $y \in \mathcal{D}$, there is an object $x \in \mathcal{C}$ together with an isomorphism $f\colon Fx \cong y$. Using that F is fully faithful and conjugation by f, we obtain bijections

$$\mathcal{C}(x,x) \xrightarrow{\cong} \mathcal{D}(Fx,Fx) \xrightarrow{\cong} \mathcal{D}(y,y), \quad (g\colon x \to x) \mapsto (f \circ Fg \circ f^{-1}\colon y \to y).$$

If all endomorphisms in $\mathcal{C}$ are identities, then $\mathcal{C}(x,x) = \{\mathrm{id}_x\}$. Hence $\mathcal{D}(y,y)$ is also a singleton and therefore $\mathcal{D}(y,y) = \{\mathrm{id}_y\}$. So all endomorphisms in $\mathcal{D}$ are also identities.

4. Universal constructions in strict categories

We now turn to universal constructions in a strict category. Rather than introducing them in full generality, we will focus on initial and terminal objects, products, coproducts, and pullbacks. In the synthetic setting, their counterparts will be introduced directly as basic constructions of synthetic categories. These examples will help us become familiar with universal properties and learn how to work with them in general.

4.1. Terminal and initial objects

Definition 4.1. Let $\mathcal{C}$ be a strict category.

- An object $t \in \mathcal{C}$ is **terminal** if, for every object $x \in \mathcal{C}$, there is a unique morphism $!\colon x \to t$ in $\mathcal{C}$.
- An object $i \in \mathcal{C}$ is **initial** if, for every object $x \in \mathcal{C}$, there is a unique morphism $!\colon i \to x$ in $\mathcal{C}$.

Example 4.2. The strict categories in Example 1.9 have the following terminal and initial objects.

- The singleton set $*$ is a terminal object in $\mathcal{S}et$, and the empty set $\emptyset$ is an initial one.
- The singleton set $*$ is both a terminal and an initial object in $\mathcal{S}et_\bullet$.
- The zero k-vector space 0 is both a terminal and an initial object in $\mathcal{V}ect_k$.
- The singleton topological space $*$ is a terminal object in $\mathcal{T}op$, and the empty topological space $\emptyset$ is an initial one.

Proposition 4.3. *Let $\mathcal{C}$ be a strict category.*

- *(Uniqueness) Terminal objects are unique up to a unique isomorphism, i.e., if $t, t' \in \mathcal{C}$ are terminal objects, then there is a unique isomorphism $t \cong t'$ in $\mathcal{C}$.*
- *(Representability) An object $t \in \mathcal{C}$ is terminal if and only if, for every object $x \in \mathcal{C}$, there is a bijection $\mathcal{C}(x,t) \cong *$.*
- *(Iso-invariance) If $t \in \mathcal{C}$ is a terminal object and $t \cong t'$ in $\mathcal{C}$, then t' is terminal in $\mathcal{C}$.*
- *(Equiv-invariance) Equivalences preserve and reflect terminal objects, i.e., if $F\colon \mathcal{C} \xrightarrow{\simeq} \mathcal{D}$ is an equivalence between strict categories, then an object $t \in \mathcal{C}$ is terminal if and only if $Ft \in \mathcal{D}$ is terminal.*

Proof. We only prove iso-invariance and equiv-invariance. The rest is left as an exercise.

By Corollary 2.6, an isomorphism $t \cong t'$ in $\mathcal{C}$ induces, for every object $x \in \mathcal{C}$, a bijection

$$\mathcal{C}(x,t) \cong \mathcal{C}(x,t').$$

Hence one is a singleton if and only if the other is. This shows that t is terminal if and only if t' is terminal.

Let $F\colon \mathcal{C} \to \mathcal{D}$ be an equivalence and $t \in \mathcal{C}$ be an object. Suppose first that Ft is terminal in $\mathcal{D}$ and let $x \in \mathcal{C}$ be an object. Using that F is fully faithful, we have bijections

$$\mathcal{C}(x,t) \cong \mathcal{D}(Fx, Ft) \cong *.$$

This shows that t is terminal in $\mathcal{C}$.

Now, suppose that t is terminal in $\mathcal{C}$ and let $y \in \mathcal{D}$ be an object. Since F is essentially surjective, there is an object $x \in \mathcal{C}$ together with an isomorphism $y \cong Fx$. Using that F is fully faithful and Corollary 2.6, we obtain bijections

$$* \cong \mathcal{C}(x,t) \cong \mathcal{D}(Fx, Ft) \cong \mathcal{D}(y, Ft).$$

This shows that Ft is terminal in $\mathcal{D}$. □

Initial objects are *dual* to terminal ones in the following sense.

Proposition 4.4. *An object $i \in \mathcal{C}$ is initial in $\mathcal{C}$ if and only if it is terminal in $\mathcal{C}^{\mathrm{op}}$.*

Proof. Exercise. □

By applying Proposition 4.3 to $\mathcal{C}^{\mathrm{op}}$, we obtain the dual results for initial objects.

Corollary 4.5. *Let $\mathcal{C}$ be a strict category.*

- *(Uniqueness) Initial objects are unique up to a unique isomorphism.*
- *(Representability) An object $i \in \mathcal{C}$ is initial if and only if, for every object $x \in \mathcal{C}$, there is a bijection $\mathcal{C}(i, x) \cong *$.*
- *(Iso-invariance) If $i \in \mathcal{C}$ is an initial object and $i \cong i'$ in $\mathcal{C}$, then i' is initial in $\mathcal{C}$.*
- *(Equiv-invariance) Equivalences preserve and reflect initial objects.*

We now prove that the strict category $\mathcal{CAT}_s$ has a terminal and an initial object.

Proposition 4.6. *The strict category $[0]$ is a terminal object in $\mathcal{CAT}_s$ and the empty strict category $\emptyset$ is an initial object in $\mathcal{CAT}_s$.*

Proof. Exercise. □

The notion of terminal object in $\mathcal{CAT}_s$ is equiv-sensitive.

Example 4.7. The functor $0\colon [0] \to \mathcal{I}$ to the walking isomorphism $\mathcal{I} = \{0 \cong 1\}$ is an equivalence of strict categories. However, the strict category $\mathcal{I}$ is not terminal. Indeed, there are two different functors $0, 1\colon [0] \to \mathcal{I}$, selecting the objects 0 and 1, respectively.

The notion of initial object is equiv-invariant, since any strict category that is equivalent to the empty strict category in $\mathcal{CAT}_s$ is itself empty.

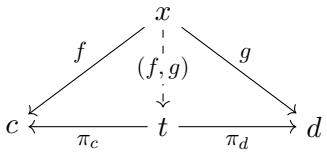
Proposition 4.8. *If $\mathcal{C}$ is equivalent to $\emptyset$, then $\mathcal{C} = \emptyset$.*

Proof. Exercise. □

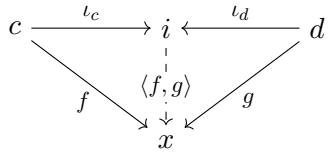
4.2. Products and coproducts

Definition 4.9. Let $\mathcal{C}$ be a strict category and $c, d \in \mathcal{C}$ be objects.

- An object $t \in \mathcal{C}$ is a **product** of c and d if there are morphisms $\pi_c\colon t \to c$ and $\pi_d\colon t \to d$ in $\mathcal{C}$ satisfying the following universal property: for every object $x \in \mathcal{C}$ and all morphisms $f\colon x \to c$ and $g\colon x \to d$ in $\mathcal{C}$, there is a unique morphism $(f, g)\colon x \to t$ such that $\pi_c \circ (f, g) = f$ and $\pi_d \circ (f, g) = g$.



  We often write $c \times d$ for a product of c and d. This notation always includes a choice of **canonical projections** $\pi_c\colon c \times d \to c$ and $\pi_d\colon c \times d \to d$.
- An object $i \in \mathcal{C}$ is a **coproduct** of c and d if there are morphisms $\iota_c\colon c \to i$ and $\iota_d\colon d \to i$ in $\mathcal{C}$ satisfying the following universal property: for every object $x \in \mathcal{C}$ and all morphisms $f\colon c \to x$ and $g\colon d \to x$ in $\mathcal{C}$, there is a unique morphism $\langle f, g\rangle\colon i \to x$ such that $\langle f, g\rangle \circ \iota_c = f$ and $\langle f, g\rangle \circ \iota_d = g$.



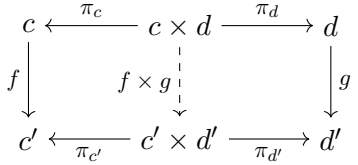
We often write $c \amalg d$ for a coproduct of c and d. This notation always includes a choice of **canonical inclusions** $\iota_c \colon c \to c \amalg d$ and $\iota_d \colon d \to c \amalg d$.

Example 4.10. The strict categories in Example 1.9 have the following (co)products.

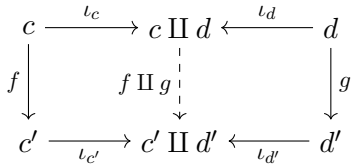
- Given sets S and T, the set $S \times T \coloneqq \{(s,t) \mid s \in S, t \in T\}$ is a product of S and T in $\mathcal{S}et$, and their disjoint union $S \amalg T$ is a coproduct of S and T in $\mathcal{S}et$.
- Given pointed sets (S,s) and (T,t), the pointed set $(S \times T, (s,t))$ is a product of (S,s) and (T,t) in $\mathcal{S}et_\bullet$, and the pointed set $\big((S \amalg T)/(s \sim t), [s]\big)$ is a coproduct of (S,s) and (T,t) in $\mathcal{S}et_\bullet$.
- Given k-vector spaces V and W, their direct sum $V \oplus W$ is both a product and a coproduct of V and W in $\mathcal{V}ect_k$.
- Given topological spaces $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ in $\mathcal{T}op$, the product $X \times Y$ equipped with the product topology is a product of $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ in $\mathcal{T}op$, and the topological space
$$\big(X \amalg Y, \{U \amalg V \mid U \in \mathcal{T}_X, V \in \mathcal{T}_Y\}\big)$$
is a coproduct of $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ in $\mathcal{T}op$.

Construction 4.11. Let $\mathcal{C}$ be a strict category and $f \colon c \to c'$, $g \colon d \to d'$ be morphisms in $\mathcal{C}$.

- If $c \times d$ and $c' \times d'$ are products of c, d and c', d', respectively, then by the universal property of the product $c' \times d'$, there is a unique morphism $f \times g \colon c \times d \to c' \times d'$ making the following diagram commute.



- If $c \amalg d$ and $c' \amalg d'$ are coproducts of c, d and c', d', respectively, then by the universal property of the coproduct $c \amalg d$, there is a unique morphism $f \amalg g \colon c \amalg d \to c' \amalg d'$ making the following diagram commute.



Proposition 4.12. *Let $\mathcal{C}$ be a strict category and $c, d \in \mathcal{C}$ be objects.*

- *(Uniqueness) Binary products are unique up to a unique isomorphism commuting with the canonical projections, i.e., if $t, t' \in \mathcal{C}$ are products of c and d with canonical projections $\pi_c \colon t \to c$, $\pi_d \colon t \to d$ and $\pi'_c \colon t' \to c$, $\pi'_d \colon t' \to d$, there is a unique isomorphism $\varphi \colon t \cong t'$ such that $\pi'_c \circ \varphi = \pi_c$ and $\pi'_d \circ \varphi = \pi_d$.*
- *(Representability) An object $t \in \mathcal{C}$ is a product of c and d if and only if, for every object $x \in \mathcal{C}$, there is a bijection*
$$\mathcal{C}(x,t) \cong \mathcal{C}(x,c) \times \mathcal{C}(x,d)$$

which is natural in x.
- *(Products with terminal objects) If d is terminal in $\mathcal{C}$, then c is a product of c and d.*
- *(Iso-invariance) If t is a product of c and d in $\mathcal{C}$ and $t \cong t'$ in $\mathcal{C}$, then t' is a product of c and d in $\mathcal{C}$.*
- *(Equiv-invariance) Equivalences preserve and reflect binary products.*

Again coproducts are *dual* to products in the following sense.

Proposition 4.13. *Let $\mathcal{C}$ be a strict category and $c, d \in \mathcal{C}$ be objects. An object $i \in \mathcal{C}$ is a coproduct of c and d in $\mathcal{C}$ if and only if it is a product of c and d in $\mathcal{C}^{\mathrm{op}}$.*

Proof. Exercise. □

By applying Proposition 4.12 to $\mathcal{C}^{\mathrm{op}}$, we obtain the dual results for binary coproducts.

Corollary 4.14. *Let $\mathcal{C}$ be a strict category and $c, d \in \mathcal{C}$ be objects.*
- *(Uniqueness) Binary coproducts are unique up to a unique isomorphism commuting with the canonical inclusions.*
- *(Representability) An object $i \in \mathcal{C}$ is a coproduct of c and d if and only if, for every object $x \in \mathcal{C}$, there is a bijection*
$$\mathcal{C}(i, x) \cong \mathcal{C}(c, x) \times \mathcal{C}(d, x)$$
which is natural in x.
- *(Coproducts with initial objects) If d is initial in $\mathcal{C}$, then c is a coproduct of c and d.*
- *(Iso-invariance) If $i \in \mathcal{C}$ is a coproduct of c and d and $i \cong i'$ in $\mathcal{C}$, then i' is a coproduct of c and d in $\mathcal{C}$.*
- *(Equiv-invariance) Equivalences preserve and reflect binary coproducts.*

We now prove that the strict category $\mathcal{CAT}_s$ has all binary products.

Construction 4.15. Let $\mathcal{C}$ and $\mathcal{D}$ be strict categories. There is a strict category $\mathcal{C} \times \mathcal{D}$ with
- object set $\mathrm{Ob}(\mathcal{C}) \times \mathrm{Ob}(\mathcal{D})$, i.e., objects are pairs (c, d) of objects $c \in \mathcal{C}$ and $d \in \mathcal{D}$;
- for all objects $(c, d), (c', d') \in \mathcal{C} \times \mathcal{D}$, hom set
$$(\mathcal{C} \times \mathcal{D})((c, d), (c', d')) \coloneqq \mathcal{C}(c, c') \times \mathcal{D}(d, d'),$$
i.e., morphisms $(c, d) \to (c', d')$ in $\mathcal{C} \times \mathcal{D}$ are pairs (f, g) of morphisms $f \colon c \to c'$ in $\mathcal{C}$ and $g \colon d \to d'$ in $\mathcal{D}$;
- for each object $(c, d) \in \mathcal{C} \times \mathcal{D}$, the identity morphism $\mathrm{id}_{(c,d)} \coloneqq (\mathrm{id}_c, \mathrm{id}_d) \colon (c, d) \to (c, d)$;
- for all morphisms $(f, g) \colon (c, d) \to (c', d')$ and $(f', g') \colon (c', d') \to (c'', d'')$ in $\mathcal{C} \times \mathcal{D}$, the composite morphism $(f', g') \circ (f, g) \coloneqq (f' \circ f, g' \circ g) \colon (c, d) \to (c'', d'')$.

This is a well-defined strict category. It comes with canonical projections $\pi_{\mathcal{C}} \colon \mathcal{C} \times \mathcal{D} \to \mathcal{C}$ and $\pi_{\mathcal{D}} \colon \mathcal{C} \times \mathcal{D} \to \mathcal{D}$ given by projecting onto the first and second components, respectively.

Proposition 4.16. *Let $\mathcal{C}$ and $\mathcal{D}$ be strict categories. The strict category $\mathcal{C} \times \mathcal{D}$ constructed above is a product of $\mathcal{C}$ and $\mathcal{D}$ in $\mathcal{CAT}_s$.*

Proof. Exercise. □

The construction of products in $\mathcal{CAT}_s$ is equiv-invariant.

Proposition 4.17. *Let $F \colon \mathcal{C} \to \mathcal{C}'$ and $G \colon \mathcal{D} \to \mathcal{D}'$ be equivalences of strict categories. Then the induced functor $F \times G \colon \mathcal{C} \times \mathcal{D} \to \mathcal{C}' \times \mathcal{D}'$ between their products is also an equivalence.*

Proof. Exercise. □

We now prove that the strict category $\mathcal{CAT}_s$ has all binary coproducts.

Construction 4.18. Let $\mathcal{C}$ and $\mathcal{D}$ be strict categories. There is a strict category $\mathcal{C} \amalg \mathcal{D}$ with

- object set $\mathrm{Ob}(\mathcal{C}) \amalg \mathrm{Ob}(\mathcal{D})$;
- for all objects $x, y \in \mathcal{C} \amalg \mathcal{D}$, hom set

$$(\mathcal{C} \amalg \mathcal{D})(x,y) \coloneqq \begin{cases} \mathcal{C}(x,y) & \text{if } x,y \in \mathcal{C} \\ \mathcal{D}(x,y) & \text{if } x,y \in \mathcal{D} \\ \emptyset & \text{else} \end{cases}$$

- identity morphisms those of $\mathcal{C}$ and $\mathcal{D}$;
- composite morphisms those of $\mathcal{C}$ and $\mathcal{D}$.

This is a well-defined strict category. It comes with canonical inclusions $\iota_{\mathcal{C}} \colon \mathcal{C} \to \mathcal{C} \amalg \mathcal{D}$ and $\iota_{\mathcal{D}} \colon \mathcal{D} \to \mathcal{C} \amalg \mathcal{D}$ given by inclusion into the first and second summands, respectively.

Proposition 4.19. *Let $\mathcal{C}$ and $\mathcal{D}$ be strict categories. The strict category $\mathcal{C} \amalg \mathcal{D}$ constructed above is a coproduct of $\mathcal{C}$ and $\mathcal{D}$ in $\mathcal{CAT}_s$.*

Proof. Exercise. □

The construction of coproducts in $\mathcal{CAT}_s$ is equiv-invariant.

Proposition 4.20. *Let $F \colon \mathcal{C} \to \mathcal{C}'$ and $G \colon \mathcal{D} \to \mathcal{D}'$ be equivalences of strict categories. Then the induced functor $F \amalg G \colon \mathcal{C} \amalg \mathcal{D} \to \mathcal{C}' \amalg \mathcal{D}'$ between their coproducts is also an equivalence.*

Proof. Exercise. □

4.3. **Pullback squares**

Definition 4.21. Let $\mathcal{C}$ be a strict category and $f \colon c \to e$ and $g \colon d \to e$ be morphisms in $\mathcal{C}$. An object $t \in \mathcal{C}$ is a **pullback** of f and g if there is a commutative square in $\mathcal{C}$

$$\begin{array}{ccc} t & \xrightarrow{\pi_d} & d \\ {\scriptstyle \pi_c}\downarrow & & \downarrow{\scriptstyle g} \\ c & \xrightarrow[f]{} & e \end{array}$$

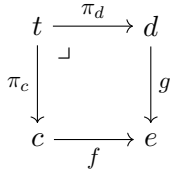
satisfying the following universal property: for every object $x \in \mathcal{C}$ and every commutative square in $\mathcal{C}$

$$\begin{array}{ccc} x & \xrightarrow{k} & d \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle g} \\ c & \xrightarrow[f]{} & e \end{array}$$

there is a unique morphism $(h,k) \colon x \to t$ such that $\pi_c \circ (h,k) = h$ and $\pi_d \circ (h,k) = k$.

We often write $c \times_e d$ for a pullback of f and g. This notation always includes a choice of **canonical projections** $\pi_c \colon c \times_e d \to c$ and $\pi_d \colon c \times_e d \to d$.

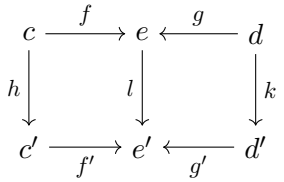
We often say that a commutative square, as below, is a **pullback square** if it satisfies the universal property of the pullback of f and g, and denote this by decorating the square with the symbol $\lrcorner$ in the top left corner.



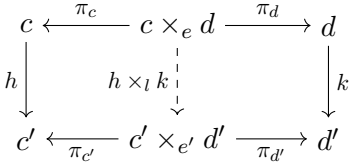
Example 4.22. The strict category $\mathcal{S}et$ has the following pullbacks. Given functions $f\colon S \to U$ and $g\colon T \to U$ of sets, a pullback of f and g in $\mathcal{S}et$ is given by the subset of $S \times T$

$$S \times_U T \coloneqq \{(s,t) \mid f(s) = g(t)\} \subseteq S \times T.$$

Construction 4.23. Let $\mathcal{C}$ be a strict category and consider a commutative diagram in $\mathcal{C}$



If $c \times_e d$ and $c' \times_{e'} d'$ are pullbacks of f, g and f', g', respectively, then by the universal property of the pullback $c' \times_{e'} d'$, there is a unique morphism $h \times_l k\colon c \times_e d \to c' \times_{e'} d'$ making the following diagram commute.



Proposition 4.24. *Let $\mathcal{C}$ be a strict category and $f\colon c \to e$ and $g\colon d \to e$ be morphisms in $\mathcal{C}$.*

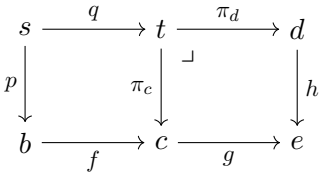
- *(Uniqueness) Pullbacks are unique up to a unique isomorphism commuting with the canonical projections.*
- *(Representability) An object $t \in \mathcal{C}$ is a pullback of f and g if and only if, for every object $x \in \mathcal{C}$, there is a bijection*

$$\mathcal{C}(x,t) \cong \mathcal{C}(x,c) \times_{\mathcal{C}(x,e)} \mathcal{C}(x,d)$$

  which is natural in x, where the right-hand side is the pullback of the functions $f_*\colon \mathcal{C}(x,c) \to \mathcal{C}(x,e)$ and $g_*\colon \mathcal{C}(x,d) \to \mathcal{C}(x,e)$.
- *(Products as pullbacks) If e is terminal in $\mathcal{C}$, then an object $t \in \mathcal{C}$ is a pullback of the unique morphisms $f = !\colon c \to e$ and $g = !\colon d \to e$ if and only if it is a product of c and d.*
- *(Iso-invariance) If $t \in \mathcal{C}$ is a pullback of f and g and $t \cong t'$ in $\mathcal{C}$, then t' is a pullback of f and g in $\mathcal{C}$.*
- *(Equiv-invariance) Equivalences preserve and reflect pullbacks.*

We also have the following useful properties of pullback squares.

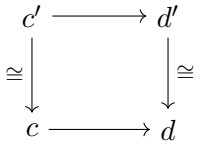
Proposition 4.25 (Pullback cancellation)**.** *Let $\mathcal{C}$ be a strict category and consider a commutative diagram in $\mathcal{C}$,*



where the right-hand square is a pullback square. Prove that the left-hand square is a pullback square if and only if the outer rectangle is a pullback square.

Proof. Exercise. □

Proposition 4.26. *Let $\mathcal{C}$ be a strict category and consider a commutative diagram in $\mathcal{C}$,*



where the vertical morphisms are isomorphisms. Then the square is a pullback square.

Proof. Exercise. □

We now prove that the strict category $\mathcal{CAT}_s$ has all pullbacks.

Construction 4.27. Let $F\colon \mathcal{C} \to \mathcal{E}$ and $G\colon \mathcal{D} \to \mathcal{E}$ be functors of strict categories. There is a strict category $\mathcal{C} \times_{\mathcal{E}} \mathcal{D}$ with

- object set $\mathrm{Ob}(\mathcal{C}) \times_{\mathrm{Ob}(\mathcal{E})} \mathrm{Ob}(\mathcal{D})$, i.e., objects are pairs (c, d) of objects $c \in \mathcal{C}$ and $d \in \mathcal{D}$ such that $Fc = Gd$ in $\mathcal{E}$;
- for all objects $(c, d), (c', d') \in \mathcal{C} \times_{\mathcal{E}} \mathcal{D}$, hom set
$$(\mathcal{C} \times_{\mathcal{E}} \mathcal{D})((c, d), (c', d')) \coloneqq \mathcal{C}(c, c') \times_{\mathcal{E}(Fc, Fc')} \mathcal{D}(d, d'),$$
i.e., morphisms $(c, d) \to (c', d')$ are pairs (f, g) of morphisms $f\colon c \to c'$ in $\mathcal{C}$ and $g\colon d \to d'$ in $\mathcal{D}$ such that $Ff = Gg$ in $\mathcal{E}$;
- for each object $(c, d) \in \mathcal{C} \times_{\mathcal{E}} \mathcal{D}$, the identity morphism $\mathrm{id}_{(c,d)} \coloneqq (\mathrm{id}_c, \mathrm{id}_d)\colon (c, d) \to (c, d)$;
- for all morphisms $(f, g)\colon (c, d) \to (c', d')$ and $(f', g')\colon (c', d') \to (c'', d'')$ in $\mathcal{C} \times_{\mathcal{E}} \mathcal{D}$, the composite morphism $(f', g') \circ (f, g) \coloneqq (f' \circ f, g' \circ g)\colon (c, d) \to (c'', d'')$.

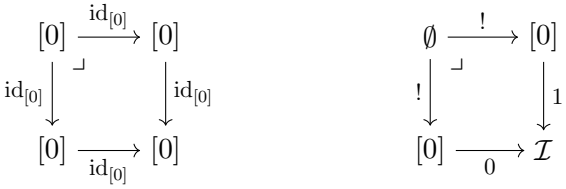
This is a well-defined strict category. It comes with canonical projections $\pi_{\mathcal{C}}\colon \mathcal{C} \times_{\mathcal{E}} \mathcal{D} \to \mathcal{C}$ and $\pi_{\mathcal{D}}\colon \mathcal{C} \times_{\mathcal{E}} \mathcal{D} \to \mathcal{D}$ given by projecting onto the first and second components, respectively.

Proposition 4.28. *Let $F\colon \mathcal{C} \to \mathcal{E}$ and $G\colon \mathcal{D} \to \mathcal{E}$ be functors of strict categories. The strict category $\mathcal{C} \times_{\mathcal{E}} \mathcal{D}$ constructed above is a pullback of F and G in $\mathcal{CAT}_s$.*

Proof. Exercise. □

Unlike products, the construction of pullbacks in $\mathcal{CAT}_s$ is equiv-sensitive.

Example 4.29. We have the following pullback squares in $\mathcal{CAT}_s$



where $\mathcal{I} = \{0 \cong 1\}$ is the walking isomorphism. Moreover, there is a commutative diagram of functors

$$\begin{array}{ccccc} [0] & \xrightarrow{0} & \mathcal{I} & \xleftarrow{1} & [0] \\ {\scriptstyle \mathrm{id}_{[0]}}\downarrow{\scriptstyle \simeq} & & \downarrow{\scriptstyle !}{\scriptstyle \simeq} & & {\scriptstyle \mathrm{id}_{[0]}}\downarrow{\scriptstyle \simeq} \\ [0] & \xrightarrow[\mathrm{id}_{[0]}]{} & [0] & \xleftarrow[\mathrm{id}_{[0]}]{} & [0] \end{array}$$

such that all vertical functors are equivalences, but the induced functor between their pullbacks

$$\emptyset = [0] \times_{\mathcal{I}} [0] \to [0] \times_{[0]} [0] \cong [0]$$

is not an equivalence.

5. Subcategories, localizations, and strict groupoids

We now introduce *strict groupoids*, namely strict categories whose morphisms are all isomorphisms. They generalize sets, regarded as strict categories with only identity morphisms, by allowing objects to be related by isomorphisms and to have nontrivial automorphisms. Thus, a strict groupoid records not only which objects are isomorphic, but also the isomorphisms between them.

In the synthetic setting, the counterparts of the notion of strict groupoids will be basic objects of the theory, playing the role traditionally played by sets. The goal of this section is to construct the strict category of strict groupoids, its canonical inclusion into $\mathcal{CAT}_s$, and both a left and a right adjoint to this inclusion.

5.1. Subcategories

Definition 5.1. A functor $F\colon \mathcal{C} \to \mathcal{D}$ between strict categories is

- **faithful** if, for all objects $x, y \in \mathcal{C}$, the induced function $\mathcal{C}(x,y) \to \mathcal{D}(Fx, Fy)$ is injective;
- **full** if, for all objects $x, y \in \mathcal{C}$, the induced function $\mathcal{C}(x,y) \to \mathcal{D}(Fx, Fy)$ is surjective.

Remark 5.2. Observe that a functor is fully faithful precisely when it is both full and faithful.

Let us fix a strict category $\mathcal{C}$.

Definition 5.3. A functor $\mathcal{D} \to \mathcal{C}$ is a **strict subcategory inclusion**, denoted by $\mathcal{D} \subseteq \mathcal{C}$, if it is injective on objects and faithful. We call $\mathcal{D}$ a **strict subcategory** of $\mathcal{C}$ if there is a strict subcategory inclusion $\mathcal{D} \subseteq \mathcal{C}$.

Remark 5.4. The data of a strict subcategory $\mathcal{D}$ of $\mathcal{C}$ can be given by specifying

- a subset $\mathrm{Ob}(\mathcal{D}) \subseteq \mathrm{Ob}(\mathcal{C})$ of objects;
- for all objects $x, y \in \mathcal{D}$, a subset $\mathcal{D}(x,y) \subseteq \mathcal{C}(x,y)$ of morphisms;

satisfying the conditions:

- identity morphisms are in $\mathcal{D}$: for every object $x \in \mathcal{D}$, the identity id_x is in $\mathcal{D}(x,x)$;
- morphisms in $\mathcal{D}$ are closed under composition: if $f \in \mathcal{D}(x,y)$ and $g \in \mathcal{D}(y,z)$, then their composite $g \circ f$ in $\mathcal{C}$ is in $\mathcal{D}(x,z)$.

The notion of a strict subcategory is equiv-sensitive.

Example 5.5. Recall that the functor $0\colon [0] \to \mathcal{I}$ and the unique functor $\mathcal{I} \to [0]$ are equivalences of strict categories, where $\mathcal{I} = \{0 \cong 1\}$ is the walking isomorphism. The identity

functor $\mathrm{id}_{[0]} \colon [0] \to [0]$ is a strict subcategory inclusion, but $\mathcal{I} \xrightarrow{\simeq} [0]$ is not. Moreover, the inclusion $\langle 0, 1 \rangle \colon [0] \amalg [0] \to \mathcal{I}$ is a strict subcategory inclusion, but the composite

$$[0] \amalg [0] \to \mathcal{I} \xrightarrow{\simeq} [0]$$

is not.

Definition 5.6. A strict subcategory inclusion $I \colon \mathcal{D} \hookrightarrow \mathcal{C}$ is **full** if the functor I is full. We call $\mathcal{D}$ a **full strict subcategory** of $\mathcal{C}$ if there is a full strict subcategory inclusion $I \colon \mathcal{D} \hookrightarrow \mathcal{C}$.

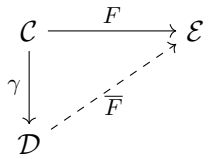
Remark 5.7. The data of a full strict subcategory $\mathcal{D}$ of $\mathcal{C}$ can be given by specifying a subset $\mathrm{Ob}(\mathcal{D}) \subseteq \mathrm{Ob}(\mathcal{C})$ of objects only. Indeed, we then take the subsets of hom sets consisting of all morphisms and the conditions from Remark 5.4 are then automatically satisfied. We say that $\mathcal{D}$ is the full strict subcategory of $\mathcal{C}$ **spanned by the objects** in $\mathrm{Ob}(\mathcal{D})$.

Definition 5.8. A strict subcategory $\mathcal{D} \subseteq \mathcal{C}$ is **wide** if it contains all isomorphisms of $\mathcal{C}$.

Remark 5.9. Since a wide strict subcategory contains all isomorphisms of C, it contains in particular all identity morphisms and so all objects of $\mathcal{C}$. Hence, the data of a wide strict subcategory $\mathcal{D}$ of $\mathcal{C}$ can be given by only specifying a subset of the morphisms of $\mathcal{C}$ which is closed under composition.

5.2. Localizations

Definition 5.10. Let $\mathcal{C}$ be a strict category and $W \subseteq \mathcal{CAT}_s([1], \mathcal{C})$ be a subset of morphisms of $\mathcal{C}$. A strict category $\mathcal{D}$ is a **localization** of $\mathcal{C}$ at W if there is a functor $\gamma \colon \mathcal{C} \to \mathcal{D}$ which sends morphisms in W to isomorphisms in $\mathcal{D}$ and satisfies the following universal property: for every strict category $\mathcal{E}$ and every functor $F \colon \mathcal{C} \to \mathcal{E}$ which sends morphisms in W to isomorphisms in $\mathcal{E}$, there is a unique functor $\overline{F} \colon \mathcal{D} \to \mathcal{E}$ such that $\overline{F} \circ \gamma = F$.



We call the functor $\gamma \colon \mathcal{C} \to \mathcal{D}$ the **localization functor**.

We now show that localizations exist by constructing an explicit strict category satisfying the universal property. We may assume that W contains all identity morphisms, since adding them does not change the universal property.

Construction 5.11. Let $\mathcal{C}$ be a strict category and $W \subseteq \mathcal{CAT}_s([1], \mathcal{C})$ be a subset of morphisms of $\mathcal{C}$ containing the identity morphisms and closed under composition, i.e., if $f, g \in W$, then $gf \in W$. There is a strict category $\mathcal{C}[W^{-1}]$ whose

- object set is $\mathrm{Ob}(\mathcal{C})$;
- for all objects $x, y \in \mathcal{C}$, hom set $\mathcal{C}[W^{-1}](x, y)$ is the large set of equivalence classes of finite zigzags

$$x = c_0 \xleftarrow{w_1} c_0' \xrightarrow{f_1} c_1 \xleftarrow{w_2} c_1' \xrightarrow{f_2} \dots \xleftarrow{w_n} c_{n-1}' \xrightarrow{f_n} c_n = y$$

  where $n \geq 0$, $w_1, \dots, w_n \in W$ and $f_1, \dots, f_n$ are morphisms in $\mathcal{C}$ modulo the relations:
  - identity morphisms pointing forwards or backwards may be removed or added;
  - adjacent morphisms pointing forwards may be composed in $\mathcal{C}$;
  - adjacent pairs $x \xrightarrow{w} y \xleftarrow{w} x$ or $y \xleftarrow{w} x \xrightarrow{w} y$ with $w \in W$ may be removed;

- identities are given by length 0 zigzags;
- composites are given by concatenation.

This is a well-defined strict category (Exercise).

It comes with a functor $\gamma\colon \mathcal{C} \to \mathcal{C}[W^{-1}]$ which

- acts as the identity on objects;
- sends a morphism $f\colon x \to y$ in $\mathcal{C}$ to the zigzag

$$x \xleftarrow{\mathrm{id}_x} x \xrightarrow{f} y.$$

This is a well-defined functor (Exercise).

Proposition 5.12. *The strict category $\mathcal{C}[W^{-1}]$ together with the functor $\gamma\colon \mathcal{C} \to \mathcal{C}[W^{-1}]$ is a localization of $\mathcal{C}$ at W.*

Proof. Exercise. □

Remark 5.13. Let $\mathcal{C}$ be a strict category and $W \subseteq \mathcal{CAT}_s([1], \mathcal{C})$ be a subset of morphisms of $\mathcal{C}$. Consider the smallest subset $\overline{W} \subseteq \mathcal{CAT}_s([1], \mathcal{C})$ containing W, the identity morphisms of $\mathcal{C}$, and closed under composition. Then the strict category $\mathcal{C}[\overline{W}^{-1}]$ together with the functor $\gamma\colon \mathcal{C} \to \mathcal{C}[\overline{W}^{-1}]$ is a localization of $\mathcal{C}$ at W. This shows that all localizations exist in $\mathcal{CAT}_s$.

Example 5.14. Consider the strict category $\mathcal{C} = \{0 \xrightarrow{s} 1 \xrightarrow{r} 0 \mid rs = \mathrm{id}_0\}$ with two objects 0 and 1, and whose morphisms are id_0, id_1, $s\colon 0 \to 1$, $r\colon 1 \to 0$, and $sr\colon 1 \to 1$, subject to the relation $rs = \mathrm{id}_0$. Then inverting the morphism s in $\mathcal{C}$ yields an isomorphism of strict categories $\mathcal{C}[\{s\}^{-1}] \cong \mathcal{I}$ with the walking isomorphism $\mathcal{I}$. Indeed, in the localization $\mathcal{C}[\{s\}^{-1}]$, we have

$$r = rss^{-1} = s^{-1}$$

since $rs = \mathrm{id}_0$. Thus r and s become mutual inverses.

5.3. **Strict groupoids**

Definition 5.15. A **strict groupoid** is a strict category $\mathcal{G}$ in which every morphism is an isomorphism.

We denote by $\mathcal{G}pd_s$ the full strict subcategory of $\mathcal{C}at_s$ spanned by the small strict groupoids, and by $\mathcal{GPD}_s$ the huge full strict subcategory of $\mathcal{CAT}_s$ spanned by the strict groupoids.

Construction 5.16. The **underlying strict groupoid** of a strict category $\mathcal{C}$ is the strict subcategory of $\mathcal{C}$ given by the strict groupoid $\mathcal{C}^{\simeq}$ whose

- objects are the objects of $\mathcal{C}$;
- morphisms $x \to y$ are the isomorphisms $x \xrightarrow{\cong} y$ in $\mathcal{C}$.

This is a well-defined strict subcategory (Exercise).

Remark 5.17. The underlying strict groupoid $\mathcal{C}^{\simeq}$ of a strict category $\mathcal{C}$ is a wide strict subcategory of $\mathcal{C}$. Moreover, every wide strict subcategory of $\mathcal{C}$ contains $\mathcal{C}^{\simeq}$.

Construction 5.18. There is a functor $(-)^{\simeq}\colon \mathcal{C}at_s \to \mathcal{G}pd_s$ sending

- a small strict category $\mathcal{C}$ to its underlying strict groupoid $\mathcal{C}^{\simeq}$;
- a functor $F\colon \mathcal{C} \to \mathcal{D}$ in $\mathcal{C}at_s$ to the induced functor $F^{\simeq}\colon \mathcal{C}^{\simeq} \to \mathcal{D}^{\simeq}$ obtained by restricting F to underlying strict groupoids. Since F preserves isomorphisms, this is well-defined.

This is a well-defined functor (Exercise).

Proposition 5.19. *The inclusion $\iota\colon \mathcal{G}pd_s \subseteq \mathcal{C}at_s$ and the underlying strict groupoid functor form an adjunction*

$$\iota\colon \mathcal{G}pd_s \rightleftarrows \mathcal{C}at_s :(-)^{\simeq}.$$

Proof. Exercise. □

Proposition 5.20. *Let $\mathcal{C}$ be a strict category and $W \coloneqq \mathcal{CAT}_s([1],\mathcal{C})$ be the set of all morphisms in $\mathcal{C}$. The localization $\Pi\mathcal{C} \coloneqq \mathcal{C}[W^{-1}]$ of $\mathcal{C}$ at W is a strict groupoid, called the* ***groupoidification*** *of $\mathcal{C}$. We denote its localization functor by $\gamma_{\mathcal{C}}\colon \mathcal{C} \to \Pi\mathcal{C}$.*

Proof. Considering the construction $\mathcal{C}[W^{-1}]$ from before, it suffices to prove that each zigzag is an isomorphism. However, this follows from taking the reversed zigzag with morphisms pointing in the opposite direction. □

Construction 5.21. There is a functor $\Pi\colon \mathcal{C}at_s \to \mathcal{G}pd_s$ sending

- a small strict category $\mathcal{C}$ to its groupoidification $\Pi\mathcal{C}$;
- a functor $F\colon \mathcal{C} \to \mathcal{D}$ in $\mathcal{C}at_s$ to the unique functor $\Pi F \coloneqq \overline{\gamma_{\mathcal{D}} \circ F}\colon \Pi\mathcal{C} \to \Pi\mathcal{D}$

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{F} & \mathcal{D} \\ {\scriptstyle \gamma_{\mathcal{C}}}\downarrow & & \downarrow{\scriptstyle \gamma_{\mathcal{D}}} \\ \Pi\mathcal{C} & \xrightarrow[\overline{\gamma_{\mathcal{D}} \circ F}]{} & \Pi\mathcal{D} \end{array}$$

  given by the universal property of the localization $\gamma_{\mathcal{C}}\colon \mathcal{C} \to \Pi\mathcal{C}$, using that the composite $\gamma_{\mathcal{D}} \circ F\colon \mathcal{C} \to \Pi\mathcal{D}$ sends all morphisms in $\mathcal{C}$ to isomorphisms in $\Pi\mathcal{D}$.

This is a well-defined functor (Exercise).

Proposition 5.22. *The groupoidification functor and the inclusion $\iota\colon \mathcal{G}pd_s \subseteq \mathcal{C}at_s$ form an adjunction*

$$\Pi\colon \mathcal{C}at_s \rightleftarrows \mathcal{G}pd_s :\iota.$$

Proof. Exercise. Prove this using the universal property of the groupoidification, as a localization at all morphisms. □

6. Discrete right fibrations and cartesian fibrations

As we have seen with the Yoneda lemma, set-valued functors play a central role in strict category theory. In particular, asking when a set-valued functor is naturally isomorphic to a representable one gives rise to universal constructions, such as terminal objects, products and pullbacks.

Definition 6.1. Let $\mathcal{C}$ be a locally small strict category. A functor $F\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{S}et$ is **representable** if there is an object $c \in \mathcal{C}$ and a natural isomorphism $\mathcal{C}(-,c) \cong F$. We say that the object c **represents** the functor F or that the functor F is **represented** by the object c.

Example 6.2. Given a locally small strict category $\mathcal{C}$, we have seen the following examples.

- An object in $\mathcal{C}$ is terminal if and only if it represents the functor $\mathrm{cst}(*)\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{S}et$ constant at the singleton set; see Proposition 4.3.
- Given objects $c, d \in \mathcal{C}$, an object in $\mathcal{C}$ is a product of c and d if and only if it represents the functor $\mathcal{C}(-,c) \times \mathcal{C}(-,d)\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{S}et$; see Proposition 4.12.
- Given morphisms $f\colon c \to e$ and $g\colon d \to e$ in $\mathcal{C}$, an object in $\mathcal{C}$ is a pullback of f and g if and only if it represents the functor $\mathcal{C}(-,c) \times_{\mathcal{C}(-,e)} \mathcal{C}(-,d)\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{S}et$; see Proposition 4.24.

Discrete left/right fibrations provide another way to encode set-valued functors. In the synthetic setting, we will describe the counterparts of set-valued functors exclusively through *left/right fibrations*, and the whole theory of representability will be developed in this language. We introduce the strict notion here to build intuition for this approach.

We will also discuss *cocartesian/cartesian fibrations* of strict categories, which play a similar role for functors valued in strict categories. The correspondence with such functors is not quite as straightforward because of the strictness of the setting, but this difficulty will disappear when we pass to the synthetic setting. There, (co)cartesian fibrations will encode category-valued functors and will in fact be used to define the category of categories itself. They will play a central role in the theory and provide tools for many constructions.

6.1. Discrete right fibrations

We focus on discrete right fibrations, but everything can be dualized to discrete left fibrations via the opposite functor $(-)^{\mathrm{op}} \colon \mathcal{CAT}_s \to \mathcal{CAT}_s$.

Definition 6.3. A functor $P \colon \mathcal{E} \to \mathcal{C}$ between strict categories is a **discrete right fibration** if, for every object $y \in \mathcal{E}$ and every morphism $f \colon c \to Py$ in $\mathcal{C}$, there is a unique morphism $\overline{f} \colon x \to y$ in $\mathcal{E}$ such that $P\overline{f} = f$. We call $\overline{f}$ a P**-lift** of f.

Construction 6.4. Let $\mathcal{C}$ be a strict category and $F \colon \mathcal{C}^{\mathrm{op}} \to \mathcal{S}et$ be a functor. We construct a strict category $\int^{\mathcal{C}} F$, called the **category of elements** or **Grothendieck construction** of F, whose

- objects are pairs (c, x) of an object $c \in \mathcal{C}$ and an element $x \in Fc$;
- morphisms $(c, x) \to (d, y)$ are morphisms $f \colon c \to d$ in $\mathcal{C}$ such that $x = Ff(y)$, where $Ff \colon Fd \to Fc$ is the function given by the image of f under F;
- identities and composites are given by those in $\mathcal{C}$.

This is a well-defined strict category. It comes with a functor $\pi^F \colon \int^{\mathcal{C}} F \to \mathcal{C}$ given by projecting onto the first component, called the **canonical projection**.

Proposition 6.5. *Let $\mathcal{C}$ be a strict category and $F \colon \mathcal{C}^{\mathrm{op}} \to \mathcal{S}et$ be a functor. The functor*

$$\pi^F \colon \int^{\mathcal{C}} F \to \mathcal{C}$$

is a discrete right fibration.

Proof. Let (d, y) be an object in $\int^{\mathcal{C}} F$ and $f \colon c \to d$ be a morphism in $\mathcal{C}$. Then the unique π^F-lift of f with target (d, y) is $f \colon (c, Ff(y)) \to (d, y)$. □

Example 6.6. Let $\mathcal{C}$ be a locally small strict category and $c \in \mathcal{C}$ be an object. Take F to be the representable functor $\mathcal{C}(-, c) \colon \mathcal{C}^{\mathrm{op}} \to \mathcal{S}et$. We write

$$\mathcal{C}_{/c} \coloneqq \int^{\mathcal{C}} \mathcal{C}(-, c)$$

and refer to it as the **slice strict category** over c. Unpacking the definition, it is the strict category whose

- objects are pairs (x, f) of an object $x \in \mathcal{C}$ and a morphism $f \colon x \to c$ in $\mathcal{C}$;
- given objects $(x, f), (y, g) \in \mathcal{C}_{/c}$, morphisms $(x, f) \to (y, g)$ are morphisms $h \colon x \to y$ in $\mathcal{C}$ such that $gh = f$;
- identities and composites are given by those in $\mathcal{C}$.

We call the canonical projection $\pi^c \coloneqq \pi^{\mathcal{C}(-,c)} \colon \mathcal{C}_{/c} \to \mathcal{C}$ the **representable** discrete right fibration at c.

Proposition 6.7. *Let $\mathcal{C}$ be a strict category. If $t \in \mathcal{C}$ is a terminal object, then $\mathcal{C}_{/t} \cong \mathcal{C}$.*

Proof. Exercise. □

Definition 6.8. A discrete right fibration $P\colon \mathcal{E} \to \mathcal{C}$ has **small fibers** if, for every object $c \in \mathcal{C}$, the fiber $\mathcal{E}_c \coloneqq \{x \in \mathcal{E} \mid Px = c\}$ is a small set.

Construction 6.9. Let $\mathcal{C}$ be a strict category and $P\colon \mathcal{E} \to \mathcal{C}$ be a discrete right fibration with small fibers. There is a functor

$$\ulcorner P \urcorner\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{S}et$$

sending:

- an object $c \in \mathcal{C}$ to the fiber $\mathcal{E}_c \coloneqq \{x \in \mathcal{E} \mid Px = c\}$;
- a morphism $f\colon c \to d$ in $\mathcal{C}$ to the function $\mathcal{E}_d \to \mathcal{E}_c$ sending an element $y \in \mathcal{E}_d$ to the element $x \in \mathcal{E}_c$, where $\overline{f}\colon x \to y$ is the unique P-lift of f with target y.

This is a well-defined functor (Exercise).

Proposition 6.10. *Let $\mathcal{C}$ be a strict category and $P\colon \mathcal{E} \to \mathcal{C}$ be a discrete right fibration with small fibers. There is an isomorphism of strict categories over $\mathcal{C}$*

$$\textstyle\int^{\mathcal{C}} \ulcorner P \urcorner \cong \mathcal{E}$$

where "over $\mathcal{C}$" means that it commutes with the canonical projection $\pi^{\ulcorner P \urcorner}$ and the functor P.

Proof. There is a functor $\int^{\mathcal{C}} \ulcorner P \urcorner \to \mathcal{E}$ over $\mathcal{C}$ sending

- an object $(c, x) \in \int^{\mathcal{C}} \ulcorner P \urcorner$ to the object $x \in \mathcal{E}$;
- a morphism $f\colon (c, x) \to (d, y)$ in $\int^{\mathcal{C}} \ulcorner P \urcorner$ to the unique P-lift $\overline{f}\colon x \to y$ of f with target y.

This is a well-defined functor over $\mathcal{C}$ (Exercise). Moreover, it is an isomorphism of strict categories with inverse functor $\mathcal{E} \to \int^{\mathcal{C}} \ulcorner P \urcorner$ over $\mathcal{C}$ sending

- an object $x \in \mathcal{E}$ to the object $(Px, x) \in \int^{\mathcal{C}} \ulcorner P \urcorner$;
- a morphism $\overline{f}\colon x \to y$ in $\mathcal{E}$ to the morphism $P\overline{f}\colon (Px, x) \to (Py, y)$ in $\int^{\mathcal{C}} \ulcorner P \urcorner$.

This is a well-defined functor over $\mathcal{C}$ and forms an inverse of the above functor, so that $\int^{\mathcal{C}} \ulcorner P \urcorner \cong \mathcal{E}$ over $\mathcal{C}$. This concludes the proof. □

Notation 6.11. Let us denote by $\mathcal{DRF}ib(\mathcal{C})$ the full strict subcategory of the slice $\mathcal{CAT}_{s/\mathcal{C}}$ spanned by the discrete right fibrations with small fibers.

Construction 6.12. Let $\mathcal{C}$ be a strict category, $F, G\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{S}et$ be functors, and $\alpha\colon F \Rightarrow G$ be a natural transformation. There is a functor

$$\textstyle\int^{\mathcal{C}} \alpha\colon \int^{\mathcal{C}} F \to \int^{\mathcal{C}} G$$

sending:

- an object $(c, x) \in \int^{\mathcal{C}} F$ to the object $(c, \alpha_c(x)) \in \int^{\mathcal{C}} G$, where we recall that $\alpha_c\colon Fc \to Gc$ is the function given by the component of α at c;
- a morphism $f\colon (c, x) \to (d, y)$ in $\int^{\mathcal{C}} F$ to the morphism $f\colon (c, \alpha_c(x)) \to (d, \alpha_d(y))$ in $\int^{\mathcal{C}} G$; this is well-defined as $Gf(\alpha_d(y)) = \alpha_c(Ff(y)) = \alpha_c(x)$ using naturality of α.

This is a well-defined functor and $\pi^G \circ \int^{\mathcal{C}} \alpha = \pi^F$ (Exercise).

Construction 6.13. Let $\mathcal{C}$ be a strict category. There is a functor

$$\textstyle\int^{\mathcal{C}}\colon \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}et) \to \mathcal{CAT}_{s/\mathcal{C}}$$

sending

- a functor $F\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{S}et$ to the functor $\pi^F\colon \int^{\mathcal{C}} F \to \mathcal{C}$;
- a natural transformation $\alpha\colon F \Rightarrow G$ in $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}et)$ to the functor $\int^{\mathcal{C}} \alpha\colon \int^{\mathcal{C}} F \to \int^{\mathcal{C}} G$ over $\mathcal{C}$.

This is a well-defined functor (Exercise).

Theorem 6.14. *The functor $\int^{\mathcal{C}}\colon \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}et) \to \mathcal{CAT}_{s/\mathcal{C}}$ is fully faithful and restricts to an equivalence of strict categories*

$$\int^{\mathcal{C}}\colon \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}et) \xrightarrow{\simeq} \mathcal{DRF}ib(\mathcal{C}).$$

Proof. We first show that $\int^{\mathcal{C}}\colon \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}et) \to \mathcal{CAT}_{s/\mathcal{C}}$ is fully faithful. Given functors $F, G\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{S}et$, we show that the induced function

$$\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}et)(F, G) \to \mathcal{CAT}_{s/\mathcal{C}}(\textstyle\int^{\mathcal{C}} F, \int^{\mathcal{C}} G), \quad \alpha \mapsto \int^{\mathcal{C}} \alpha$$

is a bijection.

Given natural transformations $\alpha, \beta\colon F \Rightarrow G$ such that $\int^{\mathcal{C}} \alpha = \int^{\mathcal{C}} \beta$, for every object $(c, x) \in \int^{\mathcal{C}} F$, we have $(c, \alpha_c(x)) = (c, \beta_c(x))$. Hence, for every object $c \in \mathcal{C}$, we have $\alpha_c = \beta_c$ and so $\alpha = \beta$. This proves injectivity.

Now, given a functor $H\colon \int^{\mathcal{C}} F \to \int^{\mathcal{C}} G$ over $\mathcal{C}$, for every object $(c, x) \in \int^{\mathcal{C}} F$, we have that $H(c, x) = (c, H_2(c, x))$. Define a natural transformation $\alpha\colon F \Rightarrow G$ by setting $\alpha_c\colon Fc \to Gc$ to be the function sending $x \in Fc$ to $\alpha_c(x) \coloneqq H_2(c, x)$, where $H(c, x) = (c, H_2(c, x))$. By functoriality of H, this is natural (Exercise). Moreover, we have that $\int^{\mathcal{C}} \alpha = H$. This proves surjectivity.

Then $\int^{\mathcal{C}}$ restricts to the full strict subcategory $\mathcal{DRF}ib(\mathcal{C})$ of the slice $\mathcal{CAT}_{s/\mathcal{C}}$ as every $\pi^F\colon \int^{\mathcal{C}} F \to \mathcal{C}$ is a discrete right fibration by Proposition 6.5. In particular, the restriction is again fully faithful. The fact that it is an equivalence then follows from Proposition 6.10, where we proved that it is essentially surjective. □

Proposition 6.15. *Let $\mathcal{C}$ be a locally small strict category and $c \in \mathcal{C}$ be an object. Then the object $(c, \mathrm{id}_c) \in \mathcal{C}_{/c}$ is terminal.*

Proof. Exercise. □

Theorem 6.16 (Representation Theorem)**.** *Let $\mathcal{C}$ be a locally small strict category. A functor $F\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{S}et$ is representable if and only if the strict category $\int^{\mathcal{C}} F$ has a terminal object.*

Proof idea. If an object $c \in \mathcal{C}$ represents F via a natural isomorphism $\alpha\colon \mathcal{C}(-, c) \cong F$, then the induced object $(c, \alpha_c(\mathrm{id}_c)) \in \int_{\mathcal{C}} F$ is terminal.

Conversely, if $(c, u) \in \int_{\mathcal{C}} F$ is a terminal object, then the induced natural transformation $\alpha_u\colon \mathcal{C}(-, c) \Rightarrow F$ (see Theorem 2.1) is a natural isomorphism.

Details are left as an exercise. □

Theorem 6.17 (Pointwise formula for adjoints)**.** *Let $L\colon \mathcal{C} \to \mathcal{D}$ be a functor between locally small strict categories. Then L admits a right adjoint $R\colon \mathcal{D} \to \mathcal{C}$ if and only if, for every object $d \in \mathcal{D}$, the strict category $\int^{\mathcal{C}} \mathcal{D}(L-, d)$ has a terminal object, where $\mathcal{D}(L-, d)$ is the composite*

$$\mathcal{D}(L-, d)\colon \mathcal{C}^{\mathrm{op}} \xrightarrow{L^{\mathrm{op}}} \mathcal{D}^{\mathrm{op}} \xrightarrow{\mathcal{D}(-, d)} \mathcal{S}et.$$

Proof. Suppose that L admits a right adjoint $R\colon \mathcal{D} \to \mathcal{C}$. Then we have bijections

$$\mathcal{C}(c, Rd) \cong \mathcal{D}(Lc, d)$$

natural in $c \in \mathcal{C}$ and $d \in \mathcal{D}$. By applying the functor $\int^{\mathcal{C}}$, we get an induced isomorphism of strict categories $\int^{\mathcal{C}} \mathcal{D}(L-, d) \cong \int^{\mathcal{C}} \mathcal{C}(-, Rd) = \mathcal{C}_{/Rd}$ and so $\int^{\mathcal{C}} \mathcal{D}(L-, d)$ admits a terminal object by Proposition 6.15.

Conversely, suppose that, for every object $d \in \mathcal{D}$, the strict category $\int^{\mathcal{C}} \mathcal{D}(L-, d)$ has a terminal object. We construct the right adjoint $R\colon \mathcal{D} \to \mathcal{C}$ as follows. By the Representation Theorem (Theorem 6.16), the functor $\mathcal{D}(L-, d)$ is representable. We set $Rd \in \mathcal{C}$ to be a representing object and the natural isomorphism $\mathcal{C}(-, Rd) \cong \mathcal{D}(L-, d)$ gives bijections

$$\mathcal{C}(c, Rd) \cong \mathcal{D}(Lc, d)$$

which are natural in $c \in \mathcal{C}$. Given a morphism $g\colon d \to e$ in $\mathcal{D}$, consider the composite of natural transformations

$$\mathcal{C}(-, Rd) \cong \mathcal{D}(L-, d) \xRightarrow{g_*} \mathcal{D}(L-, e) \cong \mathcal{C}(-, Re). \tag{6.18}$$

Using that the Yoneda functor $\mathbf{y}_{\mathcal{C}}\colon \mathcal{C} \to \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}et)$ is fully faithful, we set Rg to be the unique morphism $Rg\colon Rd \to Re$ in $\mathcal{C}$ such that the natural transformation

$$(Rg)_*\colon \mathcal{C}(-, Rd) \Rightarrow \mathcal{C}(-, Re)$$

is equal to the composite (9.54). By uniqueness of these morphisms, this makes R into a functor. Moreover, by construction, the bijections

$$\mathcal{C}(c, Rd) \cong \mathcal{D}(Lc, d)$$

are also natural in $d \in \mathcal{D}$. □

Example 6.19. Let us apply this result to $(-)_+\colon \mathcal{S}et \to \mathcal{S}et_\bullet$ to show that it is a left adjoint. For every pointed set (T, t), then $\int^{\mathcal{S}et} \mathcal{S}et_\bullet((-)_+, (T, t))$ is the strict category whose

- objects are pairs (S, f) of a set S and a basepoint-preserving function $f\colon S_+ \to (T, t)$;
- morphisms $(S, f) \to (S', f')$ are functions $g\colon S \to S'$ such that $f' \circ g_+ = f$.

Consider the function $\varepsilon_T\colon T_+ \to (T, t)$ which is the identity on T and sends the freely added element of T_+ to t. Then (T, ε_T) is a terminal object in this strict category. Indeed, for every basepoint-preserving function $f\colon S_+ \to (T, t)$, its restriction $f|_S\colon S \to T$ is the unique function such that $\varepsilon_T \circ (f|_S)_+ = f$.

$$\begin{array}{ccc} S & & S_+ \\ {\scriptstyle \exists!\, f|_S}\,\vdots & & {\scriptstyle (f|_S)_+}\downarrow \quad \searrow^{f} \\ T & & T_+ \xrightarrow[\varepsilon_T]{} (T, t) \end{array}$$

So the right adjoint of $(-)_+\colon \mathcal{S}et \to \mathcal{S}et_\bullet$ is given by sending a pointed set (T, t) to its underlying set T. To get the assignment on morphisms, observe that, for a basepoint-preserving function $f\colon (S, s) \to (T, t)$, its underlying function $f\colon S \to T$ is the unique function such that $\varepsilon_T \circ f_+ = f \circ \varepsilon_S$.

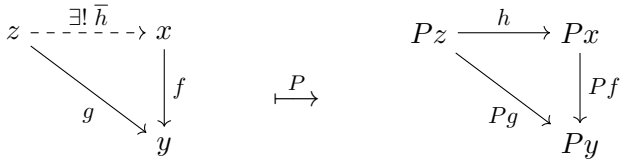
$$\begin{array}{ccc} S & & S_+ \xrightarrow{\varepsilon_S} (S, s) \\ {\scriptstyle \exists!\, f}\,\vdots & & {\scriptstyle f_+}\downarrow \qquad\quad \downarrow{\scriptstyle f} \\ T & & T_+ \xrightarrow[\varepsilon_T]{} (T, t) \end{array}$$

So the right adjoint of $(-)_+ \colon \mathcal{S}et \to \mathcal{S}et_\bullet$ is given by sending a basepoint-preserving function $f \colon (S, s) \to (T, t)$ to its underlying function $f \colon S \to T$. From the theorem, we directly know that this is functorial and a right adjoint.

6.2. Cartesian fibrations

Similarly, we focus on cartesian fibrations, but everything can be dualized to cocartesian fibrations via the opposite functor $(-)^{\mathrm{op}} \colon \mathcal{CAT}_s \to \mathcal{CAT}_s$.

Definition 6.20. Let $P \colon \mathcal{E} \to \mathcal{C}$ be a functor between strict categories. A morphism $f \colon x \to y$ in $\mathcal{E}$ is **P-cartesian** if, for all morphisms $g \colon z \to y$ in $\mathcal{E}$ and $h \colon Pz \to Px$ in $\mathcal{C}$ such that $Pf \circ h = Pg$, there is a unique morphism $\overline{h} \colon z \to x$ such that $P\overline{h} = h$ and $f\overline{h} = g$.



Proposition 6.21. *Let $P \colon \mathcal{E} \to \mathcal{C}$ be a functor between strict categories.*

(i) The composite of two P-cartesian morphisms is P-cartesian.

(ii) A morphism f in $\mathcal{E}$ is an isomorphism if and only if it is P-cartesian and Pf is an isomorphism in $\mathcal{C}$.

(iii) If $f \colon x \to z$ and $g \colon y \to z$ are two P-cartesian morphisms in $\mathcal{E}$ such that $Pf = Pg$, then there is a unique isomorphism $\varphi \colon x \xrightarrow{\cong} y$ in $\mathcal{E}$ such that $P\varphi = \mathrm{id}_{Px}$ and $g\varphi = f$.

Proof. Exercise. □

Definition 6.22. A functor $P \colon \mathcal{E} \to \mathcal{C}$ between strict categories is a **cartesian fibration** if, for every object $y \in \mathcal{E}$ and every morphism $f \colon c \to Py$ in $\mathcal{C}$, there is a P-cartesian morphism $\overline{f} \colon x \to y$ in $\mathcal{E}$ such that $P\overline{f} = f$. We call $\overline{f}$ a **P-cartesian lift** of f.

Remark 6.23. Note that, by (iii) above, the P-cartesian lift of a morphism $f \colon c \to Py$ with target y is unique up to a unique isomorphism.

Construction 6.24. Let $\mathcal{C}$ be a strict category and $F \colon \mathcal{C}^{\mathrm{op}} \to \mathcal{C}at_s$ be a functor. We construct a strict category $\int^{\mathcal{C}} F$, called the **Grothendieck construction** of F, whose

- objects are pairs (c, x) of an object $c \in \mathcal{C}$ and an object $x \in Fc$;
- morphisms $(c, x) \to (d, y)$ are pairs (f, u) of a morphism $f \colon c \to d$ in $\mathcal{C}$ and a morphism $u \colon x \to Ff(y)$ in Fc, where $Ff \colon Fd \to Fc$ is the functor given by the image of f under F;
- for an object $(c, x) \in \int^{\mathcal{C}} F$, the identity morphism is the pair $(\mathrm{id}_c, \mathrm{id}_x)$;
- given composable morphisms $(f, u) \colon (c, x) \to (d, y)$ and $(g, v) \colon (d, y) \to (e, z)$, their composite is the pair $(gf, Ff(v) \circ u)$, where the second component is given by

$$x \xrightarrow{u} Ff(y) \xrightarrow{Ff(v)} Ff(Fg(z)) = F(gf)(z).$$

This is a well-defined strict category (Exercise).

It comes with a projection functor $\pi^F \colon \int^{\mathcal{C}} F \to \mathcal{C}$ given by projecting onto the first component, called the **canonical projection**.

Proposition 6.25. *Let $\mathcal{C}$ be a strict category and $F \colon \mathcal{C}^{\mathrm{op}} \to \mathcal{C}at_s$ be a functor. Then the functor*

$$\pi^F \colon \int^{\mathcal{C}} F \to \mathcal{C}$$

is a cartesian fibration.

Proof. The π^F-cartesian morphisms are precisely those morphisms $(f,u)\colon (c,x)\to(d,y)$ in $\int^{\mathcal{C}} F$ such that $u\colon x\to Ff(y)$ is an isomorphism in Fc (Exercise). Let (d,y) be an object in $\int^{\mathcal{C}} F$ and $f\colon c\to d$ be a morphism in $\mathcal{C}$. Then a π^F-cartesian lift of f is given by $(f,\mathrm{id}_{Ff(y)})\colon (c,Ff(y))\to(d,y)$. □

Construction 6.26. Let $P\colon \mathcal{E}\to\mathcal{C}$ be a functor between strict categories and $c\in\mathcal{C}$ be an object. There is a strict category $\mathcal{E}_c$, called the **fiber** of P at c, whose

- objects are objects $x\in\mathcal{E}$ such that $Px=c$;
- morphisms $x\to y$ are morphisms $u\colon x\to y$ in $\mathcal{E}$ such that $Pu=\mathrm{id}_c$;
- identities and composites are given by those in $\mathcal{E}$.

This is a well-defined strict category.

Proposition 6.27. *A functor $P\colon\mathcal{E}\to\mathcal{C}$ is a discrete right fibration if and only if it is a cartesian fibration such that, for every object $c\in\mathcal{C}$, the fiber $\mathcal{E}_c$ is a (large) set.*

Proof. Exercise. □

Remark 6.28. Since we saw that the property for a strict category to be a set is equiv-sensitive, the property for a cartesian fibration to have fibers that are sets is also equiv-sensitive.

Construction 6.29. Let $P\colon\mathcal{E}\to\mathcal{C}$ be a cartesian fibration and $f\colon c\to d$ be a morphism in $\mathcal{C}$. For every object $y\in\mathcal{E}_d$, choose a P-cartesian lift $\overline{f}_y\colon f^*(y)\to y$ of $f\colon c\to d=Py$ with target y. Then there is an induced functor

$$f^*\colon \mathcal{E}_d\to\mathcal{E}_c,$$

called **cartesian transport** along f, sending:

- an object $y\in\mathcal{E}_d$ to the object $f^*(y)\in\mathcal{E}_c$;
- a morphism $u\colon x\to y$ in $\mathcal{E}_d$ to the unique morphism $\overline{u}\colon f^*(x)\to f^*(y)$ such that $P\overline{u}=\mathrm{id}_c$ and the following diagram commutes

$$\begin{array}{ccc} f^*(x) & \overset{\overline{u}}{\dashrightarrow} & f^*(y) \\ {\scriptstyle \overline{f}_x}\downarrow & & \downarrow{\scriptstyle \overline{f}_y} \\ x & \xrightarrow[u]{} & y \end{array} \qquad \overset{P}{\longmapsto} \qquad \begin{array}{ccc} c & \xrightarrow{\mathrm{id}_c} & c \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f} \\ d & \xrightarrow[\mathrm{id}_d]{} & d \end{array}$$

  given by the universal property of the P-cartesian morphism $\overline{f}_y\colon f^*(y)\to y$.

This is a well-defined functor (Exercise).

Remark 6.30. Cartesian transports need not be strictly functorial. Indeed, given composable morphisms $f\colon c\to d$ and $g\colon d\to e$ in $\mathcal{C}$, in general, we do not have an equality between the functors $(g\circ f)^*$ and $f^*\circ g^*$, but only a natural isomorphism.

This suggests that, unlike discrete right fibrations, cartesian fibrations whose fibers are small strict categories capture more than just functors $\mathcal{C}^{\mathrm{op}}\to\mathcal{C}at_s$. This is illustrated by the following example.

Example 6.31. Let $\mathcal{I}=\{0\cong 1\}$ be the walking isomorphism, and let $\mathcal{E}$ be the strict groupoid with three objects x,y,y' and exactly one morphism between every two objects. The functor

$$P\colon\mathcal{E}\to\mathcal{I},\qquad x\mapsto 0,\quad y,y'\mapsto 1,$$

is a cartesian fibration: every morphism in $\mathcal{E}$ is an isomorphism and hence P-cartesian, and every morphism in $\mathcal{I}$ admits a lift with any prescribed target.

Observe that its fibers are given by $\mathcal{E}_0 \cong [0]$ and $\mathcal{E}_1 \cong \mathcal{I}$ and so are not isomorphic. However, every functor $F\colon \mathcal{I}^{\mathrm{op}} \cong \mathcal{I} \to \mathcal{C}at_s$ takes the isomorphism $0 \cong 1$ to an isomorphism $F(0) \cong F(1)$. Consequently, the cartesian fibration P cannot be isomorphic over $\mathcal{I}$ to $\int^{\mathcal{I}} F$ for any such functor F.

Remark 6.32. The Grothendieck construction for functors $F\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{C}at_s$ also extends to a functor

$$\int^{\mathcal{C}}\colon \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{C}at_s) \to \mathcal{CAT}_{s/\mathcal{C}}.$$

This functor is faithful but is not full in general. Its morphisms preserve the canonical choices of cartesian lifts. Moreover, it is not in general essentially surjective on cartesian fibrations with small fibers, as shown by Example 6.31.

Definition 6.33. Let $P\colon \mathcal{E} \to \mathcal{C}$ be a functor. A **left adjoint section** of P is a left adjoint $S\colon \mathcal{C} \to \mathcal{E}$ of P such that $PS = \mathrm{id}_{\mathcal{C}}$ and the unit $\eta\colon \mathrm{id}_{\mathcal{C}} \Rightarrow PS$ is the identity natural transformation.

Theorem 6.34 (Functoriality of universals)**.** *Let $P\colon \mathcal{E} \to \mathcal{C}$ be a cartesian fibration. Suppose that, for every object $c \in \mathcal{C}$, the fiber $\mathcal{E}_c$ has an initial object. Then P admits a left adjoint section $S\colon \mathcal{C} \to \mathcal{E}$.*

Proof. Choose, for every object $c \in \mathcal{C}$, an initial object $i_c \in \mathcal{E}_c$, and for every morphism $f\colon c \to d = P(i_d)$ in $\mathcal{C}$, a P-cartesian lift $\overline{f}_{i_d}\colon f^*(i_d) \to i_d$ in $\mathcal{E}$. The functor $S\colon \mathcal{C} \to \mathcal{E}$ is defined by sending:

- an object $c \in \mathcal{C}$ to the object $S(c) \coloneqq i_c \in \mathcal{E}$;
- a morphism $f\colon c \to d$ in $\mathcal{C}$ to the composite

$$S(f) \coloneqq i_c \xrightarrow{!} f^*(i_d) \xrightarrow{\overline{f}_{i_d}} i_d,$$

  where $!\colon i_c \to f^*(i_d)$ is the unique morphism in $\mathcal{E}_c$ given by the universal property of the initial object i_c.

The fact that S is a well-defined functor and a left adjoint section of P are left as exercises. □

Example 6.35. We apply this result to obtain the tensor product functor for $\mathcal{V}ect_k$. Consider the strict category $\mathcal{V}ect_k^{\otimes}$ whose

- objects are tuples (V, W, U, f) of k-vector spaces V, W, U and a k-bilinear map

$$f\colon V \times W \to U;$$

- morphisms $(V, W, U, f) \to (V', W', U', f')$ are tuples (v, w, u) of k-linear maps $v\colon V \to V'$, $w\colon W \to W'$, and $u\colon U \to U'$ making the following diagram commute.

$$\begin{array}{ccc} V \times W & \xrightarrow{f} & U \\ {\scriptstyle v \times w}\downarrow & & \downarrow{\scriptstyle u} \\ V' \times W' & \xrightarrow[f']{} & U' \end{array}$$

It comes with a projection functor $\pi_{1,2}\colon \mathcal{V}ect_k^{\otimes} \to \mathcal{V}ect_k \times \mathcal{V}ect_k$ given by sending (V, W, U, f) to (V, W) and (v, w, u) to (v, w). This is a cartesian fibration. Indeed, a $\pi_{1,2}$-cartesian

morphism is a tuple (v, w, u) such that u is an isomorphism. Let $v\colon V \to V'$ and $w\colon W \to W'$ be k-linear maps. A $\pi_{1,2}$-cartesian lift of (v, w) with target $(V', W', U', f') \in \mathcal{V}ect_k^{\otimes}$ is given by

$$(v, w, \mathrm{id}_{U'})\colon (V, W, U', f'(v \times w)) \to (V', W', U', f').$$

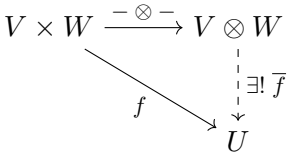
Let V and W be k-vector spaces. The fiber of $\pi_{1,2}$ at (V, W) is the strict category whose

- objects are pairs (U, f) of a k-vector space U and a bilinear map $f\colon V \times W \to U$;
- morphisms $(U, f) \to (U', f')$ are morphisms $u\colon U \to U'$ such that $uf = f'$.

Consider the tensor product $V \otimes W$ and the bilinear function

$$- \otimes -\colon V \times W \to V \otimes W, \quad (v, w) \mapsto v \otimes w.$$

Then $(V \otimes W, - \otimes -)$ is an initial object in the fiber of $\pi_{1,2}$ at (V, W). Indeed, for every bilinear map $f\colon V \times W \to U$, there is a unique k-linear map $\overline{f}\colon V \otimes W \to U$ such that $\overline{f}(- \otimes -) = f$.



Hence, the projection $\pi_{1,2}$ admits a left adjoint section $S\colon \mathcal{V}ect_k \times \mathcal{V}ect_k \to \mathcal{V}ect_k^{\otimes}$ sending a pair (V, W) of k-vector spaces to the tuple $(V, W, V \otimes W, - \otimes -)$. By projecting onto the third component, we obtain the tensor functor

$$- \otimes - \coloneqq \pi_3 S\colon \mathcal{V}ect_k \times \mathcal{V}ect_k \to \mathcal{V}ect_k.$$

II. Synthetic category theory - Axioms

In this second part, we lay the foundations of synthetic category theory and introduce the axioms needed for its development. We work in an ambient strict category whose objects we call *categories* or ∞-*categories*. We equip this ambient strict category with additional structure and impose axioms that allow us to develop the theory of (∞-)categories internally to it. Although we work in a strict category, the approach remains synthetic because the objects of that strict category are treated as primitive (∞-)categories whose behaviour is specified axiomatically, without choosing a concrete model for them.

7. Different approaches to foundations

There are (at least) three approaches to setting up the foundations of synthetic category theory:

- a *naive category theory*, presented in the book [CCNW21];
- a *functorial type theory*, as developed in the lecture notes of Walde;
- a *strict categorical interpretation* of the structure underlying such a functorial type theory, which is the approach taken here.

They all have in common the following structures:

- some basic mathematical objects called *categories* (or ∞-*categories*);
- distinguished categories called *animae*, which will behave like ∞-groupoids;
- functors between categories;
- a composition for these functors.

In the naive approach, we have witnesses for the associativity of composition and for some of the higher coherences. As acknowledged in [CCNW21] when discussing the epistemic status of the theory, it remains unclear whether the current axioms include all the coherence conditions needed for the intended development. In the type-theoretic approach, composition is associative by definition and, in our strict categorical interpretation, composition is associative on the nose (an equality in a set).

Once the foundational frameworks are in place, these approaches develop synthetic category theory from essentially the same axioms, formulated in different languages.

Each approach has its advantages and drawbacks. The naive approach stays close to familiar higher-categorical practice, and may also admit models built directly from fully weak structures, such as the category of categories constructed internally to the theory, without first choosing a strict presentation. However, it requires explicit bookkeeping of higher coherence data, which are not particularly relevant to the main ideas underlying the development of the theory.

The type-theoretic approach makes the rules governing constructions, substitution, and equality explicit. It also provides a framework particularly well suited for formalization in a proof assistant. Its practical drawback is that it introduces a new language and formalism that readers may need time to become familiar with before they can comfortably develop category theory within it.

We adopt a strict categorical interpretation of the type-theoretic approach. This allows us to use the familiar language of strict category theory while retaining the organizational principles of type theory. Our aim is to develop the theory using constructions and statements with type-theoretic counterparts. This approach should also provide a concrete framework for showing that established models of ∞-categories, such as quasi-categories and complete Segal spaces, satisfy the axioms of synthetic category theory. The main drawback is that it

retains an external set-theoretic metatheory. Although the development of category theory is synthetic, the ambient strict category is still described in set-theoretic terms. Thus, this approach allows us to reason synthetically within familiar foundations, but does not itself provide an alternative to those foundations.

The following table summarizes the correspondence between basic structures in the type theory and its strict categorical interpretation.

Functorial type theory	Strict categorical model
category C	object C
anima X	distinguished object X
closed term $c : C$	morphism $[0] \to C$
dependent family $x : C \vdash D(x)$	distinguished morphism $D \twoheadrightarrow C$
constant family $x : C \vdash D$	projection $C \times D \twoheadrightarrow C$
dependent term $x : C \vdash f(x) : D(x)$	section of $D \twoheadrightarrow C$
dependent term $x : C \vdash f(x) : D$	morphism $f\colon C \to D$
substituting $x : C \vdash f(x) : D$ in $y : D \vdash E(y)$	pullback of $E \twoheadrightarrow D$ along $f\colon C \to D$

We therefore work in an ambient strict category, whose objects we call *categories* or ∞-*categories* and whose morphisms we call *functors*. We distinguish certain objects as *animae* and certain morphisms as *family projections*. To interpret closed terms, we require a terminal object $[0]$; to interpret substitution, we require family projections to admit pullbacks along arbitrary functors, with the resulting projections again being family projections; and so on.

This approach is inspired by Joyal's theory of clans and tribes, by Cartmell's notion of a contextual category, and by some of the earliest versions of the class "Synthetic category theory" taught by Cisinski at Universität Regensburg.

8. Basic structure of clans for category theory

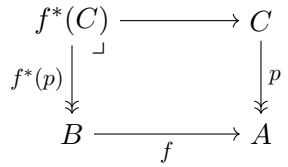
8.1. Clans for category theory

Definition 8.1. A **clan for category theory** consists of

- (Cat) a strict category $\mathcal{S}yn\mathcal{C}at$, whose objects are called **categories** or **∞-categories** and are denoted by $A, B, C, D, E, \ldots$, and whose morphisms are called **functors** and are denoted by $f\colon C \to D, g\colon D \to E, \ldots$;
- (An) a full strict subcategory $\mathcal{A}n \subseteq \mathcal{S}yn\mathcal{C}at$, whose objects are called **animae** and are denoted by $\Gamma, \Delta, X, Y, Z, \ldots$;
- (Fam) a wide strict subcategory $\mathcal{F}am\mathcal{P}roj \subseteq \mathcal{S}yn\mathcal{C}at$, whose morphisms are called **family projections** and are denoted by $p\colon C \twoheadrightarrow A, q\colon D \twoheadrightarrow B, \ldots$;

satisfying the following conditions:

- (C1) if $X \cong C$ are isomorphic categories and X is an anima, then C is an anima;
- (C2) $\mathcal{S}yn\mathcal{C}at$ has a terminal object $[0]$ and $[0]$ is an anima;
- (C3) for every category C, the unique morphism $C \to [0]$ is a family projection;
- (C4) for every family projection $p\colon C \twoheadrightarrow A$ and every functor $f\colon B \to A$, the pullback of p along f exists



and its pullback $f^*(p)\colon f^*(C) \twoheadrightarrow B$ is a family projection;

(C5) for every family projection $p\colon X \twoheadrightarrow \Gamma$ between animae and every functor $f\colon \Delta \to \Gamma$ between animae, the pullback $f^*(X)$ of p along f is an anima.

Remark 8.2. Since $\mathcal{F}am\mathcal{P}roj \subseteq \mathcal{S}yn\mathcal{C}at$ is a wide strict subcategory, every isomorphism is a family projection and the composite of two family projections is a family projection.

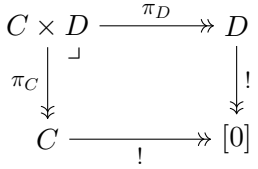
Definition 8.3. An **object** of a category C is a functor $c\colon [0] \to C$. We write $c \in C$.

A **term** of a family projection $p\colon C \twoheadrightarrow A$ is a section $s\colon A \to C$ of p, i.e., a functor $s\colon A \to C$ such that $ps = \mathrm{id}_A$.

Remark 8.4. Observe that a term of the family projection $C \twoheadrightarrow [0]$ is precisely the data of a functor $[0] \to C$, i.e., an object of C, since $[0]$ is terminal.

Lemma 8.5. *Let C and D be categories. Their product $C \times D$ exists and the canonical projections $\pi_C\colon C \times D \to C$, $\pi_D\colon C \times D \to D$ are family projections.*

Proof. Since the unique functors $C \twoheadrightarrow [0]$ and $D \twoheadrightarrow [0]$ are family projections, by (C4), there is a pullback square



and the pullbacks π_C and π_D of the unique morphisms are family projections as well. □

8.2. Families over a category as clans for category theory

Notation 8.6. Let A be a category.

- Let $\mathcal{F}am(A)$ denote the full strict subcategory of the slice $\mathcal{S}yn\mathcal{C}at_{/A}$ spanned by the family projections over A. We refer to its objects as **families of categories over** A.
- Let $\mathcal{A}n\mathcal{F}am(A)$ denote the full strict subcategory of $\mathcal{F}am(A)$ spanned by the family projections $p\colon C \twoheadrightarrow A$ such that, for every functor $f\colon \Gamma \to A$ with Γ an anima, the pullback $f^*(C)$ of p along f is an anima. We refer to its objects as **families of animae over** A.
- Let $\mathcal{F}am\mathcal{P}roj(A)$ denote the wide strict subcategory of $\mathcal{F}am(A)$ whose morphisms are the family projections over A.

Example 8.7. If A is a category and Γ is an anima, the family projection $\pi_A\colon A \times \Gamma \twoheadrightarrow A$ is a family of animae over A.

If $A = \Gamma$ is an anima, the objects of $\mathcal{A}n\mathcal{F}am(\Gamma)$ are simply the family projections $p\colon X \twoheadrightarrow \Gamma$ with X an anima, as we now show.

Lemma 8.8. *Let Γ be an anima. A family projection $p\colon C \twoheadrightarrow \Gamma$ is a family of animae over Γ, i.e., for every functor $f\colon \Delta \to \Gamma$ between animae, the pullback $f^*(C)$ of p along f is an anima, if and only if C is an anima.*

Proof. If $p\colon C \twoheadrightarrow \Gamma$ is a family of animae over Γ, taking $f = \mathrm{id}_\Gamma$, we get that $C \cong \mathrm{id}_\Gamma^*(C)$ is an anima by (C1). The converse holds by (C5). □

Remark 8.9. Since $\mathcal{S}yn\mathcal{C}at_{/[0]} \cong \mathcal{S}yn\mathcal{C}at$, $[0]$ is an anima by (C2), and every functor to the terminal object is a family projection by (C3), using Lemma 8.8, we have isomorphisms of strict categories

$$\mathcal{F}am([0]) \cong \mathcal{S}yn\mathcal{C}at, \quad \mathcal{A}n\mathcal{F}am([0]) \cong \mathcal{A}n \quad \text{and} \quad \mathcal{F}am\mathcal{P}roj([0]) \cong \mathcal{F}am\mathcal{P}roj.$$

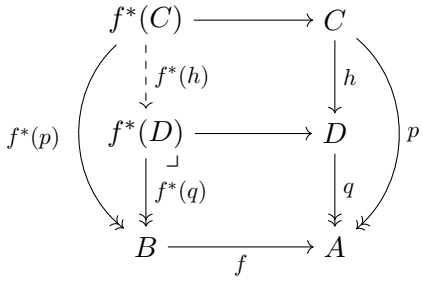
We will show that this data satisfies the conditions of a clan for category theory. But first, we need some preparatory constructions and results.

Construction 8.10. Given a functor $f\colon B \to A$, there is a functor between strict categories

$$f^*\colon \mathcal{F}am(A) \to \mathcal{F}am(B)$$

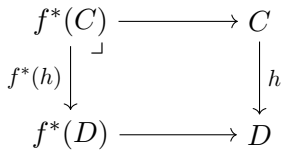
called the **pullback functor** along f, sending:

- a family projection $p\colon C \twoheadrightarrow A$ to a chosen pullback family projection $f^*(p)\colon f^*(C) \twoheadrightarrow B$, which exists by (C4);
- family projections $p\colon C \twoheadrightarrow A$ and $q\colon D \twoheadrightarrow A$, a functor $h\colon C \to D$ over A to the unique functor $f^*(h)\colon f^*(C) \to f^*(D)$ between their pullbacks along f.



This is a well-defined functor.

Remark 8.11. Observe that by the cancellation property of pullbacks, the top square in the above diagram is also a pullback.



In particular, if $h\colon C \twoheadrightarrow D$ is a family projection, then so is $f^*(h)\colon f^*(C) \twoheadrightarrow f^*(D)$ by (C4).

Remark 8.12. If $f\colon B \to [0]$ is the unique functor to the terminal object, using Remark 8.9, the pullback functor f^* is given by the functor between strict categories

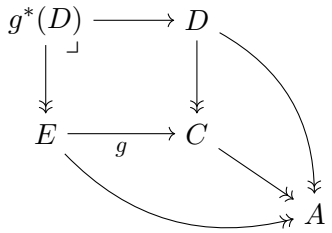
$$B \times (-)\colon \mathcal{S}yn\mathcal{C}at \to \mathcal{F}am(B).$$

Lemma 8.13. *Given a category A, pullbacks along family projections over A exist in $\mathcal{F}am(A)$ and are family projections. Moreover, given a functor $f\colon B \to A$, the pullback functor*

$$f^*\colon \mathcal{F}am(A) \to \mathcal{F}am(B)$$

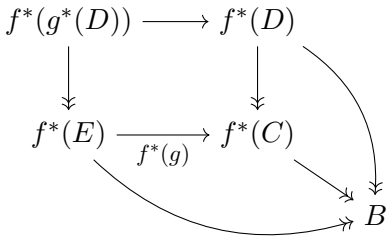
preserves pullbacks along family projections.

Proof. Given a family projection $D \twoheadrightarrow C$ and a functor $g\colon E \to C$ between families of categories over A, their pullback in $\mathcal{F}am(A)$ is given by the pullback $g^*(D)$ in $\mathcal{S}yn\mathcal{C}at$, which exists by (C4) for $\mathcal{S}yn\mathcal{C}at$,

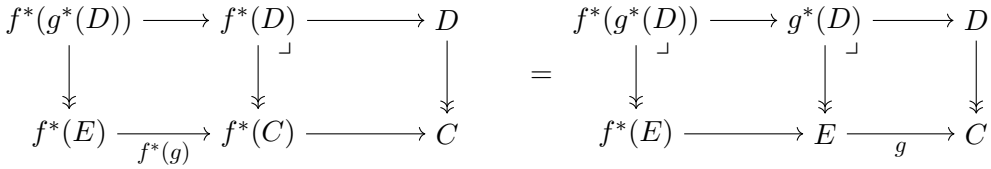


where $g^*(D)$ is regarded as an object of $\mathcal{F}am(A)$ through the composite of family projections $g^*(D) \twoheadrightarrow E \twoheadrightarrow A$. Observe that, using (C4) for $\mathcal{S}yn\mathcal{C}at$, the functor $g^*(D) \twoheadrightarrow E$ is a family projection over A.

Next, we need to show that the induced commutative square in $\mathcal{F}am(B)$



is a pullback square, where the vertical functors are family projections by Remark 8.11. Consider the following equality of commutative diagrams



where the middle squares are pullback squares by Remark 8.11 and the right-hand square is a pullback square by construction. By the cancellation property of pullbacks, we get that the left-hand square is a pullback square as well, as desired. □

We now prove that this gives a clan for category theory.

Proposition 8.14. *Let A be a category. The following data form a clan for category theory:*

(Cat) the strict category $\mathcal{F}am(A)$ of families of categories over A;
(An) its full strict subcategory $\mathcal{A}n\mathcal{F}am(A) \subseteq \mathcal{F}am(A)$ of families of animae over A;
(Fam) its wide strict subcategory $\mathcal{F}am\mathcal{P}roj(A) \subseteq \mathcal{F}am(A)$ of family projections over A.

Proof. We need to check the conditions of a clan for category theory:

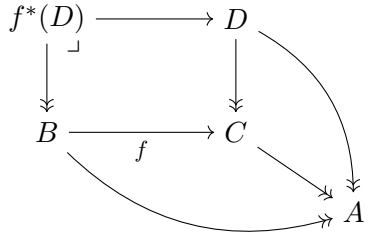
(C1) Let $C \twoheadrightarrow A$ and $D \twoheadrightarrow A$ be family projections such that $C \cong D$ over A, and suppose that C is a family of animae over A. Given a functor $f\colon \Gamma \to A$ with Γ an anima, we get an induced isomorphism between pullbacks $f^*(C) \cong f^*(D)$. Since $f^*(C)$ is an anima, we deduce from (C1) for $\mathcal{S}yn\mathcal{C}at$ that $f^*(D)$ is an anima as well. Hence $D \twoheadrightarrow A$ is a family of animae over A.

(C2) The terminal object in $\mathcal{F}am(A)$ is given by the identity functor $\mathrm{id}_A \colon A \twoheadrightarrow A$. Given a functor $f \colon \Gamma \to A$ with Γ an anima, then the pullback $f^*(A) \cong \Gamma$ is an anima. Hence $\mathrm{id}_A \colon A \twoheadrightarrow A$ is a family of animae over A.

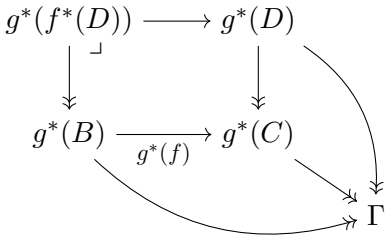
(C3) For every family projection $p \colon C \twoheadrightarrow A$, the unique functor to the terminal object $\mathrm{id}_A \colon A \twoheadrightarrow A$ is p itself, and so it is a family projection over A.

(C4) This is Lemma 8.13.

(C5) Let $D \twoheadrightarrow C$ be a family projection between families of animae over A and $f \colon B \to C$ be a functor between families of animae over A. Consider the pullback square in $\mathcal{F}am(A)$



Given a functor $g \colon \Gamma \to A$ with Γ an anima, by Lemma 8.13, there is an induced pullback square



Since $g^*(B)$, $g^*(C)$, and $g^*(D)$ are animae, we deduce from (C5) for $\mathcal{S}yn\mathcal{C}at$ that $g^*(f^*(D))$ is also an anima. Hence $f^*(D) \twoheadrightarrow A$ is a family of animae over A. □

Remark 8.15. When we do a construction for a category C or make a statement for an object $[0] \to C$ in a clan for category theory, by interpreting this in the clan $\mathcal{F}am(A)$, this amounts to doing a construction for a family projection $C \twoheadrightarrow A$ or making a statement for a term $A \to C$. Hence, every construction or statement can also be done for families of categories over a category A and for terms of these families.

Remark 8.16. For an anima Γ, the clan $\mathcal{F}am(\Gamma)$ will satisfy all the axioms of the synthetic theory of category theory and so it will form a model for category theory.

For a category A that is not an anima, the clan $\mathcal{F}am(A)$ will fail to inherit the structure of a model for category theory; for instance, the anima of sections axiom will not hold.

Definition 8.17. Let $(\mathcal{S}yn\mathcal{C}at, \mathcal{A}n, \mathcal{F}am\mathcal{P}roj)$ and $(\mathcal{S}yn\mathcal{C}at', \mathcal{A}n', \mathcal{F}am\mathcal{P}roj')$ be clans for category theory. A **clan morphism** between them is a functor $F \colon \mathcal{S}yn\mathcal{C}at \to \mathcal{S}yn\mathcal{C}at'$ between the underlying strict categories that preserves animae, family projections, terminal objects, and pullbacks along family projections.

Proposition 8.18. *Given a functor $f \colon B \to A$, the pullback functor between strict categories*

$$f^* \colon \mathcal{F}am(A) \to \mathcal{F}am(B)$$

is a clan morphism.

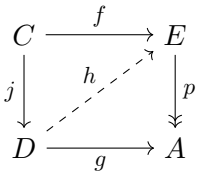
Proof. Exercise. □

9. First structural axioms

9.1. Equality families

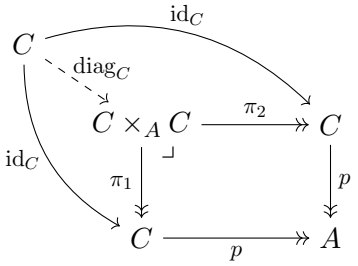
We are now going to introduce an internal notion of equality in a category.

Definition 9.1. A functor $j\colon C \to D$ is **anodyne** if, for every family projection $p\colon E \twoheadrightarrow A$, there is a lift in every commutative square

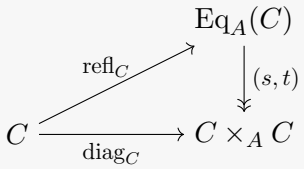


i.e., a functor $h\colon D \to E$ such that $hj = f$ and $ph = g$.

Construction 9.2. Let $p\colon C \twoheadrightarrow A$ be a family projection. Then the **diagonal functor** $\mathrm{diag}_C\colon C \to C \times_A C$ is the unique functor induced by the universal property of the pullback

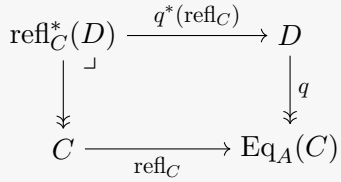


Axiom A (Equality families).
For every family projection $p\colon C \twoheadrightarrow A$, there is a factorization of the diagonal functor



into an anodyne functor $\mathrm{refl}_C\colon C \to \mathrm{Eq}_A(C)$, called the **reflexivity functor**, and a family projection $(s,t)\colon \mathrm{Eq}_A(C) \twoheadrightarrow C \times_A C$ such that:

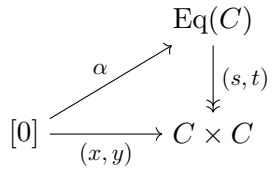
- the family projection $(s,t)\colon \mathrm{Eq}_A(C) \twoheadrightarrow C \times_A C$ is a family of animae over $C \times_A C$, i.e., for every functor $(f,g)\colon \Gamma \to C \times_A C$ with Γ an anima, the pullback $(f,g)^*(\mathrm{Eq}_A(C))$ is an anima;
- for every family projection $q\colon D \twoheadrightarrow \mathrm{Eq}_A(C)$, the pullback $q^*(\mathrm{refl}_C)\colon \mathrm{refl}_C^*(D) \to D$ of refl_C along q is also anodyne.



Remark 9.3. If a clan for category theory $\mathcal{S}yn\mathcal{C}at$ satisfies Axiom A, then, for every category A, the clan for category theory $\mathcal{F}am(A)$ satisfies Axiom A.

Notation 9.4. When $A = [0]$, we simply write $\mathrm{Eq}(C) \coloneqq \mathrm{Eq}_{[0]}(C)$.

Remark 9.5. Given objects $x, y\colon [0] \to C$, we think of an object $\alpha\colon [0] \to \mathrm{Eq}(C)$ making the following diagram commute



as an equality $\alpha\colon x =_C y$ between x and y in C.

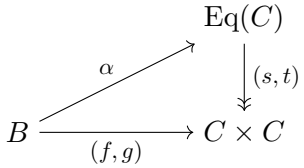
In contrast to the strict equality $x = y$ of functors $[0] \to C$, which is a property, an equality $\alpha\colon x =_C y$ is structure: there could be many different equalities between x and y. The anima $(x, y)^*(\mathrm{Eq}(C))$ remembers all the ways in which x and y can be identified, and how these identifications relate to one another. For every object $x\colon [0] \to C$, we have a functor

$$\mathrm{refl}_x\colon [0] \xrightarrow{x} C \xrightarrow{\mathrm{refl}_C} \mathrm{Eq}(C)$$

which gives an equality $\mathrm{refl}_x\colon x =_C x$. So a strict equality $x = y$ yields an equality $\mathrm{refl}_x\colon x =_C y$.

In what follows, we focus on family projections $C \twoheadrightarrow [0]$. By interpreting the definitions, constructions, and results in $\mathcal{F}am(A)$, we get the same definitions, constructions, and results for family projections $C \twoheadrightarrow A$.

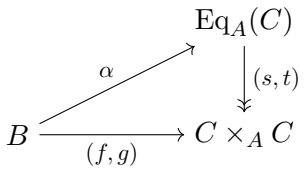
Definition 9.6. Given functors $f, g\colon B \to C$, a **homotopy** between f and g is a functor $\alpha\colon B \to \mathrm{Eq}(C)$ such that the following diagram commutes.



We write $\alpha\colon f \sim g$.

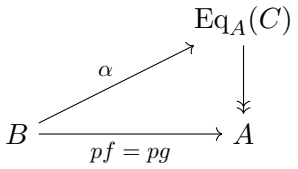
Remark 9.7. Observe that $\mathrm{id}_{\mathrm{Eq}(C)}\colon \mathrm{Eq}(C) \to \mathrm{Eq}(C)$ gives a homotopy $\mathrm{id}_{\mathrm{Eq}(C)}\colon s \sim t$.

Remark 9.8. Given a category A, the interpretation of this definition in $\mathcal{F}am(A)$ reads as follows. Given family projections $B \twoheadrightarrow A$ and $p\colon C \twoheadrightarrow A$ and functors $f, g\colon B \to C$ over A, a homotopy between f and g in $\mathcal{F}am(A)$ is a functor $\alpha\colon B \to \mathrm{Eq}_A(C)$ such that the following diagram commutes.



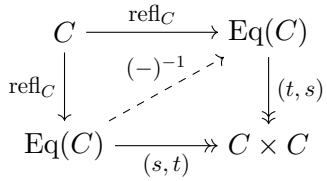
We write $\alpha\colon f \sim_A g$, and call it a **homotopy relative to** A.

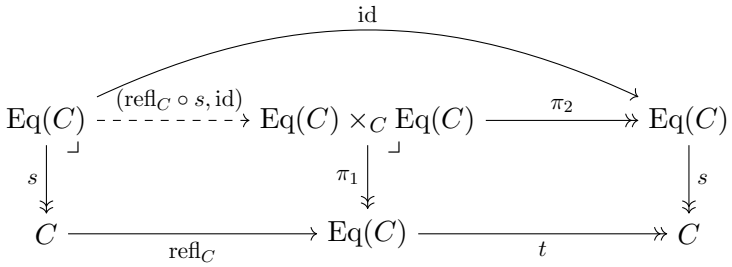
In particular, a homotopy $\alpha\colon f \sim_A g$ fixes the part that is over A, in the sense that the following diagram commutes.

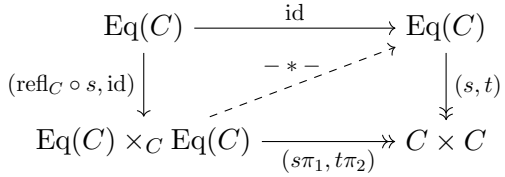


Homotopies are reflexive, symmetric, and transitive in a structural sense. We can build the structure as follows.

Construction 9.9. Let C be a category.

- (Reflexivity) The reflexivity functor induces, for every functor $f\colon B \to C$, a functor
$$\mathrm{refl}_f\colon B \xrightarrow{f} C \xrightarrow{\mathrm{refl}_C} \mathrm{Eq}(C)$$
which gives a homotopy $\mathrm{refl}_f\colon f \sim f$.
- (Symmetry) Since $\mathrm{refl}_C\colon C \to \mathrm{Eq}(C)$ is anodyne, there is a lift in the following commutative diagram.


For every homotopy $\alpha\colon f \sim g$ between functors $f, g\colon B \to C$, we get a functor
$$\alpha^{-1}\colon B \xrightarrow{\alpha} \mathrm{Eq}(C) \xrightarrow{(-)^{-1}} \mathrm{Eq}(C)$$
which gives a homotopy $\alpha^{-1}\colon g \sim f$.
- (Transitivity) Consider the following commutative diagram


where the right-hand square is a pullback square by construction, and the left-hand square is a pullback square by cancellation. Since $(\mathrm{refl}_C \circ s, \mathrm{id})\colon \mathrm{Eq}(C) \to \mathrm{Eq}(C) \times_C \mathrm{Eq}(C)$ is a pullback of refl_C along a family projection, it is anodyne, and so there is a lift in the following commutative diagram.


For all homotopies $\alpha\colon f \sim g$ and $\beta\colon g \sim h$ between functors $f, g, h\colon B \to C$, we get a functor
$$\alpha * \beta\colon B \xrightarrow{(\alpha,\beta)} \mathrm{Eq}(C) \times_C \mathrm{Eq}(C) \xrightarrow{-*-} \mathrm{Eq}(C)$$

which gives a homotopy $\alpha * \beta \colon f \sim h$.

Remark 9.10. By interpreting the above constructions in $\mathcal{F}am(A)$, we also get the structure of reflexivity, symmetry, and transitivity for homotopies relative to A. This will be true for the remainder of the constructions below as well, but we will not mention it again.

Definition 9.11. A **commutative triangle** consists of functors f, g, h and a homotopy $\alpha \colon gf \sim h$, as depicted below left. A **commutative square** consists of functors f, g, h, k and a homotopy $\alpha \colon hf \sim kg$, as depicted below right.



Proposition 9.12. *Let C be a category. We have that:*

- *(Left unitality) The following triangle commutes strictly.*

$$\begin{array}{ccc} \mathrm{Eq}(C) & \xrightarrow{(\mathrm{refl} \circ s, \mathrm{id})} & \mathrm{Eq}(C) \times_C \mathrm{Eq}(C) \\ & \searrow^{\mathrm{id}} & \downarrow{\scriptstyle - * -} \\ & & \mathrm{Eq}(C) \xrightarrow{(s,t)} C \times C \end{array}$$

  For every homotopy $\alpha \colon f \sim g$ between functors $f, g \colon B \to C$, we have that $\mathrm{refl}_f * \alpha = \alpha$. In particular, we get a relative homotopy

$$\mathtt{lunit}_\alpha \coloneqq \mathrm{refl}_\alpha \colon \mathrm{refl}_f * \alpha \sim_{C \times C} \alpha.$$

- *(Right unitality) There is a commutative triangle over $C \times C$,*

$$\begin{array}{ccc} \mathrm{Eq}(C) & \xrightarrow{(\mathrm{id}, \mathrm{refl} \circ t)} & \mathrm{Eq}(C) \times_C \mathrm{Eq}(C) \\ & \searrow^{\mathrm{id}} \quad \mathtt{runit} & \downarrow{\scriptstyle - * -} \\ & & \mathrm{Eq}(C) \xrightarrow{(s,t)} C \times C \end{array}$$

  i.e., `runit` *is a homotopy relative to $C \times C$ between the two composites. For every homotopy $\alpha \colon f \sim g$ between functors $f, g \colon B \to C$, we get a relative homotopy*

$$\mathtt{runit}_\alpha \colon \alpha * \mathrm{refl}_g \sim_{C \times C} \alpha.$$

- *(Associativity) There is a commutative square over $C \times C$*

$$\begin{array}{ccc} \mathrm{Eq}(C) \times_C \mathrm{Eq}(C) \times_C \mathrm{Eq}(C) & \xrightarrow{\mathrm{id} \times_C (- * -)} & \mathrm{Eq}(C) \times_C \mathrm{Eq}(C) \\ \downarrow{\scriptstyle (- * -) \times_C \mathrm{id}} \quad \mathtt{ass} & & \downarrow{\scriptstyle - * -} \\ \mathrm{Eq}(C) \times_C \mathrm{Eq}(C) & \xrightarrow[- * -]{} & \mathrm{Eq}(C) \xrightarrow{(s,t)} C \times C \end{array}$$

  For all homotopies $\alpha \colon f \sim g$, $\beta \colon g \sim h$, $\gamma \colon h \sim k$ between functors $f, g, h, k \colon B \to C$, we get a relative homotopy

$$\mathtt{ass}_{\alpha,\beta,\gamma} \colon \alpha * (\beta * \gamma) \sim_{C \times C} (\alpha * \beta) * \gamma.$$

- *(Right inverse) There is a commutative square over $C \times C$*

$$\begin{array}{ccccc} \mathrm{Eq}(C) & \xrightarrow{(\mathrm{id},(-)^{-1})} & \mathrm{Eq}(C) \times_C \mathrm{Eq}(C) & & \\ {\scriptstyle s}\downarrow & \texttt{rinv} & \downarrow{\scriptstyle -*-} & & \\ C & \xrightarrow[\mathrm{refl}_C]{} & \mathrm{Eq}(C) & \xrightarrow{(s,t)} & C \times C \end{array}$$

For every homotopy $\alpha\colon f \sim g$ between functors $f, g\colon B \to C$, we get a relative homotopy

$$\texttt{rinv}_\alpha\colon \alpha * \alpha^{-1} \sim_{C\times C} \mathrm{refl}_f.$$

- *(Left inverse) There is a commutative square over $C \times C$*

$$\begin{array}{ccccc} \mathrm{Eq}(C) & \xrightarrow{((-)^{-1},\mathrm{id})} & \mathrm{Eq}(C) \times_C \mathrm{Eq}(C) & & \\ {\scriptstyle t}\downarrow & \texttt{linv} & \downarrow{\scriptstyle -*-} & & \\ C & \xrightarrow[\mathrm{refl}_C]{} & \mathrm{Eq}(C) & \xrightarrow{(s,t)} & C \times C \end{array}$$

For every homotopy $\alpha\colon f \sim g$ between functors $f, g\colon B \to C$, we get a relative homotopy

$$\texttt{linv}_\alpha\colon \alpha^{-1} * \alpha \sim_{C\times C} \mathrm{refl}_g.$$

- *(Inverse of inverse) There is a commutative triangle over $C \times C$.*

$$\begin{array}{ccccc} \mathrm{Eq}(C) & \xrightarrow{(-)^{-1}} & \mathrm{Eq}(C) & & \\ & \searrow^{\mathrm{id}} & \downarrow{\scriptstyle (-)^{-1}} \ \texttt{invinv} & & \\ & & \mathrm{Eq}(C) & \xrightarrow{(s,t)} & C \times C \end{array}$$

For every homotopy $\alpha\colon f \sim g$ between functors $f, g\colon B \to C$, we get a relative homotopy

$$\texttt{invinv}_\alpha\colon (\alpha^{-1})^{-1} \sim_{C\times C} \alpha.$$

Proof. Left unitality holds by construction of $- * -$.

The rest is left as an exercise. For this, use the lifting property of anodyne functors with respect to family projections to construct homotopies between relevant composites. □

Functors also preserve homotopies in a structural sense. We can build the structure as follows.

Construction 9.13. Let $f\colon C \to D$ be a functor. Since $\mathrm{refl}_C\colon C \to \mathrm{Eq}(C)$ is anodyne, there is a lift in the following commutative diagram.

$$\begin{array}{ccccc} C & \xrightarrow{f} & D & \xrightarrow{\mathrm{refl}_D} & \mathrm{Eq}(D) \\ {\scriptstyle \mathrm{refl}_C}\downarrow & \nearrow_{\mathrm{Eq}(f)} & & & \downarrow{\scriptstyle (s,t)} \\ \mathrm{Eq}(C) & \xrightarrow[(s,t)]{} & C \times C & \xrightarrow[f\times f]{} & D \times D \end{array}$$

For every homotopy $\alpha\colon g \sim h$ between functors $g, h\colon B \to C$, we get a functor

$$f\alpha\colon B \xrightarrow{\alpha} \mathrm{Eq}(C) \xrightarrow{\mathrm{Eq}(f)} \mathrm{Eq}(D)$$

which gives a homotopy $f\alpha\colon fg \sim fh$.

Proposition 9.14. *Let $f\colon C \to D$ be a functor.*

- *(Unital) The following square commutes strictly.*

$$\begin{array}{ccccc} C & & \xrightarrow{\quad f \quad} & & D \\ {\scriptstyle \mathrm{refl}_C}\downarrow & & & & \downarrow{\scriptstyle \mathrm{refl}_D} \\ \mathrm{Eq}(C) & & \xrightarrow[\mathrm{Eq}(f)]{} & \mathrm{Eq}(D) & \xrightarrow{(s,t)} \!\!\!\rightarrow D \times D \end{array}$$

  For every functor $g\colon B \to C$, we have that $\mathrm{refl}_{fg} = f\mathrm{refl}_g$. In particular, we get a relative homotopy

$$\mathtt{unit}^f_g \coloneqq \mathrm{refl}_{\mathrm{refl}_{fg}}\colon \mathrm{refl}_{fg} \sim_{D\times D} f\mathrm{refl}_g.$$

- *(Functorial) There is a commutative square over $D \times D$*

$$\begin{array}{ccccc} \mathrm{Eq}(C) \times_C \mathrm{Eq}(C) & & \xrightarrow{\mathrm{Eq}(f) \times_f \mathrm{Eq}(f)} & & \mathrm{Eq}(D) \times_D \mathrm{Eq}(D) \\ {\scriptstyle - * -}\downarrow & & \mathtt{funct}^f & & \downarrow{\scriptstyle - * -} \\ \mathrm{Eq}(C) & & \xrightarrow[\mathrm{Eq}(f)]{} & \mathrm{Eq}(D) & \xrightarrow{(s,t)} \!\!\!\rightarrow D \times D \end{array}$$

  For all homotopies $\alpha\colon g \sim h$ and $\beta\colon h \sim k$ between functors $g, h, k\colon B \to C$, we get a relative homotopy

$$\mathtt{funct}^f_{\alpha,\beta}\colon f\alpha * f\beta \sim_{D\times D} f(\alpha * \beta).$$

Proof. Unitality holds by construction of $\mathrm{Eq}(f)$. Functoriality is left as an exercise. □

Construction 9.15. Let $f\colon B \to C$ be a functor. For every homotopy $\alpha\colon g \sim h$ between functors $g, h\colon C \to D$, we get a functor

$$\alpha f\colon B \xrightarrow{f} C \xrightarrow{\alpha} \mathrm{Eq}(D)$$

which gives a homotopy $\alpha f\colon gf \sim hf$.

9.2. **Equivalences**

Definition 9.16. A functor $f\colon C \to D$ is an **equivalence** if there is a functor $g\colon D \to C$ and homotopies $\alpha\colon gf \sim \mathrm{id}_C$ and $\beta\colon fg \sim \mathrm{id}_D$.

Remark 9.17. Given a category A, the interpretation of this definition in $\mathcal{F}am(A)$ reads as follows. Given family projections $p\colon C \twoheadrightarrow A$ and $q\colon D \twoheadrightarrow A$, a functor over A

$$\begin{array}{ccc} C & \xrightarrow{\quad f \quad} & D \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle q} & \\ & A & \end{array}$$

is an equivalence in $\mathcal{F}am(A)$ if there is a functor $g\colon D \to C$ over A, i.e., $pg = q$, and relative homotopies $\alpha\colon gf \sim_A \mathrm{id}_C$ and $\beta\colon fg \sim_A \mathrm{id}_D$. We say that f is an **equivalence over** A.

Proposition 9.18. *Every anodyne functor is an equivalence.*

Proof. Exercise. □

Remark 9.19. Given a category A, observe that a functor in $\mathcal{F}am(A)$ is anodyne if and only if it is so in $\mathcal{S}yn\mathcal{C}at$.

As a special case, we obtain the following.

Corollary 9.20. *Let C be a category. The reflexivity functor $\mathrm{refl}_C \colon C \to \mathrm{Eq}(C)$ is an equivalence.*

Remark 9.21. Given a category A, the interpretation of this result in $\mathcal{F}am(A)$ reads as follows. Given a family projection $p \colon C \twoheadrightarrow A$, the reflexivity functor

$$\begin{array}{ccc} C & \xrightarrow{\mathrm{refl}_C} & \mathrm{Eq}_A(C) \\ & {\scriptstyle p}\searrow \quad \swarrow & \\ & A & \end{array}$$

is an equivalence over A.

Proposition 9.22. *Let $f \colon C \to D$ be an equivalence. Then f reflects homotopies.*

Proof. Let $g \colon D \to C$ and $\alpha \colon gf \sim \mathrm{id}_C$. Let $h, k \colon B \to C$ be functors and $\beta \colon fh \sim fk$ be a homotopy. We have homotopies

$$h \overset{\alpha^{-1}h}{\sim} gfh \overset{g\beta}{\sim} gfk \overset{\alpha k}{\sim} k$$

so their concatenation provides a homotopy $\alpha^{-1}h * g\beta * \alpha k \colon h \sim k$. □

Proposition 9.23 (2-out-of-3 property)**.** *Consider a commutative triangle*

$$\begin{array}{ccc} C & \xrightarrow[h]{} & E \\ & {\scriptstyle f}\nearrow \quad \downarrow{\scriptstyle g} & \\ & \alpha \quad D & \end{array}$$

with homotopy $\alpha \colon gf \sim h$. Then, if two out of f, g, h are equivalences, so is the third.

Proof. We prove one case, the other two are left as an exercise.

Suppose that f and h are equivalences, and denote by $f' \colon D \to C$, $h' \colon E \to C$ and

$$\beta \colon f'f \sim \mathrm{id}_C, \quad \gamma \colon ff' \sim \mathrm{id}_D, \quad \delta \colon h'h \sim \mathrm{id}_C, \quad \varepsilon \colon hh' \sim \mathrm{id}_E$$

the data making them into equivalences. We prove that the composite $fh' \colon E \to D$ is an inverse of g. We have homotopies

$$gfh' \overset{\alpha h'}{\sim} hh' \overset{\varepsilon}{\sim} \mathrm{id}_E$$

so their concatenation provides a homotopy $\alpha h' * \varepsilon \colon gfh' \sim \mathrm{id}_E$. We also have homotopies

$$fh'g \overset{fh'g\gamma^{-1}}{\sim} fh'gff' \overset{fh'\alpha f'}{\sim} fh'hf' \overset{f\delta f'}{\sim} ff' \overset{\gamma}{\sim} \mathrm{id}_D$$

so their concatenation provides a homotopy $fh'g\gamma^{-1} * fh'\alpha f' * f\delta f' * \gamma \colon fh'g \sim \mathrm{id}_D$. □

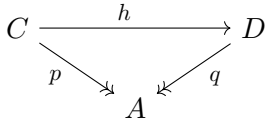
Proposition 9.24 (Rectification)**.** *Let $p \colon C \twoheadrightarrow A$ and $q \colon D \twoheadrightarrow A$ be family projections, and $f \colon C \to D$ be a functor over A.*

$$\begin{array}{ccc} C & \xrightarrow{\quad f \quad} & D \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle q} & \\ & A & \end{array}$$

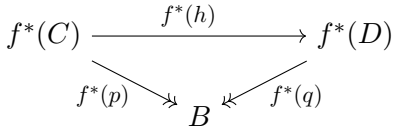
Then f is an equivalence over A if and only if it is an equivalence.

Proof. Exercise. □

Proposition 9.25 (Pullback stability)**.** *Let $p\colon C \twoheadrightarrow A$ and $q\colon D \twoheadrightarrow A$ be family projections. A functor over A*



is an equivalence over A if and only if, for every functor $f\colon B \to A$, the induced functor over B

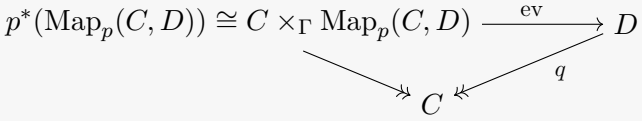


is an equivalence over B.

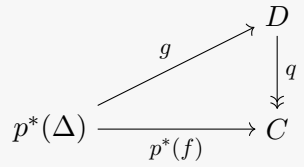
Proof. Exercise. □

9.3. **Anima of sections and functor extensionality**

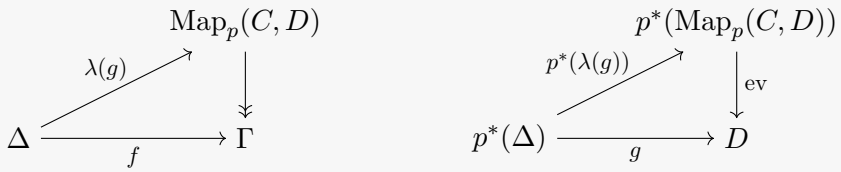
Axiom B (Anima of sections).
Let Γ be an anima and $p\colon C \twoheadrightarrow \Gamma$ be a family projection. For every family projection $q\colon D \twoheadrightarrow C$, there is a family projection $\mathrm{Map}_p(C,D) \twoheadrightarrow \Gamma$ with $\mathrm{Map}_p(C,D)$ an anima together with a functor over C



called the **evaluation functor**, satisfying the following property: for every functor $f\colon \Delta \to \Gamma$ between animae and every functor $g\colon p^*(\Delta) \cong C \times_\Gamma \Delta \to D$ making the following diagram commute strictly



there is a unique functor $\lambda(g)\colon \Delta \to \mathrm{Map}_p(C,D)$ making the following diagrams commute strictly.



Remark 9.26. If a clan for category theory $\mathcal{S}yn\mathcal{C}at$ satisfies Axiom B, then, for every anima Γ, the clan for category theory $\mathcal{F}am(\Gamma)$ satisfies Axiom B. This is generally not the case for the clan of families over an arbitrary category.

Remark 9.27. The universal property implies that there is a bijection

$$\mathcal{A}n_{/\Gamma}(\Delta, \mathrm{Map}_p(C,D)) \cong \mathcal{S}yn\mathcal{C}at_{/C}(p^*(\Delta), D)$$
$$\Delta \xrightarrow{h} \mathrm{Map}_p(C,D) \mapsto p^*(\Delta) \xrightarrow{p^*(h)} p^*(\mathrm{Map}_p(C,D)) \xrightarrow{\mathrm{ev}} D$$

which is natural in Δ. In particular, a term $\Gamma \to \mathrm{Map}_p(C,D)$ of $\mathrm{Map}_p(C,D) \twoheadrightarrow \Gamma$ corresponds to a section $C \to D$ of $D \twoheadrightarrow C$, since $p^*(\Gamma) \cong C$.

By restricting to the cases where f is a family projection, we get:

Proposition 9.28. *For every anima Γ and every family projection $p\colon C \twoheadrightarrow \Gamma$, the functor between strict categories $p^*\colon \mathcal{A}n\mathcal{F}am(\Gamma) \to \mathcal{F}am(C)$ admits a right adjoint:*

$$p^*\colon \mathcal{A}n\mathcal{F}am(\Gamma) \rightleftarrows \mathcal{F}am(C) : \mathrm{Map}_p(C,-).$$

In particular, the right adjoint $\mathrm{Map}_p(C,-)\colon \mathcal{F}am(C) \to \mathcal{A}n\mathcal{F}am(\Gamma)$ preserves terminal objects and pullbacks that exist in $\mathcal{F}am(C)$.

Proof. This follows directly from the pointwise formula for adjunction using the universal property of the evaluation functor. □

Notation 9.29. When $\Gamma = [0]$, given a family projection $D \twoheadrightarrow C$, we simply write

$$\mathrm{Map}_C(C,D) \coloneqq \mathrm{Map}_!(C,D),$$

where $!\colon C \twoheadrightarrow [0]$ denotes the unique functor.

Remark 9.30. An object $[0] \to \mathrm{Map}_C(C,D)$ corresponds to a section $C \to D$ of $D \twoheadrightarrow C$.

Translating Proposition 9.28 in the case of the family projection $C \twoheadrightarrow [0]$, we get:

Corollary 9.31. *Let C be a category. There is an adjunction*

$$C \times (-)\colon \mathcal{A}n \rightleftarrows \mathcal{F}am(C) : \mathrm{Map}_C(C,-).$$

To state the functor extensionality axiom, we first need the following technical lemma.

Lemma 9.32. *For every anima Γ and every family projection $p\colon C \twoheadrightarrow \Gamma$, the functor $\mathrm{Map}_p(C,-)\colon \mathcal{F}am(C) \to \mathcal{A}n\mathcal{F}am(\Gamma)$ preserves family projections.*

Proof. Let $D \twoheadrightarrow C$ and $E \twoheadrightarrow C$ be family projections and consider a family projection $q\colon E \twoheadrightarrow D$ over C. We need to prove that $\mathrm{Map}_p(C,q)\colon \mathrm{Map}_p(C,E) \to \mathrm{Map}_p(C,D)$ is a family projection over Γ.

Consider the following pullback squares

$$\begin{array}{ccc} p^*(\mathrm{Map}_p(C,D)) & \longrightarrow & C \\ {\scriptstyle p'}\downarrow & \lrcorner & \downarrow{\scriptstyle p} \\ \mathrm{Map}_p(C,D) & \longrightarrow & \Gamma \end{array} \qquad \begin{array}{ccc} \mathrm{ev}^*(E) & \longrightarrow & E \\ \downarrow & \lrcorner & \downarrow{\scriptstyle q} \\ p^*(\mathrm{Map}_p(C,D)) & \xrightarrow[\mathrm{ev}]{} & D \\ {\scriptstyle p'}\downarrow & & \\ \mathrm{Map}_p(C,D) & & \end{array}$$

Then we can form the anima of sections $\mathrm{Map}_{p'}(p^*(\mathrm{Map}_p(C,D)), \mathrm{ev}^*(E)) \twoheadrightarrow \mathrm{Map}_p(C,D)$. We prove that there is an isomorphism over $\mathrm{Map}_p(C,D)$

$$\begin{array}{ccc} \mathrm{Map}_p(C,E) & \xrightarrow{\cong} & \mathrm{Map}_{p'}(p^*(\mathrm{Map}_p(C,D)),\mathrm{ev}^*(E)) \\ & {\scriptstyle \mathrm{Map}_p(C,q)} \searrow \quad \swarrow & \\ & \mathrm{Map}_p(C,D) & \end{array}$$

which would prove that $\mathrm{Map}_p(C,q)$ is a family projection as it is isomorphic to one.

We show that, for every functor $X \to \mathrm{Map}_p(C,D)$, we have a natural bijection

$$\mathcal{A}n_{/\,\mathrm{Map}_p(C,D)}(X, \mathrm{Map}_{p'}(p^*(\mathrm{Map}_p(C,D)),\mathrm{ev}^*(E))) \cong \mathcal{A}n_{/\,\mathrm{Map}_p(C,D)}(X, \mathrm{Map}_p(C,E))$$

and conclude using the Yoneda lemma.

Let $X \to \mathrm{Map}_p(C,D)$ be a functor between animae. A functor over $\mathrm{Map}_p(C,D)$

$$\begin{array}{ccc} X & \longrightarrow & \mathrm{Map}_{p'}(p^*(\mathrm{Map}_p(C,D)),\mathrm{ev}^*(E)) \\ & \searrow \quad \swarrow & \\ & \mathrm{Map}_p(C,D) & \end{array}$$

corresponds by the universal property of $\mathrm{Map}_{p'}(p^*(\mathrm{Map}_p(C,D)),-)$ to a unique functor over $p^*(\mathrm{Map}_p(C,D))$

$$\begin{array}{ccc} (p')^*X \cong p^*(X) & \longrightarrow & \mathrm{ev}^*(E) \\ & \searrow \quad \swarrow & \\ & p^*(\mathrm{Map}_p(C,D)) & \end{array}$$

Such a functor corresponds, by the universal property of the pullback $\mathrm{ev}^*(E)$ to a unique functor making the following square commute strictly

$$\begin{array}{ccccc} p^*(X) & \longrightarrow & E & & \\ \downarrow & & \downarrow{\scriptstyle q} & \searrow & \\ p^*(\mathrm{Map}_p(C,D)) & \xrightarrow[\mathrm{ev}]{} & D & \twoheadrightarrow & C \end{array}$$

Observe that this square further lives over C. Then, such a functor corresponds, by the universal property of $\mathrm{Map}_p(C,-)$ to a unique functor over $\mathrm{Map}_p(C,D)$.

$$\begin{array}{ccccc} X & \longrightarrow & \mathrm{Map}_p(C,E) & & \\ \downarrow & & {\scriptstyle \mathrm{Map}_p(C,q)}\downarrow & \searrow & \\ \mathrm{Map}_p(C,D) & \xrightarrow[\mathrm{id}]{} & \mathrm{Map}_p(C,D) & \twoheadrightarrow & \Gamma \end{array}$$

This concludes the proof. □

Construction 9.33. Let Γ be an anima, and $p\colon C \twoheadrightarrow \Gamma$ and $D \twoheadrightarrow C$ be family projections. Since $\mathrm{refl}_{\mathrm{Map}_p(C,D)}\colon \mathrm{Map}_p(C,D) \to \mathrm{Eq}_\Gamma(\mathrm{Map}_p(C,D))$ is anodyne, there is a lift in the following strictly commutative diagram

$$\begin{array}{ccc} \mathrm{Map}_p(C,D) & \xrightarrow{\mathrm{Map}_p(C,\mathrm{refl}_D)} & \mathrm{Map}_p(C,\mathrm{Eq}_C(D)) \\ {\scriptstyle \mathrm{refl}_{\mathrm{Map}_p(C,D)}}\downarrow & \nearrow & \downarrow{\scriptstyle (\mathrm{Map}_p(C,s),\mathrm{Map}_p(C,t))} \\ \mathrm{Eq}_\Gamma(\mathrm{Map}_p(C,D)) & \xrightarrow[(s,t)]{} & \mathrm{Map}_p(C,D)\times_\Gamma \mathrm{Map}_p(C,D) \end{array}$$

where we use that $\mathrm{Map}_p(C,-)$ preserves terminal objects, pullbacks, and family projections.

Axiom C (Functor extensionality).
For every anima Γ and all family projections $p\colon C \twoheadrightarrow \Gamma$ and $D \twoheadrightarrow C$, the functor from Construction 9.33

$$\begin{array}{ccc} \mathrm{Eq}_\Gamma(\mathrm{Map}_p(C,D)) & \longrightarrow & \mathrm{Map}_p(C,\mathrm{Eq}_C(D)) \\ & {\scriptstyle (s,t)}\searrow \quad \swarrow{\scriptstyle (\mathrm{Map}_p(C,s),\mathrm{Map}_p(C,t))} & \\ & \mathrm{Map}_p(C,D)\times_\Gamma \mathrm{Map}_p(C,D) & \end{array}$$

is an equivalence over $\mathrm{Map}_p(C,D)\times_\Gamma \mathrm{Map}_p(C,D)$.

9.4. **Mapping animae and homotopies**

Definition 9.34. Let C and D be categories. The **mapping anima** is the anima

$$\mathrm{Map}(C,D) \coloneqq \mathrm{Map}_C(C, C\times D)$$

where $C\times D$ is regarded as a family projection over C through the canonical projection $\pi_C\colon C\times D \twoheadrightarrow C$.

Remark 9.35. Given categories C and D, an object $[0]\to \mathrm{Map}(C,D)$ corresponds to a section $C\to C\times D$ of the canonical projection $\pi_C\colon C\times D\twoheadrightarrow C$, corresponding to a functor $C\to D$. If $f\colon C\to D$ is a functor, we also denote by $f\in \mathrm{Map}(C,D)$ the corresponding object.

Proposition 9.36. *For every category C, there is an adjunction:*

$$C\times(-)\colon \mathcal{A}n \rightleftarrows \mathcal{S}yn\mathcal{C}at : \mathrm{Map}(C,-).$$

Proof. This adjunction is obtained by composing the adjunctions

$$C\times(-)\colon \mathcal{A}n \rightleftarrows \mathcal{F}am(C) : \mathrm{Map}_C(C,-) \quad \text{and} \quad \pi\colon \mathcal{F}am(C) \rightleftarrows \mathcal{S}yn\mathcal{C}at : C\times(-)$$

where the composite of left adjoints is

$$C\times(-)\colon \mathcal{A}n \xrightarrow{C\times(-)} \mathcal{F}am(C) \xrightarrow{\pi} \mathcal{S}yn\mathcal{C}at$$

and the composite of right adjoints is

$$\mathrm{Map}(C,-)\colon \mathcal{S}yn\mathcal{C}at \xrightarrow{C\times(-)} \mathcal{F}am(C) \xrightarrow{\mathrm{Map}_C(C,-)} \mathcal{A}n. \qquad \square$$

Definition 9.37. The **underlying anima** of a category C is

$$C^{\simeq} \coloneqq \mathrm{Map}([0],C).$$

Construction 9.38. Let $f\colon C\to D$ be a functor. Given a category A, the functor

$$A\times \mathrm{Map}(A,C) \xrightarrow{\mathrm{ev}} C \xrightarrow{f} D$$

corresponds under the adjunction $A \times (-) \dashv \mathrm{Map}(A, -)$ to the functor

$$f_* \coloneqq \mathrm{Map}(A, f) = \lambda(f \circ \mathrm{ev})\colon \mathrm{Map}(A, C) \to \mathrm{Map}(A, D),$$

called the **postcomposition functor** with f.

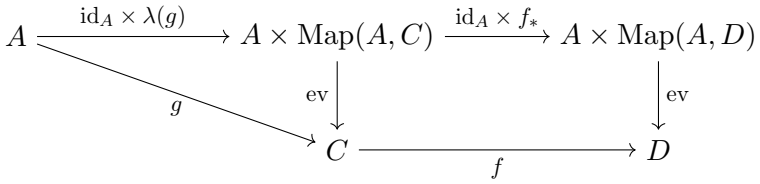
Lemma 9.39. *Let $g\colon A \to C$ be a functor, and $\lambda(g)\colon [0] \to \mathrm{Map}(A, C)$ be the corresponding object. Then*

$$[0] \xrightarrow{\lambda(g)} \mathrm{Map}(A, C) \xrightarrow{f_*} \mathrm{Map}(A, D)$$

corresponds under the adjunction $A \times (-) \dashv \mathrm{Map}(A, -)$ to the functor

$$fg\colon A \to D.$$

Proof. By construction, the following diagram commutes strictly.



Hence $\lambda(fg) = f_*\lambda(g)$, and by unicity fg is the unique functor corresponding to $f_*\lambda(g)$. □

We now give a mapping anima version of functor extensionality.

Proposition 9.40. *Given categories C and D, the functor*

$$\begin{array}{ccc} \mathrm{Eq}(\mathrm{Map}(C, D)) & \longrightarrow & \mathrm{Map}(C, \mathrm{Eq}(D)) \\ & {\scriptstyle (s,t)} \searrow \quad \swarrow {\scriptstyle (s_*, t_*)} & \\ & \mathrm{Map}(C, D) \times \mathrm{Map}(C, D) & \end{array}$$

is an equivalence over $\mathrm{Map}(C, D) \times \mathrm{Map}(C, D)$.

Proof. Exercise. □

Remark 9.41. Given functors $f, g\colon C \to D$, also regarded as objects $f, g\colon [0] \to \mathrm{Map}(C, D)$, the above result implies that an equality $\alpha\colon f =_{\mathrm{Map}(C,D)} g$ in $\mathrm{Map}(C, D)$ corresponds up to equivalence to a homotopy $\alpha\colon f \sim g$.

Construction 9.42. Given categories A and D, the inverse $\mathrm{Map}(A, \mathrm{Eq}(D)) \xrightarrow{\simeq} \mathrm{Eq}(\mathrm{Map}(A, D))$ of the equivalence from Proposition 9.40 induces a homotopy

$$\mathtt{post}\colon s_* \sim t_*\colon \mathrm{Map}(A, \mathrm{Eq}(D)) \to \mathrm{Map}(A, D).$$

For every homotopy $\alpha\colon f \sim g$ between functors $f, g\colon C \to D$, we get a homotopy

$$f_* = (s\alpha)_* = s_*\alpha_* \overset{\mathtt{post}\,\alpha_*}{\sim} t_*\alpha_* = (t\alpha)_* = g_*$$

between functors $f_*, g_*\colon \mathrm{Map}(A, C) \to \mathrm{Map}(A, D)$.

Construction 9.43. Let $f\colon C \to D$ be a functor. Given a category E, the functor

$$C \times \mathrm{Map}(D, E) \xrightarrow{f \times \mathrm{id}} D \times \mathrm{Map}(D, E) \xrightarrow{\mathrm{ev}} E$$

corresponds under the adjunction $C \times (-) \dashv \mathrm{Map}(C, -)$ to a functor

$$f^* \coloneqq \lambda(\mathrm{ev}(f \times \mathrm{id}))\colon \mathrm{Map}(D, E) \to \mathrm{Map}(C, E),$$

called the **precomposition functor** with f.

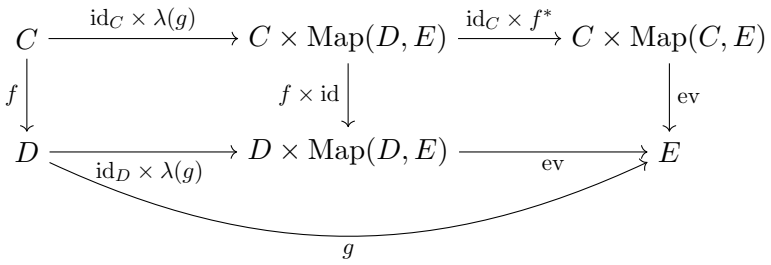
Lemma 9.44. *Let $g\colon D \to E$ be a functor, and $\lambda(g)\colon [0] \to \mathrm{Map}(D,E)$ be the corresponding object. Then*

$$[0] \xrightarrow{\lambda(g)} \mathrm{Map}(D,E) \xrightarrow{f^*} \mathrm{Map}(C,E)$$

corresponds under the adjunction $C \times (-) \dashv \mathrm{Map}(C,-)$ to the functor

$$gf\colon C \to E.$$

Proof. By construction, the following diagram commutes strictly.

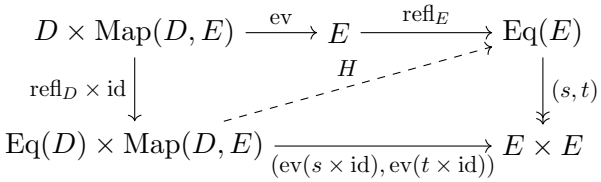


Hence $\lambda(gf) = f^*\lambda(g)$, and by unicity gf is the unique functor corresponding to $f^*\lambda(g)$. □

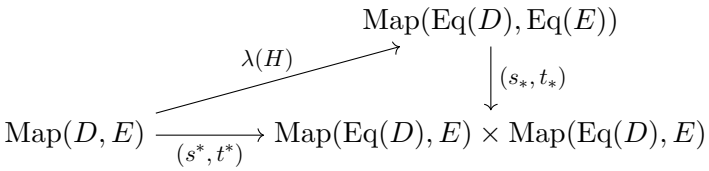
Lemma 9.45. *Let D and E be categories. There is a homotopy*

$$\mathtt{pre}\colon s^* \sim t^*\colon \mathrm{Map}(D,E) \to \mathrm{Map}(\mathrm{Eq}(D),E).$$

Proof. Since the left hand functor in the following commutative diagram is anodyne, it admits a lift



By the universal property of the mapping anima, the bottom commutative triangle corresponds to a triangle



where we use that $\lambda(\mathrm{ev}(s\times\mathrm{id})) = s^*$ and $\lambda(\mathrm{ev}(t\times\mathrm{id})) = t^*$, by definition. By Proposition 9.40, we get the desired homotopy as the composite

$$\mathtt{pre}\colon \mathrm{Map}(D,E) \xrightarrow{\lambda(H)} \mathrm{Map}(\mathrm{Eq}(D),\mathrm{Eq}(E)) \xrightarrow{\simeq} \mathrm{Eq}(\mathrm{Map}(\mathrm{Eq}(D),E)).$$ □

Construction 9.46. Given categories D and E, the homotopy

$$\mathtt{pre}\colon s^* \sim t^*\colon \mathrm{Map}(D,E) \to \mathrm{Map}(\mathrm{Eq}(D),E)$$

induces, for every homotopy $\alpha\colon f \sim g$ between functors $f, g\colon C \to D$, a homotopy

$$f^* = (s\alpha)^* = \alpha^* s^* \overset{\alpha^*\mathtt{pre}}{\sim} \alpha^* t^* = (t\alpha)^* = g^*$$

between functors $f^*, g^*\colon \mathrm{Map}(D,E) \to \mathrm{Map}(C,E)$.

Proposition 9.47. *Let $f\colon C \to D$ be a functor. The following are equivalent:*

(i) the functor f is an equivalence;
(ii) for every category A, the postcomposition functor $f_*\colon \mathrm{Map}(A,C) \to \mathrm{Map}(A,D)$ is an equivalence;
(iii) for every category E, the precomposition functor $f^*\colon \mathrm{Map}(D,E) \to \mathrm{Map}(C,E)$ is an equivalence.

Proof. The fact that (i) implies (ii) follows from Construction 9.42, and the fact that (i) implies (iii) from Construction 9.46.

We prove that (ii) implies (i). Taking $A = D$, we have that $f_*\colon \mathrm{Map}(D,C) \to \mathrm{Map}(D,D)$ is an equivalence. Let $\varphi\colon \mathrm{Map}(D,D) \to \mathrm{Map}(D,C)$ be an inverse of f_* and $\alpha\colon f_*\varphi \sim \mathrm{id}_{\mathrm{Map}(D,D)}$ be a homotopy. Then

$$[0] \xrightarrow{\mathrm{id}_D} \mathrm{Map}(D,D) \xrightarrow{\varphi} \mathrm{Map}(D,C)$$

corresponds to a functor $g\colon D \to C$. Moreover, the composite

$$[0] \xrightarrow{\mathrm{id}_D} \mathrm{Map}(D,D) \xrightarrow{\alpha} \mathrm{Eq}(\mathrm{Map}(D,D)) \xrightarrow{\simeq} \mathrm{Map}(D, \mathrm{Eq}(D))$$

corresponds to a homotopy

$$fg = f_*(g) = f_*\varphi(\mathrm{id}_D) \sim \mathrm{id}_D.$$

Now consider the equivalence $f_*\colon \mathrm{Map}(C,C) \to \mathrm{Map}(C,D)$. Since we have a homotopy

$$f_*(gf) = fgf \sim f = f_*(\mathrm{id}_C).$$

by Proposition 9.22, we get a homotopy $gf \sim \mathrm{id}_C$. This proves that f is an equivalence.

The fact that (iii) implies (i) works similarly, setting $g\colon D \to C$ to be the unique functor corresponding to

$$[0] \xrightarrow{\mathrm{id}_C} \mathrm{Map}(C,C) \xrightarrow{\varphi} \mathrm{Map}(D,C)$$

where φ denotes an inverse of $f^*\colon \mathrm{Map}(D,C) \to \mathrm{Map}(C,C)$. □

9.5. **Homotopy pullbacks**

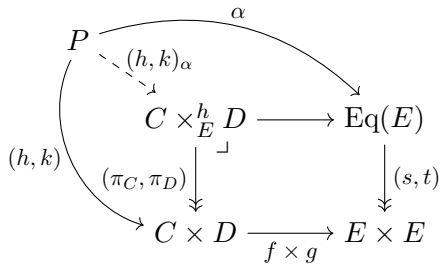
Construction 9.48. Let $f\colon C \to E$ and $g\colon D \to E$ be functors. Consider the following (strict) pullback

$$\begin{array}{ccc} C \times^h_E D & \longrightarrow & \mathrm{Eq}(E) \\ {\scriptstyle (\pi_C, \pi_D)}\downarrow & \lrcorner & \downarrow{\scriptstyle (s,t)} \\ C \times D & \xrightarrow[f \times g]{} & E \times E \end{array}$$

Given a commutative square

$$\begin{array}{ccc} P & \xrightarrow{h} & C \\ {\scriptstyle k}\downarrow & \alpha & \downarrow{\scriptstyle f} \\ D & \xrightarrow[g]{} & E \end{array}$$

with homotopy $\alpha\colon fh \sim gk$, we get an induced functor $(h,k)_\alpha\colon P \to C \times^h_E D$.

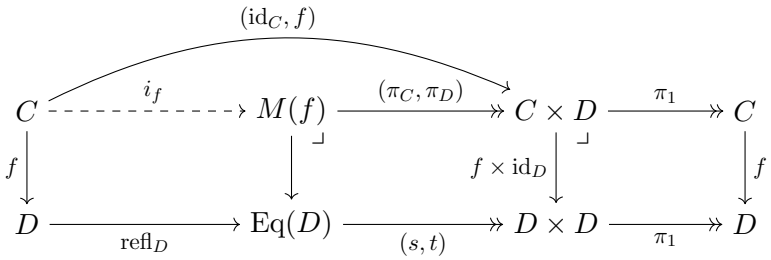


Definition 9.49. Let $f\colon C \to E$ and $g\colon D \to E$ be functors. A commutative square

$$\begin{array}{ccc} P & \xrightarrow{h} & C \\ {\scriptstyle k}\downarrow & \alpha & \downarrow{\scriptstyle f} \\ D & \xrightarrow[g]{} & E \end{array}$$

with homotopy $\alpha\colon fh \sim gk$ is a **(homotopy) pullback square** if the induced functor $(h,k)_\alpha\colon P \to C \times^h_E D$ is an equivalence of categories.

Construction 9.50. Let $f\colon C \to D$ be a functor. Consider the following strictly commutative diagram.



By cancellation, the left-hand square is a pullback square as well.

Hence, the equivalence over D

$$\begin{array}{ccc} D & \xrightarrow[\mathrm{refl}_D]{\simeq} & \mathrm{Eq}(D) \\ & {\scriptstyle \mathrm{id}_D}\searrow \quad \swarrow{\scriptstyle s} & \\ & D & \end{array}$$

induces, by pulling back along $f\colon C \to D$, the functor

$$\begin{array}{ccc} C & \xrightarrow[i_f]{\simeq} & M(f) \\ & {\scriptstyle \mathrm{id}_C}\searrow \quad \swarrow{\scriptstyle \pi_C} & \\ & C & \end{array}$$

which is an equivalence over C by Proposition 9.25. Using Proposition 9.24, since id_C and i_f are equivalences, then so is $\pi_C\colon M(f) \twoheadrightarrow C$ by 2-out-of-3. Moreover, the functor f factors as

$$C \xrightarrow{i_f} M(f) \xrightarrow{\pi_D} D$$

where i_f is an equivalence and π_D is a family projection.

Proposition 9.51. *A functor $f\colon C \to D$ is an equivalence if and only if $\pi_D\colon M(f) \twoheadrightarrow D$ is an equivalence.*

Proof. This is a direct consequence of the 2-out-of-3 property applied to $f = \pi_D \circ i_f$. □

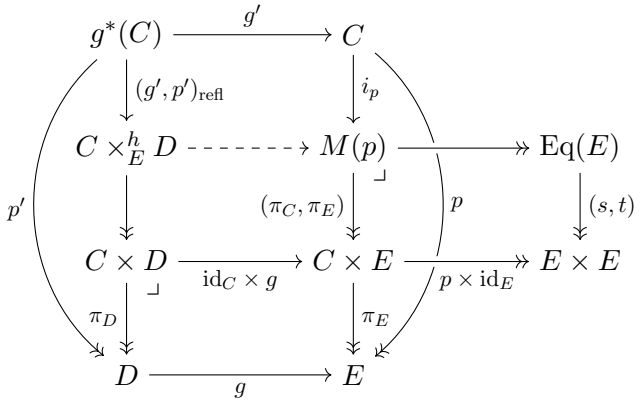
Proposition 9.52. *Every strict pullback along a family projection is a homotopy pullback.*

Proof. Let $p\colon C \twoheadrightarrow E$ be a family projection and $g\colon D \to E$ be a functor. Consider the strict pullback square

$$\begin{array}{ccc} g^*(C) & \xrightarrow{g'} & C \\ {\scriptstyle p'}\downarrow & \lrcorner & \downarrow{\scriptstyle p} \\ D & \xrightarrow[g]{} & E \end{array}$$

We show that the induced functor $(g', p')_{\mathrm{refl}}\colon g^*(C) \to C \times^h_E D$ is an equivalence.

Consider the following commutative diagram.



By cancellation, the middle left square is a pullback square, and so $C \times^h_E D \cong g^*M(p)$ over D. Hence, by Proposition 9.24, the functor

$$\begin{array}{ccc} C & \xrightarrow[i_p]{\simeq} & M(p) \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle \pi_E} & \\ & E & \end{array}$$

is an equivalence over E, and it induces, by pulling back along $g\colon D \to E$, the functor

$$\begin{array}{ccc} g^*(C) & \xrightarrow[(g', p')_{\mathrm{refl}}]{\simeq} & C \times^h_E D \\ & {\scriptstyle p'}\searrow \quad \swarrow{\scriptstyle \pi_D} & \\ & D & \end{array}$$

which is an equivalence over D by Proposition 9.25. □

Proposition 9.53. *A commutative square*

$$\begin{array}{ccc} P & \xrightarrow{h} & C \\ {\scriptstyle k}\downarrow & \alpha & \downarrow{\scriptstyle f} \\ D & \xrightarrow[g]{} & E \end{array}$$

is a pullback square if and only if, for every category A, the induced functor

$$(h_*, k_*)_{\mathtt{post}\,\alpha_*} \colon \mathrm{Map}(A, P) \to \mathrm{Map}(A, C) \times^h_{\mathrm{Map}(A,E)} \mathrm{Map}(A, D) \tag{9.54}$$

is an equivalence of animae.

Proof. First observe that we have canonical equivalences

$$\begin{aligned}
\mathrm{Map}(A, C \times^h_E D) &= \mathrm{Map}(A, (C \times D) \times_{E\times E} \mathrm{Eq}(E)) \\
&\cong (\mathrm{Map}(A, C) \times \mathrm{Map}(A, D)) \times_{\mathrm{Map}(A,E)\times\mathrm{Map}(A,E)} \mathrm{Map}(A, \mathrm{Eq}(E)) \\
&\simeq (\mathrm{Map}(A, C) \times \mathrm{Map}(A, D)) \times_{\mathrm{Map}(A,E)\times\mathrm{Map}(A,E)} \mathrm{Eq}(\mathrm{Map}(A, E)) \\
&= \mathrm{Map}(A, C) \times^h_{\mathrm{Map}(A,E)} \mathrm{Map}(A, D).
\end{aligned}$$

where we use that $\mathrm{Map}(A, -)$ preserves strict pullbacks and products, and Axiom C.

Then the square is a pullback square if and only if, by definition, the functor

$$(h, k)_\alpha \colon P \to C \times^h_E D$$

is an equivalence. By Proposition 9.47, this holds if and only if, for every category A, the induced functor $((h, k)_\alpha)_* \colon \mathrm{Map}(A, P) \to \mathrm{Map}(A, C \times^h_E D)$ is an equivalence. By the above computation and applying 2-out-of-3 to the following commutative triangle

$$\begin{array}{ccc}
\mathrm{Map}(A, P) & \xrightarrow{((h,k)_\alpha)_*} & \mathrm{Map}(A, C \times^h_E D) \\
& \searrow^{(h_*, k_*)_{\mathtt{post}\,\alpha_*}} & \downarrow \simeq \\
& & \mathrm{Map}(A, C) \times^h_{\mathrm{Map}(A,E)} \mathrm{Map}(A, D)
\end{array}$$

this holds if and only if, for every category A, the functor (9.54) is an equivalence. □

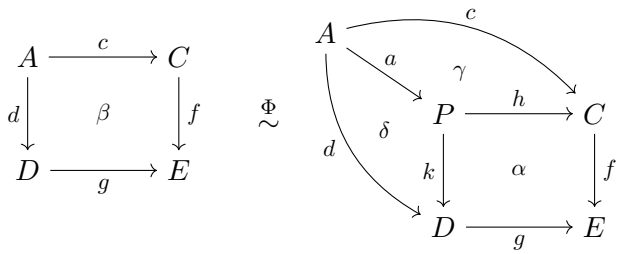
Remark 9.55. Unpacking the above universal property, for every commutative square

$$\begin{array}{ccc}
A & \xrightarrow{c} & C \\
{\scriptstyle d}\downarrow & \beta & \downarrow{\scriptstyle f} \\
D & \xrightarrow[g]{} & E
\end{array}$$

by applying the inverse of (9.54), we get a functor $a \colon A \to P$. By using the homotopy relating their composite with the identity, we get homotopies $\gamma \colon c \sim ha$, $\delta \colon ka \sim d$, and $\Phi \colon \beta \sim f\gamma * \alpha a * g\delta$, where

$$fc \overset{f\gamma}{\sim} fha \overset{\alpha a}{\sim} gka \overset{g\delta}{\sim} gd$$

Pictorially, this can be represented as:



Since the equivalence (9.54) reflects homotopies, given another functor $a' \colon A \to P$ as well as homotopies $\sigma \colon ha \sim ha'$, $\tau \colon ka \sim ka'$, and $\Psi \colon f\sigma * \alpha a' \sim \alpha a * g\tau$, where

$$fha \overset{f\sigma}{\sim} fha' \overset{\alpha a'}{\sim} gka' \quad \text{and} \quad fha \overset{\alpha a}{\sim} gka \overset{g\tau}{\sim} gka'$$

we get a homotopy $\theta \colon a \sim a'$. Higher coherences further imply that there are homotopies $h\theta \sim \sigma$, $k\theta \sim \tau$. Etc.

Proposition 9.56. *Consider a commutative square*

$$\begin{array}{ccc} P & \xrightarrow{h} & C \\ {\scriptstyle k}\downarrow & \alpha \quad \simeq & \downarrow{\scriptstyle f} \\ D & \xrightarrow[g]{} & E \end{array}$$

where f is an equivalence. Then it is a pullback square if and only if k is an equivalence.

Proof. Consider the following strictly commutative diagram.

$$\begin{array}{ccccc} C \times_E^h D & \dashrightarrow & M(f) & \longrightarrow & \mathrm{Eq}(E) \\ {\scriptstyle (\pi_C, \pi_D)}\downarrow & & {\scriptstyle (\pi_C, \pi_E)}\downarrow & & \downarrow{\scriptstyle (s,t)} \\ C \times D & \xrightarrow[\mathrm{id}_C \times g]{} & C \times E & \xrightarrow[f \times \mathrm{id}_E]{} & E \times E \\ {\scriptstyle \pi_2}\downarrow & & \downarrow{\scriptstyle \pi_2} & & \\ D & \xrightarrow[g]{} & E & & \end{array}$$

By cancellation, the top left square is a strict pullback square. Hence the composite of the two vertical squares is a strict pullback square. By Propositions 9.24 and 9.51, the functor

$$\begin{array}{ccc} M(f) & \xrightarrow[\pi_E]{\simeq} & E \\ & {\scriptstyle \pi_E}\searrow \quad \swarrow{\scriptstyle \mathrm{id}_E} & \\ & E & \end{array}$$

is an equivalence over E, and it induces, by pulling back along $g \colon D \to E$, the functor

$$\begin{array}{ccc} C \times_E^h D & \xrightarrow[\pi_D]{\simeq} & D \\ & {\scriptstyle \pi_D}\searrow \quad \swarrow{\scriptstyle \mathrm{id}_D} & \\ & D & \end{array}$$

which is an equivalence over D by Proposition 9.25.

By applying 2-out-of-3 to the following commutative triangle

$$\begin{array}{ccc} P & \xrightarrow{(h,k)_\alpha} & C \times_E^h D \\ & {\scriptstyle k}\searrow & \downarrow{\scriptstyle \simeq} \\ & & D \end{array}$$

we see that k is an equivalence if and only if $(h,k)_\alpha$, i.e., the square is a pullback square. □

Proposition 9.57 (Pullback cancellation)**.** *Consider commutative squares*

$$\begin{array}{ccccc} R & \longrightarrow & P & \longrightarrow & C \\ \downarrow & \beta & \downarrow{\scriptstyle k} & \alpha & \downarrow{\scriptstyle f} \\ B & \xrightarrow[h]{} & D & \xrightarrow[g]{} & E \end{array}$$

where the right-hand square is a pullback square. Then the left-hand square is a pullback square if and only if the total rectangle is a pullback square.

Proof. Exercise. □

Proposition 9.58. *Consider a commutative diagram*

$$\begin{array}{ccccc} C & \xrightarrow{f} & E & \xleftarrow{g} & D \\ \simeq\downarrow & \beta & \simeq\downarrow & \gamma & \downarrow\simeq \\ C' & \xrightarrow[f']{} & E' & \xleftarrow[g']{} & D' \end{array}$$

where the vertical functors are equivalences. Consider pullback squares

$$\begin{array}{ccc} P & \xrightarrow{h} & C \\ {\scriptstyle k}\downarrow & \alpha & \downarrow{\scriptstyle f} \\ D & \xrightarrow[g]{} & E \end{array} \qquad\qquad \begin{array}{ccc} P' & \xrightarrow{h'} & C' \\ {\scriptstyle k'}\downarrow & \alpha' & \downarrow{\scriptstyle f'} \\ D' & \xrightarrow[g']{} & E' \end{array}$$

Then there is an equivalence $P \xrightarrow{\simeq} P'$.

Proof. Exercise. □

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